Mathematical Concepts (G6012)

Lecture 16

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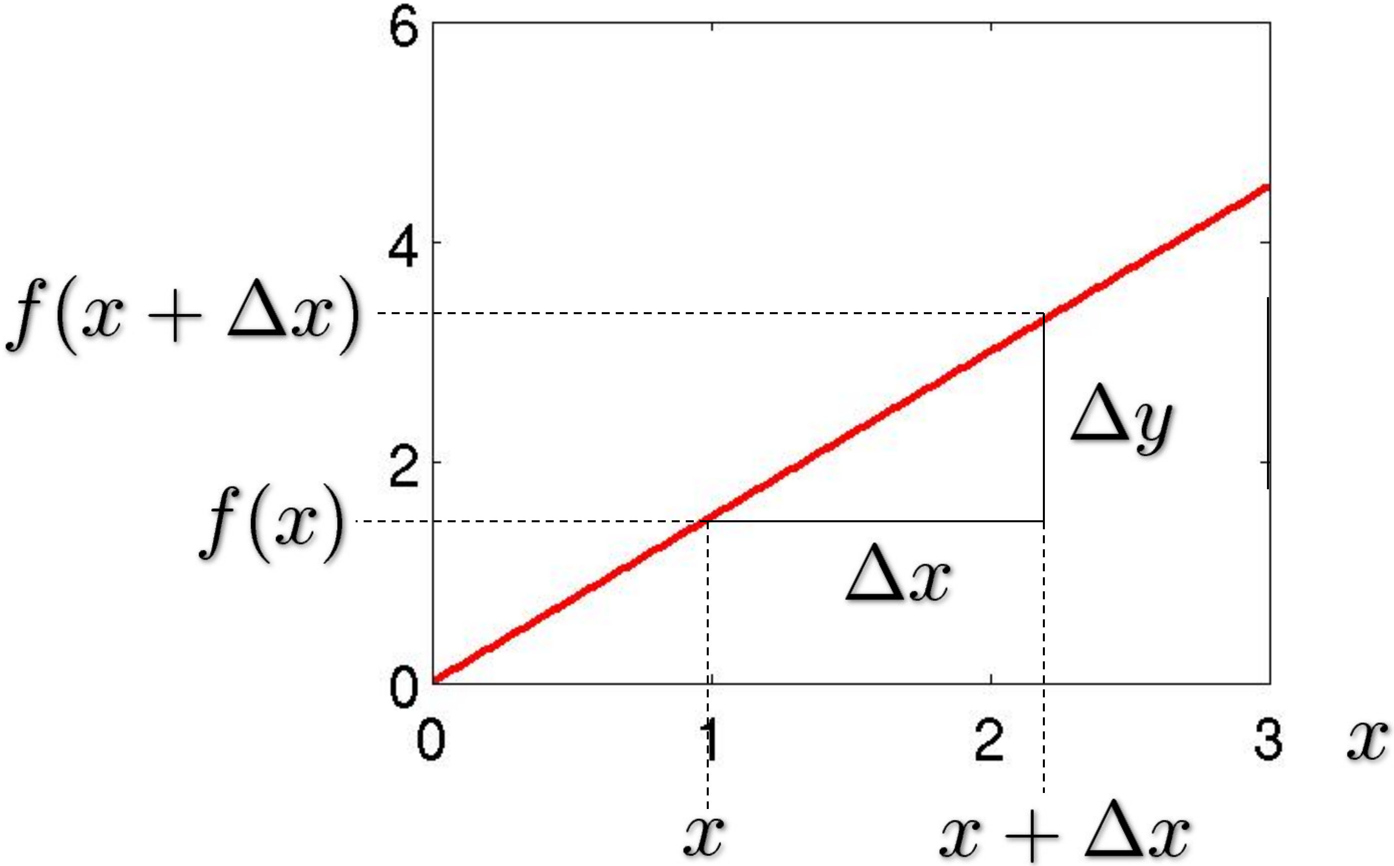
DERIVATIVES

BB

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$f(x) = 1.5 \cdot x$



Derivative of a smooth function

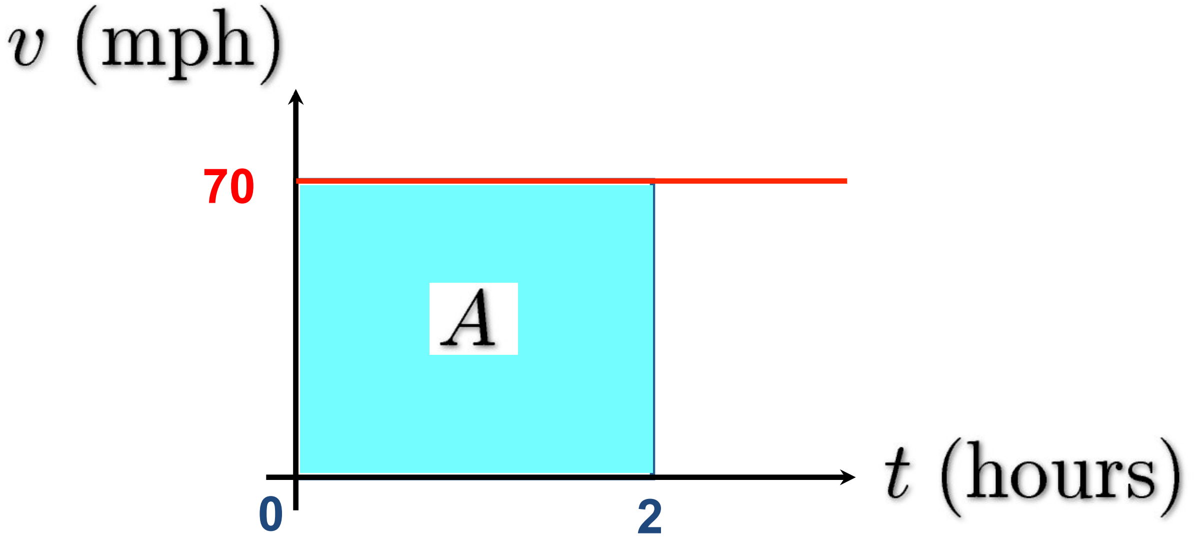
- The derivative of a smooth function is the value the ratio $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ converges to for smaller and smaller Δx, mathematicians write

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

INTEGRATION

Area under a graph

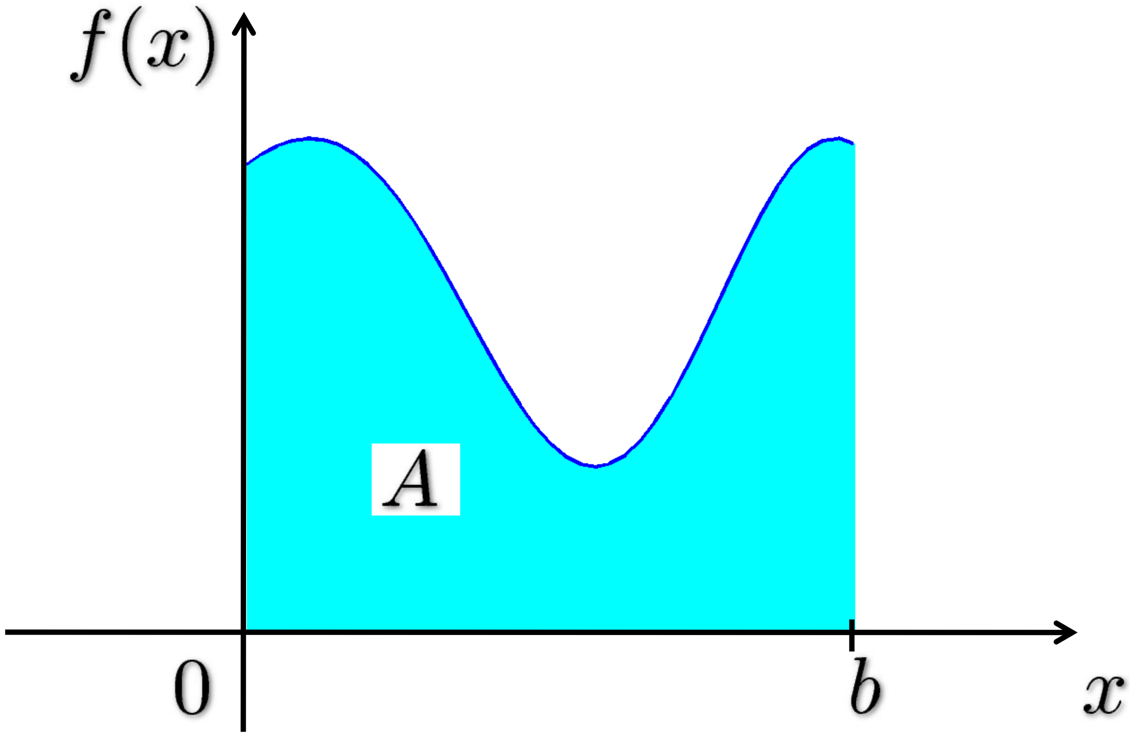
- Car travelling at 70 mph



Area = distance traveled:

$$A = v \cdot t = 70 \cdot 2 \text{ miles} = 140 \text{ miles}$$

What if we are interested in the area under this curve:

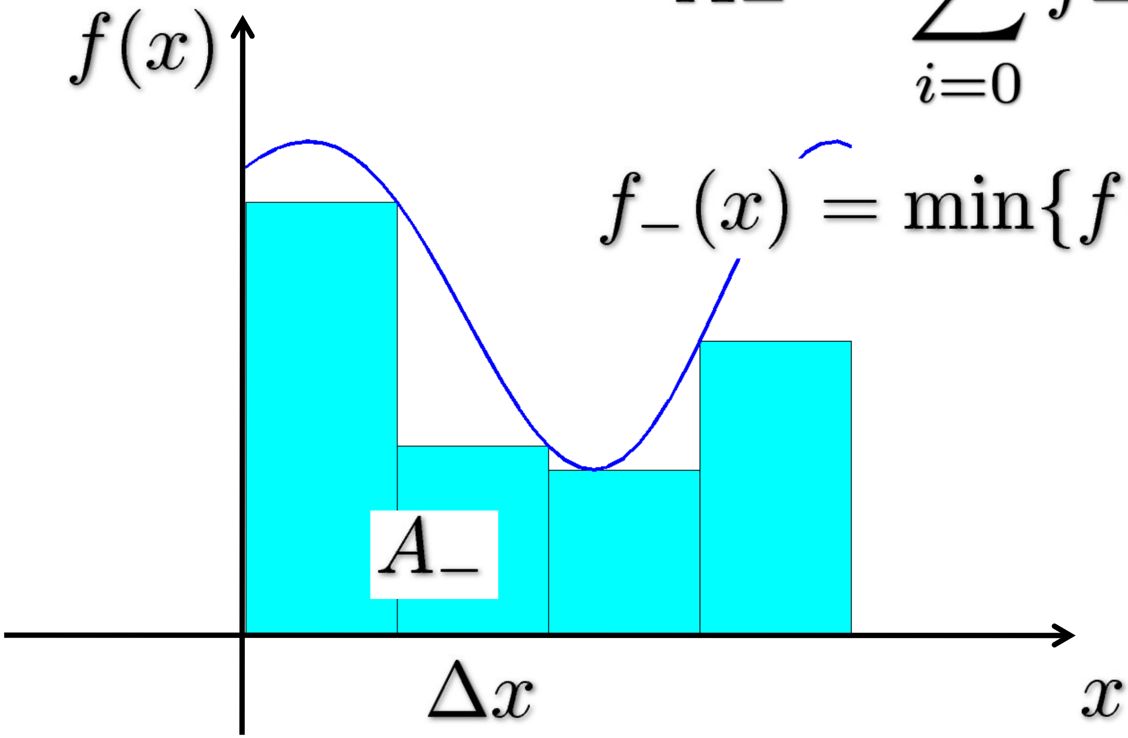


$A = ?$

Try something we know about

$$A_- = \sum_{i=0}^{4} f_-(i \cdot \Delta x) \cdot \Delta x$$

$$f_-(x) = \min\{f(z) : z \in [x, x + \Delta x]\}$$

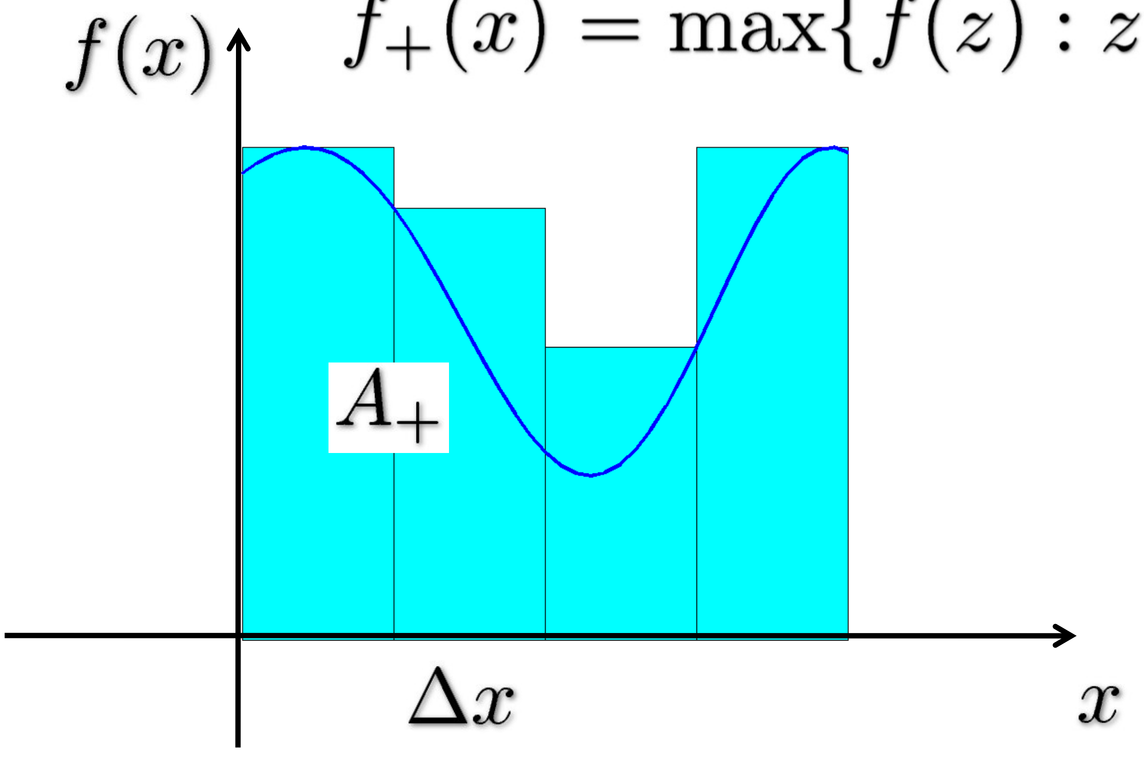


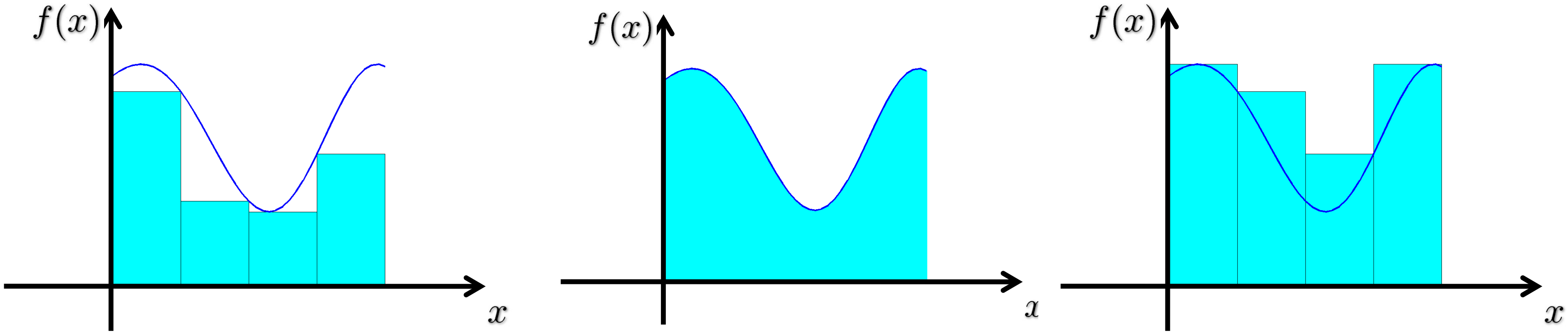
This is called “lower Riemann sum”

Upper Riemann sum

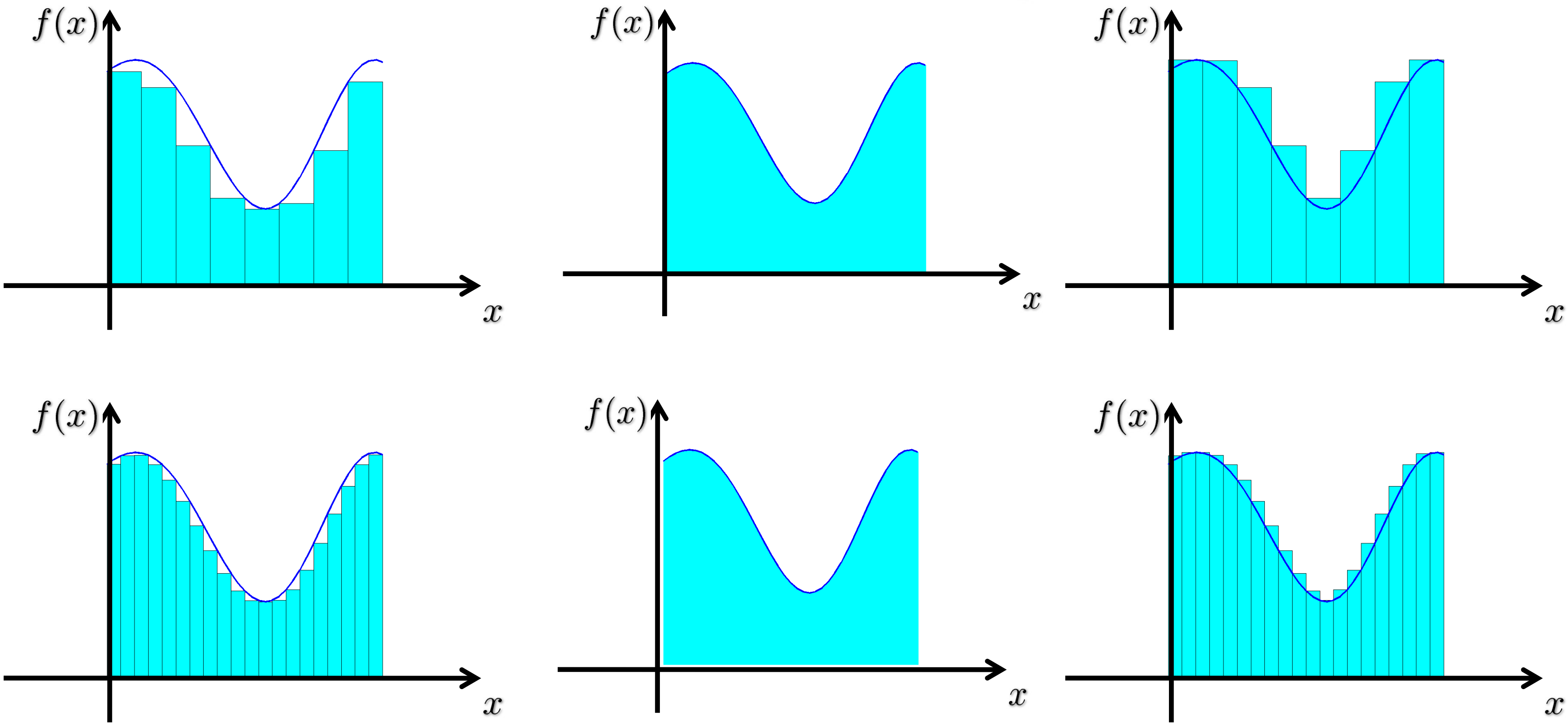
$$A_+ = \sum_{i=0}^{4} f_+(i \cdot \Delta x) \cdot \Delta x$$

$$f_+(x) = \max\{f(z) : z \in [x, x + \Delta x]\}$$





$$A_- \leq A \leq A_+$$



Riemann integral

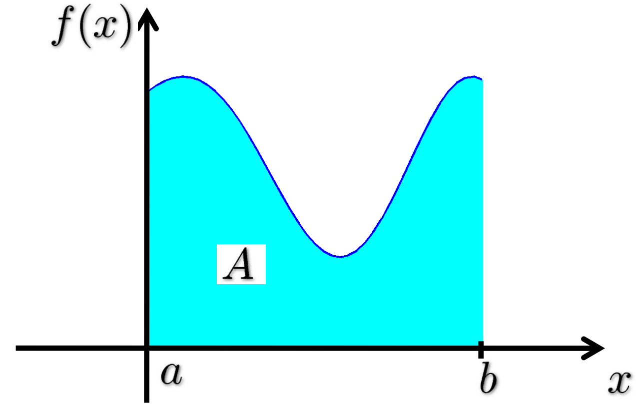
For many functions

$$\lim_{\Delta x \to 0} A_{-} = \lim_{\Delta x \to 0} A_{+}$$

The upper and lower Riemann sum become the same for small steps.

Such functions are called “Riemann integrable”, and

(Riemann) integral



$$A = \lim_{\Delta x \to 0} A_- = \lim_{\Delta x \to 0} A_+$$

is called “(Riemann) integral”

Notation:

upper limit

$$A = \int_a^b f(x)\,dx$$

integration variable (dummy variable)

Integral sign

lower limit

Example from first principles

$$\int_0^T x\,dx = \lim_{\Delta x \to 0} \sum_{i=0}^{T/\Delta x} (i \cdot \Delta x) \cdot \Delta x$$

$$= \lim_{N \to \infty} \sum_{i=0}^{NT} \frac{i}{N} \cdot \frac{1}{N} = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i=0}^{NT} i$$

$$= \lim_{N \to \infty} \frac{1}{N^2} \Big(\frac{NT(NT+1)}{2} \Big)$$

$$= \lim_{N \to \infty} \frac{1}{2} T^2 + \frac{T}{2N} = \frac{1}{2} T^2$$

Main theorem of differential and integral calculus

In principle, one could calculate integrals from first principles, but fortunately…

Integration is the opposite of differentiation!

$$\frac{d}{db}\left(\int_a^b f(x)\,dx\right) = f(b)$$

Plausibility argument

Take a difference

Differentiation

$$f'(x) = \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

divide by Δx

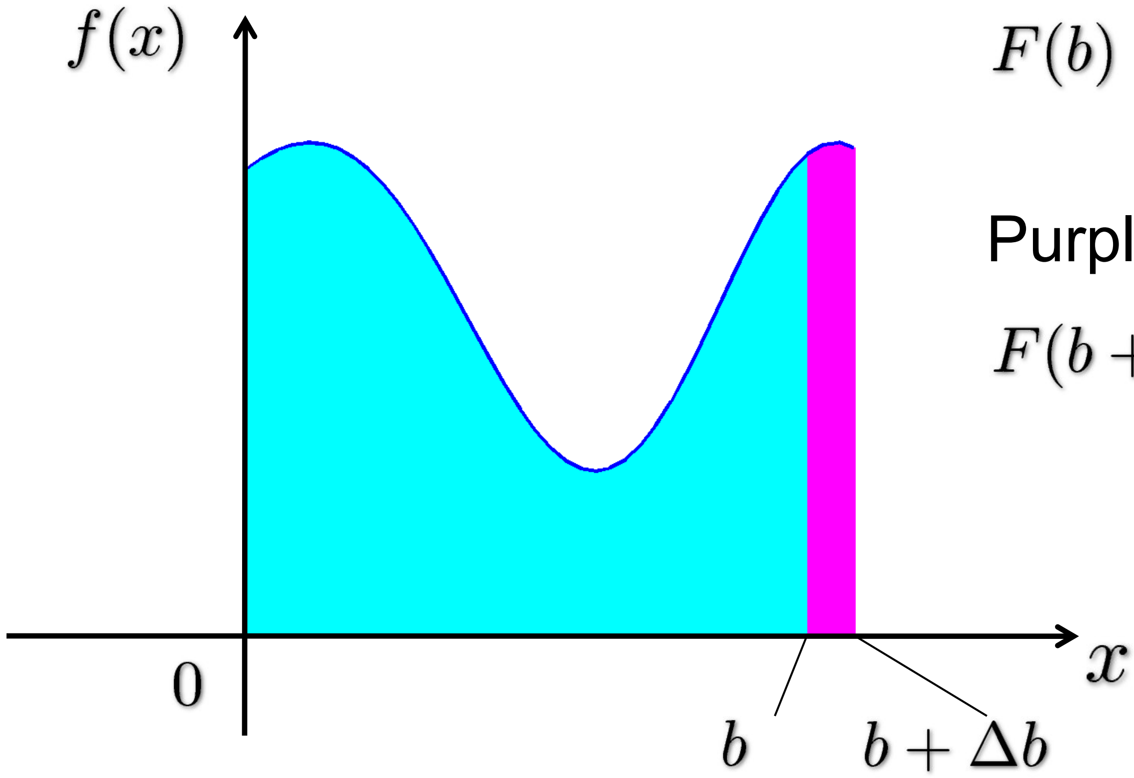
Integration

Sum it up

$$\int_0^T f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=0}^{T/\Delta x} f(i \cdot \Delta x) \cdot \Delta x$$

multiply with Δx

Differentiation inverts integration



Area from 0 to b (cyan):

$$F(b) = \int_0^b f(x)\, dx$$

Purple area from b to b+Δb:

$$F(b + \Delta b) - F(b)$$

$$\approx f(b) \Delta b$$

$$\frac{F(b + \Delta b) - F(b)}{\Delta b} \approx f(b)$$

$$\frac{d}{db} F(b) = f(b)$$

[Khan Academy proof]

Rules of Differentiation

Rule name	Function	Derivative
Polynomials	$f(x) = x^n$	$f'(x) = n\, x^{n-1}$
Constant factor	$g(x) = a\, f(x)$	$g'(x) = a\, f'(x)$
Sum and Difference	$h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$

Become rules of integration

Rule name	~~Function~~ Integral	~~Derivative~~ Function
Polynomials	$\int_0^x f(t)dt = x^n + C$	$f(x) = nx^{n-1}$
Constant factor	$\int_0^x g(t)dt = a \int_0^x f(t)dt$	$g(x) = a\, f(x)$
Sum and Difference	$\int_0^x h(t)dt = \int_0^x f(t)dt + \int_0^x g(t)dt$	$h(x) = f(x) + g(x)$

Special functions

Function	Integral
$f(x) = x^n$	$\int_0^x t^n dt = \frac{1}{n+1} x^{n+1}$
$\exp(x) = e^x$	$\exp(x) = e^x$
$\frac{1}{x}$	$\log(x) = \ln(x)$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$

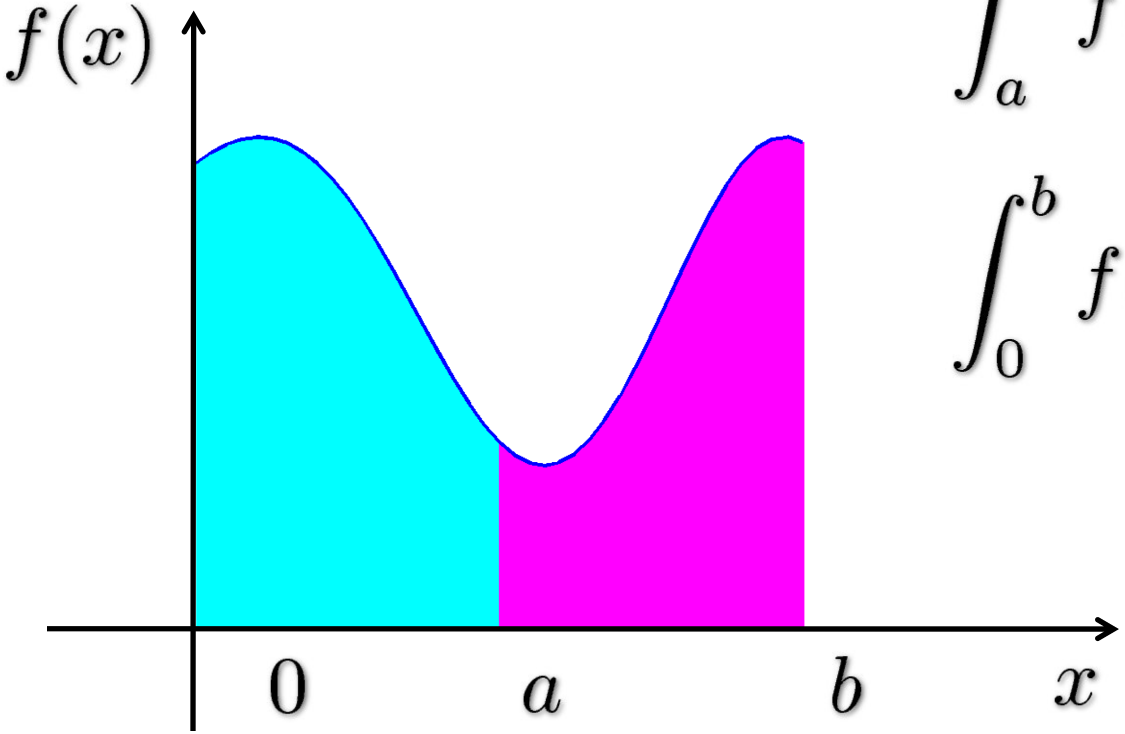
Integration constant

The “antiderivative”, “primitive integral” or “indefinite integral” is only defined up to a constant:

$$\frac{d}{dx}F(x) = f(x)$$

$$\frac{d}{dx}F(x) + C = f(x)$$

Practical tips



$$\int_a^b f(x)\,dx =$$

$$\int_0^b f(x)\,dx - \int_0^a f(x)\,dx$$

$$= F(b) - F(a)$$

Note how the integration constant does not matter here.

Practical tips II

$$\int_a^b f(x)\,dx = \big[\,F(x)\,\big]_a^b = F(b) - F(a)$$

"anti-derivative"

Example:

$$\int_2^3 x^3\,dx = \Big[\,\frac{1}{4}x^4\,\Big]_2^3 = \frac{1}{4}3^4 - \frac{1}{4}2^4 =$$

$$\frac{81-16}{4} = \frac{65}{4}$$

Example

$$\int_1^2 x^3 + 2x\,dx = [\frac{1}{4}x^4 + 2 \cdot \frac{1}{2}x^2]_1^2$$

$$= 4 + 4 - (\frac{1}{4} + 1) = \frac{27}{4}$$

More Examples

$$\int_1^2 2x\,\exp\left(x^2\right)\,dx = \left[\exp\left(x^2\right)\right]_1^2 = \exp(4) - \exp(1)$$

$$\int_1^2 \frac{1}{x}\,dx = \left[\,\log(x)\right]_1^2 = \log(2) - \log(1) = \log(2)$$

$$\begin{aligned}\int_1^2 \sin(x)\,dx = \left[\,-\cos x\right]_1^2 &= -\cos(2) - (-\cos(1)) \\ &= \cos(1) - \cos(2)\end{aligned}$$