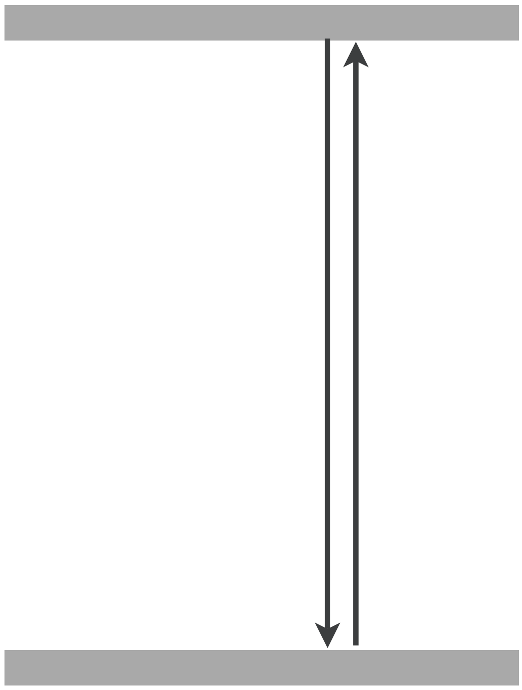
Derivation of Time Dilation



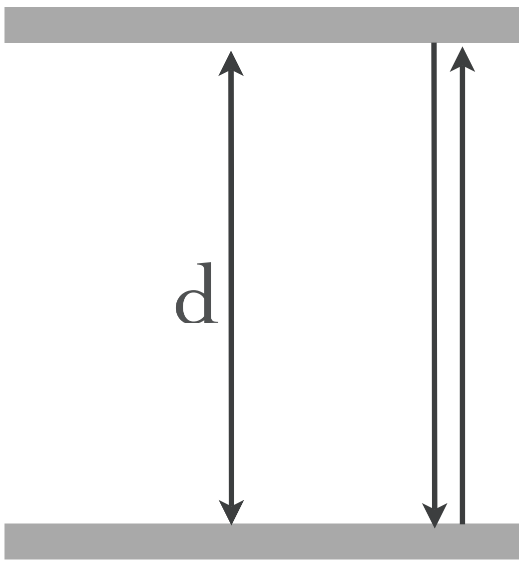
- Imagine a light clock which consists of two mirrors and beam of light reflecting back and forth between the mirror

- One "tick" is when the light goes from one mirror to the other and back again

- Scenario 1

  - You are in the same inertial frame as the light clock

  - You are therefore measuring the proper time, denoted Δt_0



- The Mirrors are separated by distance "d"

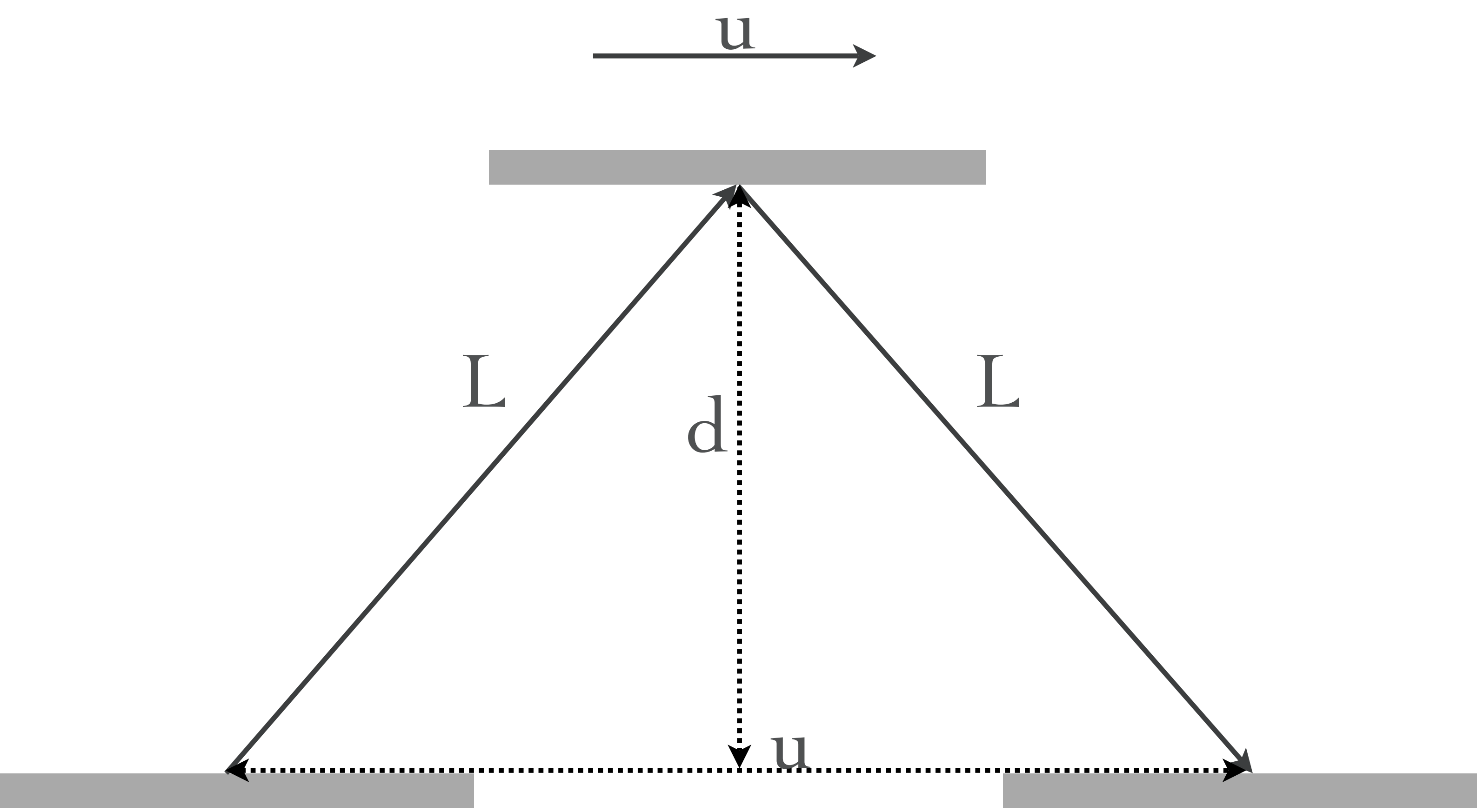
- The light moves with a speed "c"

- The proper time for one tick is given by:

$$\Delta t_0 = \frac{2d}{c}$$

- Scenario 2

  - You are in a different inertial frame to the light clock
  - The light clock is moving with velocity u ms^{-1} in the x direction
  - The path the light takes is now different as it has only a finite speed
  - The time taken for one tick is denoted Δt, the observed time



- Applying Pythagorus' theorem the length L is given by:

$$L = \sqrt{d^2 + \left(\frac{u\Delta t}{2}\right)^2}$$

- The value for the observed time is:

$$\Delta t = \frac{2L}{c}$$

- Combine the two equations above

$$\Delta t = \frac{2}{c}\sqrt{d^2 + \left(\frac{u\Delta t}{2}\right)^2}$$

- Rearrange the equation for proper time to make "d" the subject of the equation

$$d = \frac{c\Delta t_0}{2}$$

- Combine the two equations above

$$\Delta t = \frac{2}{c}\sqrt{\left(\frac{c\Delta t_0}{2}\right)^2 + \left(\frac{u\Delta t}{2}\right)^2}$$

- This equation can now be solved to make the observed time the subject of the formula

- Square both sides and simplify

$$\Delta t^2 = \frac{4}{c^2}\left[\frac{c^2\Delta t_0^2}{4} + \frac{u^2\Delta t^2}{4}\right]$$

$$\Delta t^2 = \Delta t_0^2 + \frac{u^2\Delta t^2}{c^2}$$

- Collect like terms on the same side of the equation

$$\Delta t^2 - \frac{u^2\Delta t^2}{c^2} = \Delta t_0^2$$

- Factorise the left hand side

$$\Delta t^2\left(1 - \frac{u^2}{c^2}\right) = \Delta t_0^2$$

- Rearrange to make Δt the subject of the formula

$$\Delta t^2 = \frac{\Delta t_0^2}{\left(1 - \frac{u^2}{c^2}\right)}$$

- Take the square root of both sides

$$\Delta t = \frac{\Delta t_0}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}$$

Quod erat demonstrandum