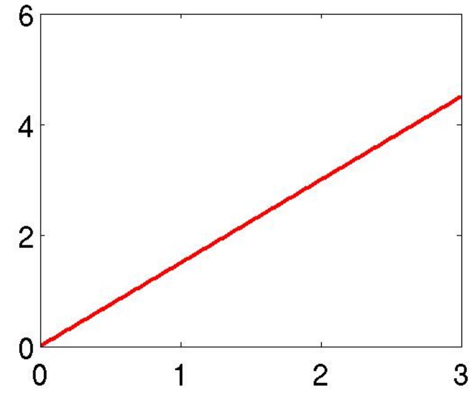
Mathematical Concepts (G6012)

Lecture 16

Thomas Nowotny
Chichester I, Room CI-105
Office hours: Tuesdays 15:00-16:45
T.Nowotny@sussex.ac.uk

DERIVATIVES

Slope, Derivative



First for **linear functions**:

$$f(x) = a \cdot x$$

The slope or derivative is the ratio of the change of f(x) and the change of x.

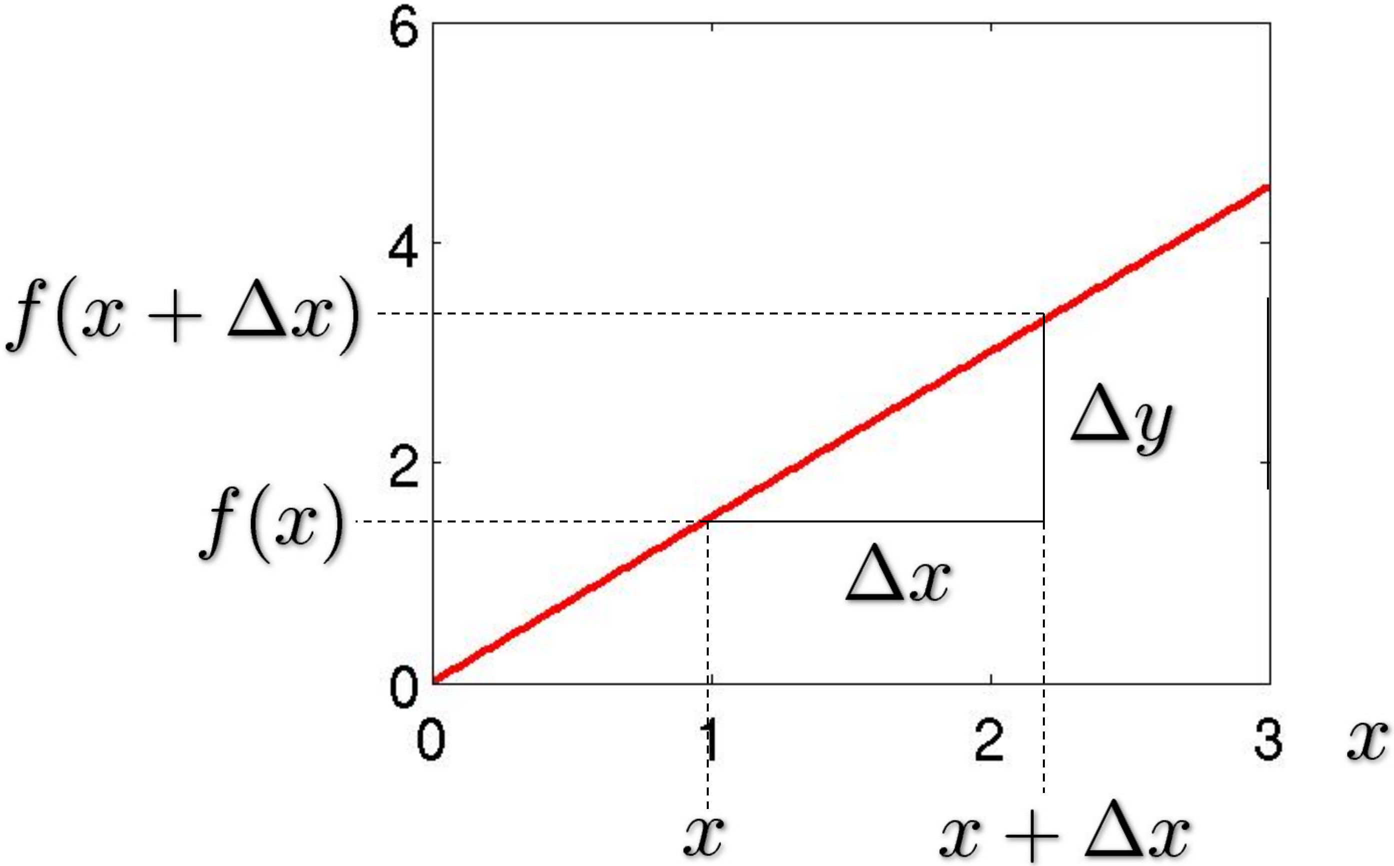
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

BB

BB

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f(x) = 1.5 \cdot x$$



BB Calculating the linear example

$$f(x) = 1.5 \cdot x$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

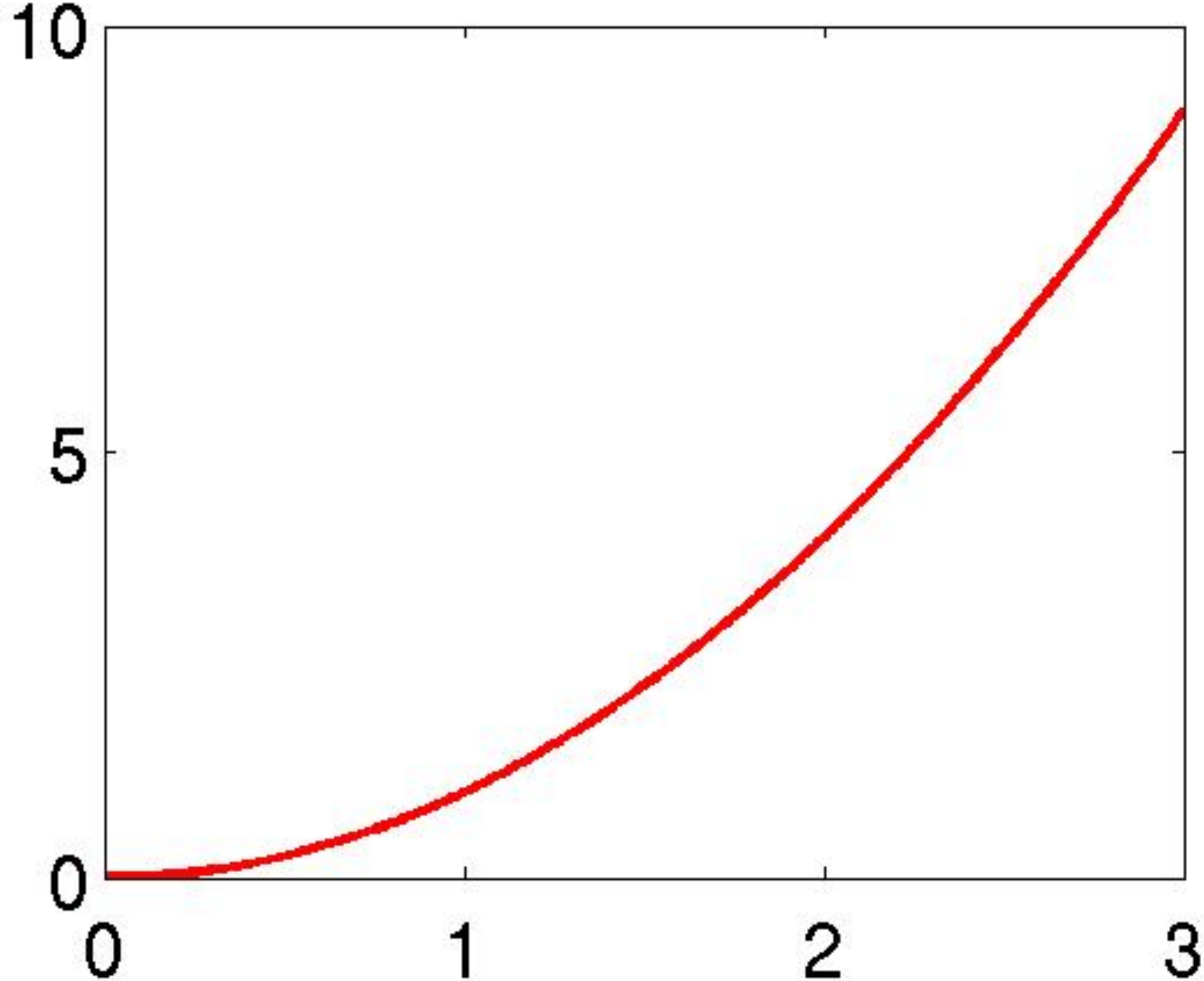
$$= \frac{1.5 \cdot (x + \Delta x) - 1.5 \cdot x}{\Delta x} = 1.5$$

Linear functions are easy …

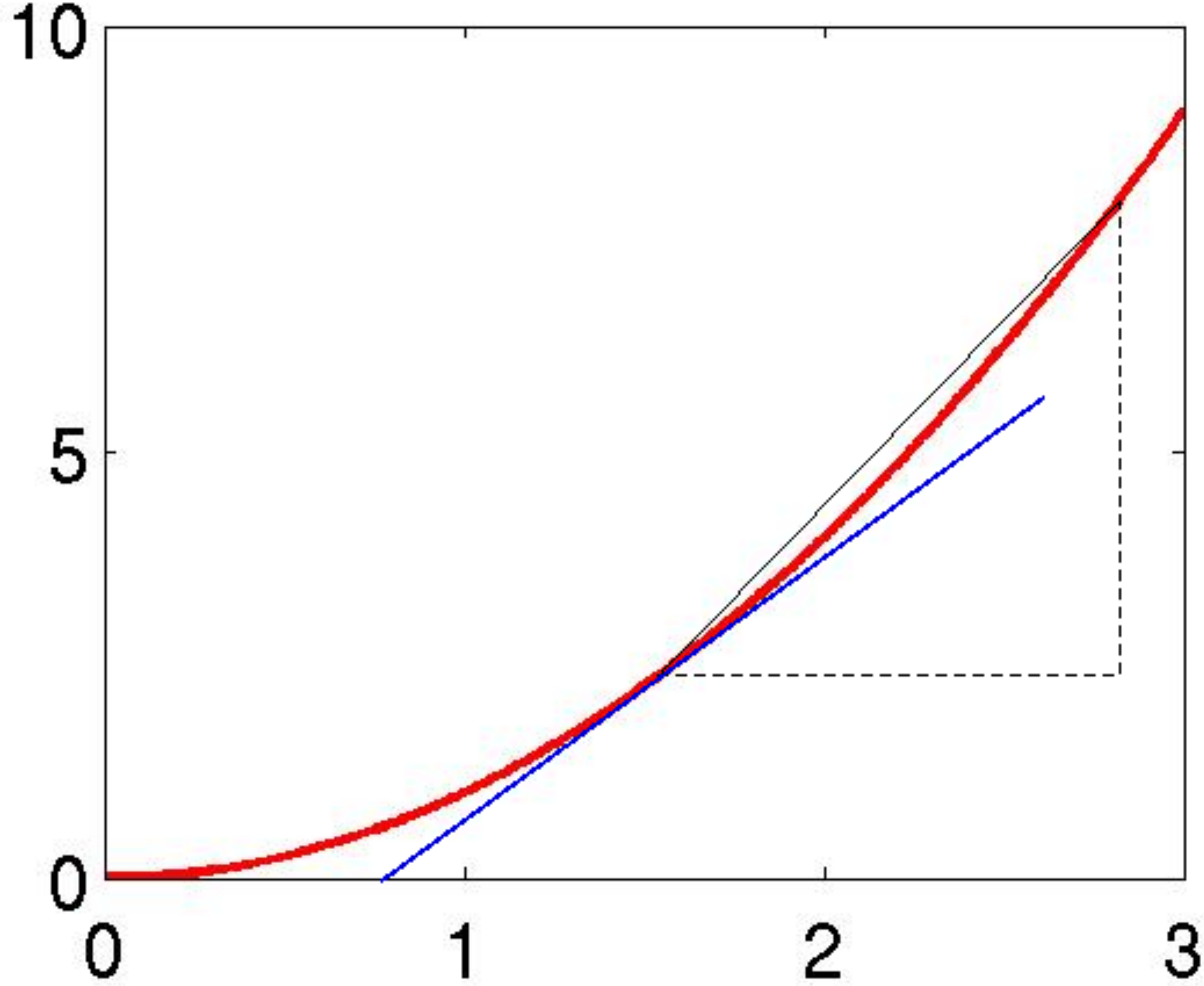
- Because the slope is the same everywhere
- We can make Δx any size we want and get the same value

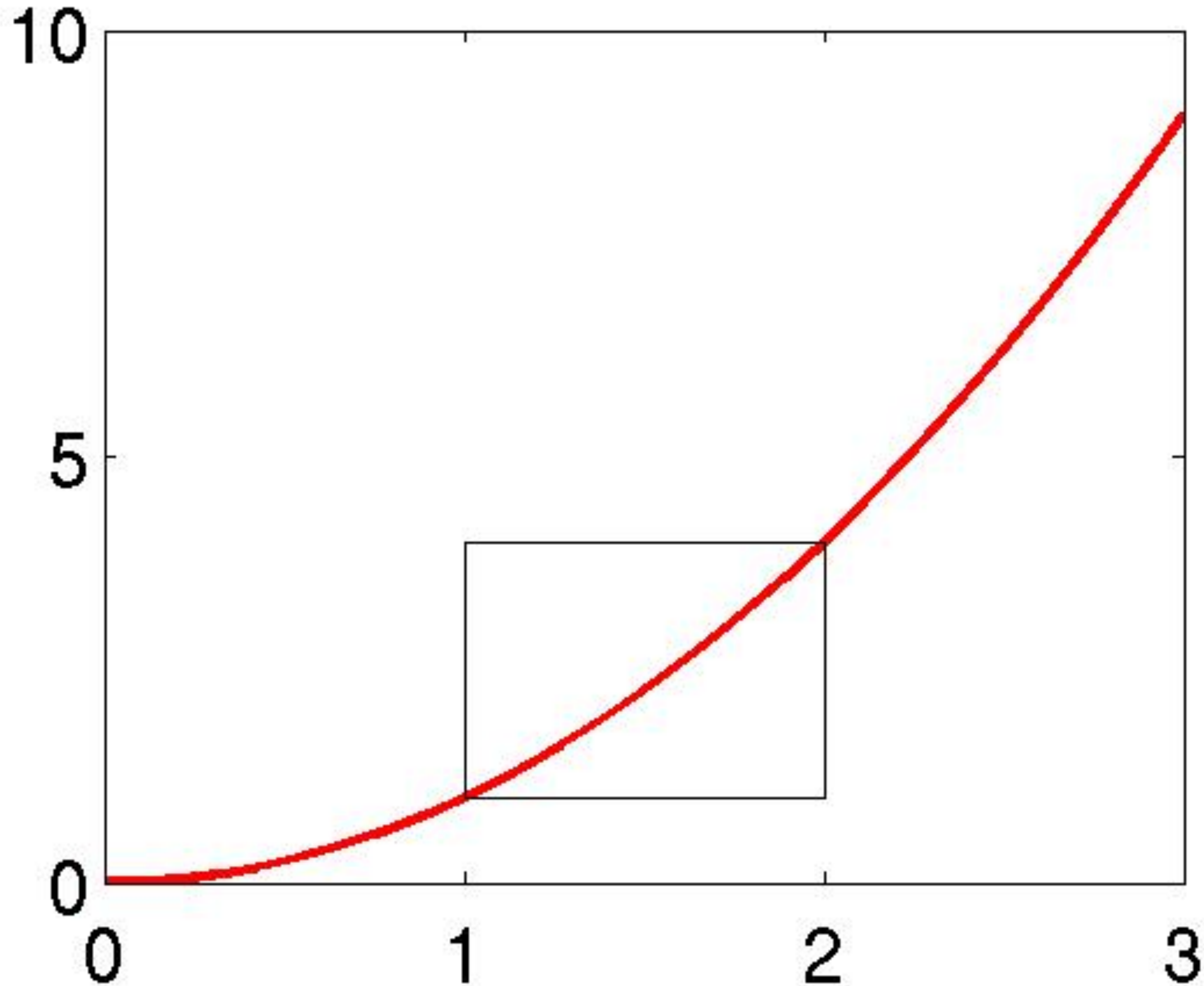
Generalisation for any smooth function

- Locally **any** smooth function looks more and more linear the further we zoom in:

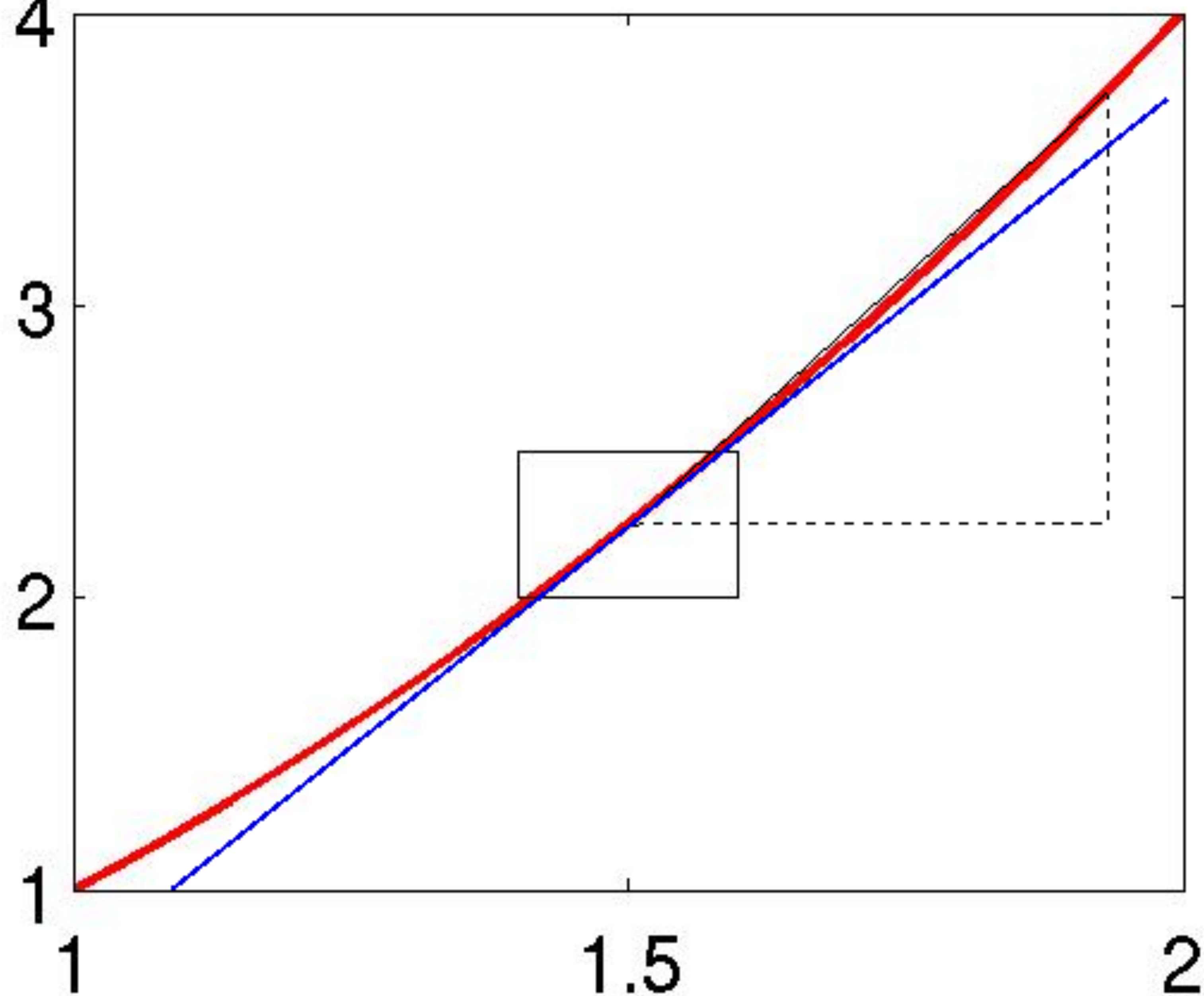


10
5
0
0
1
2
3

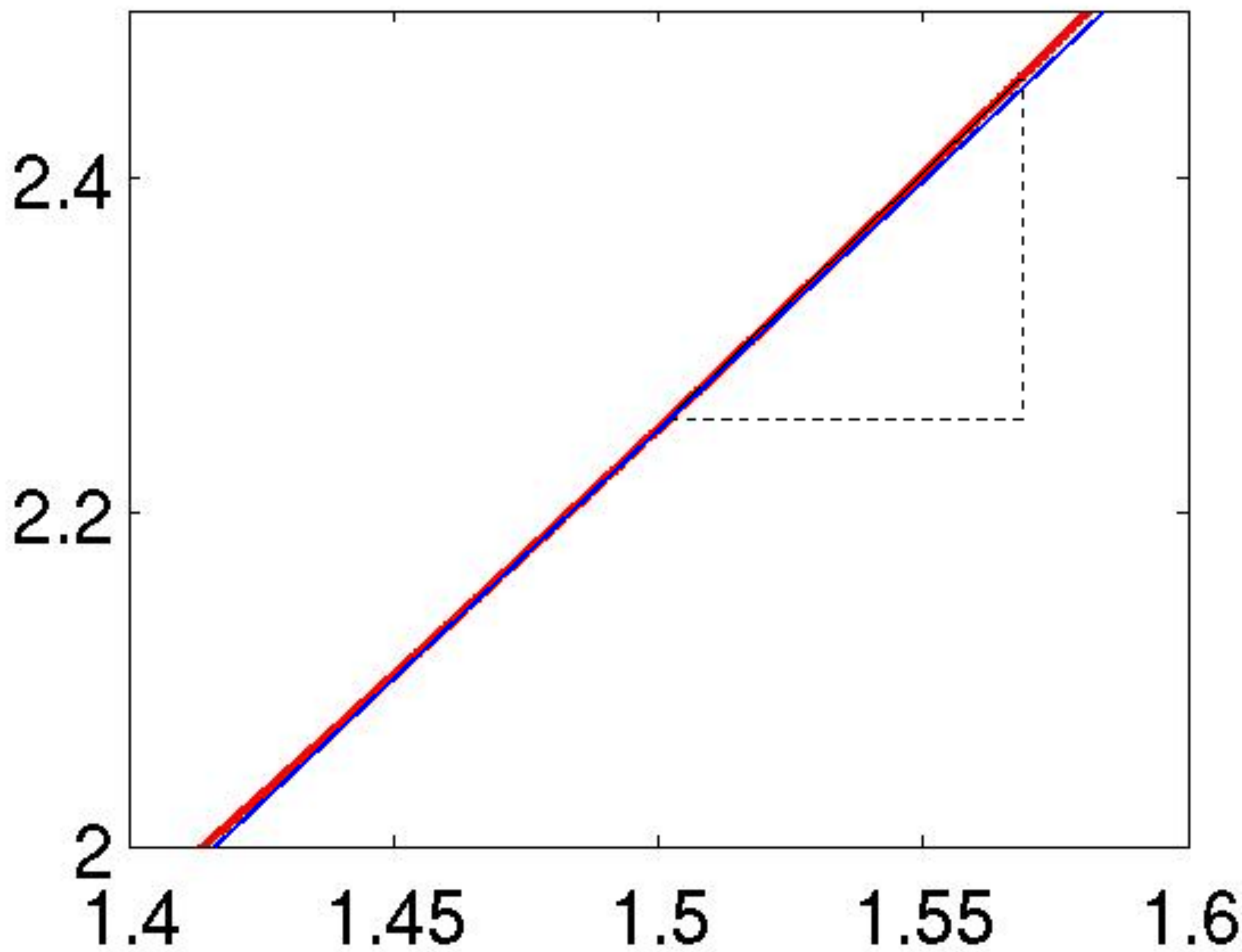

10
5
0
0
1
2
3



10
5
0
0
1
2
3



4
3
2
1
1
1.5
2



2.4
2.2
2
1.4
1.45
1.5
1.55
1.6

Derivative of a smooth function

- The derivative of a smooth function is the value the ratio $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ converges to for smaller and smaller Δx, mathematicians write

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

In the form as for the limits before

Sequence $a_n = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$

$$f'(x) = \lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$$

Alternative notations

If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function

$$x \mapsto f(x)$$

Then the derivative of f is denoted as

$$f'(x) = \frac{df(x)}{dx} = \frac{df}{dx} = \frac{d}{dx}f$$

Note …

The derivative $f'(x)$ of a function is again a function because we can calculate it for any point x.

BB

Example - Derivative of $f(x) = x^2$

$$\frac{d}{dx}x^2 = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Applications

- If $f(x)$ is your **distance** from home as a function of the time x. Then $f'(x)$ is the **speed** you are driving towards (or away from) home.
- If you take the derivative of the derivative $f''(x)$, that would be your **acceleration.**

(These are important for animating things!)

More Applications

- If *f(x)* describes the **height** of a hill, then *f'(x)* is the **steepness.**
- *f(x)* is your **total money** as a function of time, *f'(x)* is your instantaneous **spending rate.**
- (your example here)

Derivative of a polynomial

- We saw:

$f(x) = ax$ then $f'(x) = \frac{d}{dx} f(x) = a$

- For $f(x) = x^2$ we saw just now $f'(x) = 2x$

- Generally, for $f(x) = x^n$
  the derivative is $f'(x) = nx^{n-1}$

BG For those interested: General case

$$f(x) = x^n$$

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\,\Delta x + \mathcal{O}((\Delta x)^2) - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} nx^{n-1} + \mathcal{O}(\Delta x) = nx^{n-1}
\end{aligned}$$

Derivatives: Basic rules

Rule name	Function	Derivative
Polynomials	$f(x) = x^n$	$f'(x) = n\,x^{n-1}$
Constant factor	$g(x) = a\,f(x)$	$g'(x) = a\,f'(x)$
Sum and Difference	$h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$

Examples: Polynomial rule

Example 1

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}}$$

Example 2

$$g(x) = \frac{1}{x^n} = x^{-n}$$

$$g'(x) = -n\,x^{-n-1} = \frac{-n}{x^{n+1}}$$

Special functions

Function	Derivative
$\exp(x) = e^x$	$\exp(x) = e^x$
$\log(x) = \ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

Derivatives: Product rule

Function

$$h(x) = f(x) \cdot g(x)$$

Derivative

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

BG For those interested: Proof of the product rule

$$
\begin{aligned}
\frac{d}{dx} f(x)g(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \right. \\
&\qquad\qquad \left. + \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[\underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\to f'(x)} \underbrace{g(x)}_{\to g(x)} + \underbrace{f(x+\Delta x)}_{\to f(x)} \underbrace{\frac{g(x+\Delta x) - g(x)}{\Delta x}}_{\to g'(x)} \right] \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

Examples for product rule

Example 1

$$f(x) = \sin(x) \cdot \cos(x)$$

$$f'(x) = \cos(x)^2 - \sin(x)^2$$

Example 2

$$f(x) = x^2 \cdot \exp(x)$$

$$f'(x) =$$

$$2x \, \exp(x) + x^2 \exp(x)$$

Example 3

$$f(x) = x^{-1} \cdot \sin(x)$$

$$f'(x) =$$

$$-x^{-2} \cdot \sin(x) + x^{-1} \cos(x)$$

Example 4

$$f(x) = 2 \, \cos(x) \cdot \cos(x)$$

$$f'(x) = -4 \sin(x) \cos(x)$$

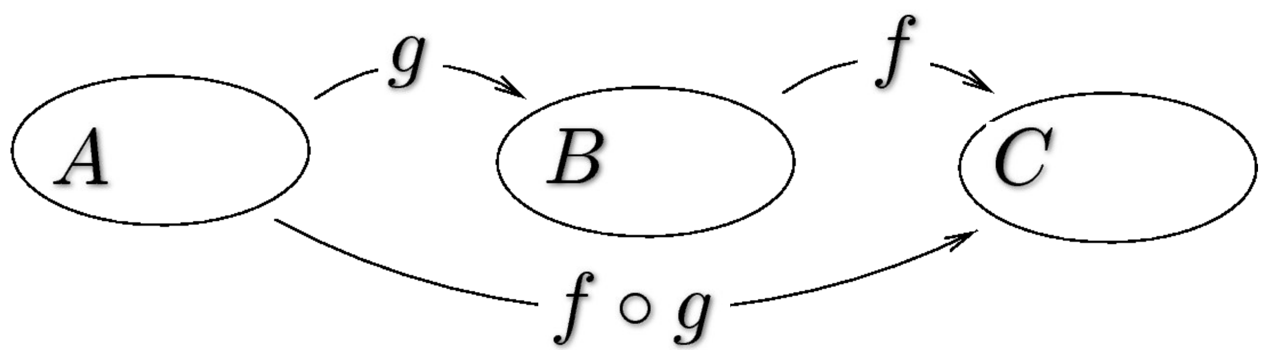
Function composition

$$g : A \to B$$
$$x \mapsto g(x)$$

$$f : B \to C$$
$$x \mapsto f(x)$$

then

$$f \circ g : A \to C$$
$$x \mapsto f(g(x))$$



Examples of composed functions

$$f(x) = \sin(x^2)$$

$$g : \mathbb{R} \to \mathbb{R}_+$$
$$x \mapsto x^2$$

$$h : \mathbb{R}_+ \to \mathbb{R}$$
$$x \mapsto \sin(x)$$

so …

$$f = h \circ g$$

$$\cancel{f = g \circ h}$$

! ! !

$$g \circ h(x) = (\sin(x))^2$$

Chain rule

Function

$$h(x) = f\big(g(x)\big) \qquad h = f \circ g$$

Derivative

$$h'(x) = f'\big(g(x)\big) \cdot g'(x) \qquad h' = f' \circ g \cdot g'$$

Example for chain rule

Example 1

$$f(x) = \sin(x^3)$$
$$f'(x) = \cos(x^3)3x^2$$

Example 2

$$f(x) = \log(2x^2)$$
$$f'(x) = \frac{1}{2x^2}4x$$

Example 3

$$f(x) = \exp(x^{-1})$$
$$f'(x) = \exp(x^{-1})(-x^{-2})$$

Example 4

$$f(x) = \left(\exp(x)\right)^{-1}$$
$$f'(x) = -\left(\exp(x)\right)^{-2}\exp(x)$$
$$= -\exp(x)^{-1}$$

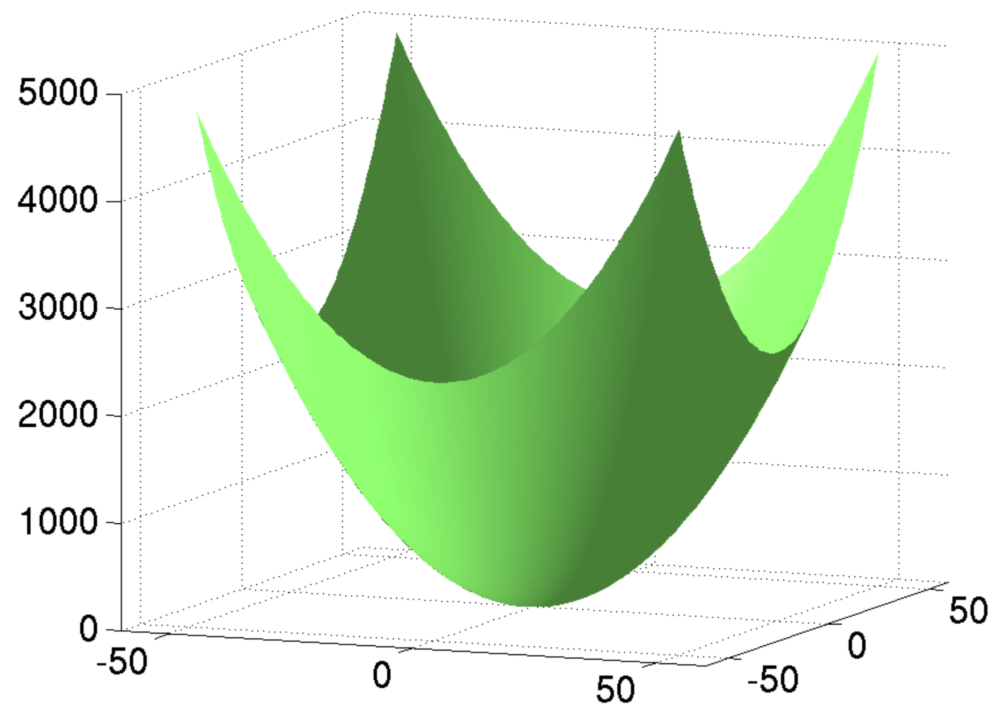
Derivatives in more than 1 dimension

$$f(x_1, x_2) = (x_1)^2 + (x_2)^2$$

Partial derivative

$\dfrac{\partial f}{\partial x_1}$ is taking

the derivative and treat x_1 as constant.



BB Partial derivative

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}\left((x_1)^2 + (x_2)^2\right)$$

$$= \frac{\partial}{\partial x_1}(x_1)^2 + \frac{\partial}{\partial x_1}(x_2)^2$$

$$= 2\,x_1 + 0 = 2\,x_1$$

$$\frac{\partial f}{\partial x_2} = 2\,x_2$$

Interpretation

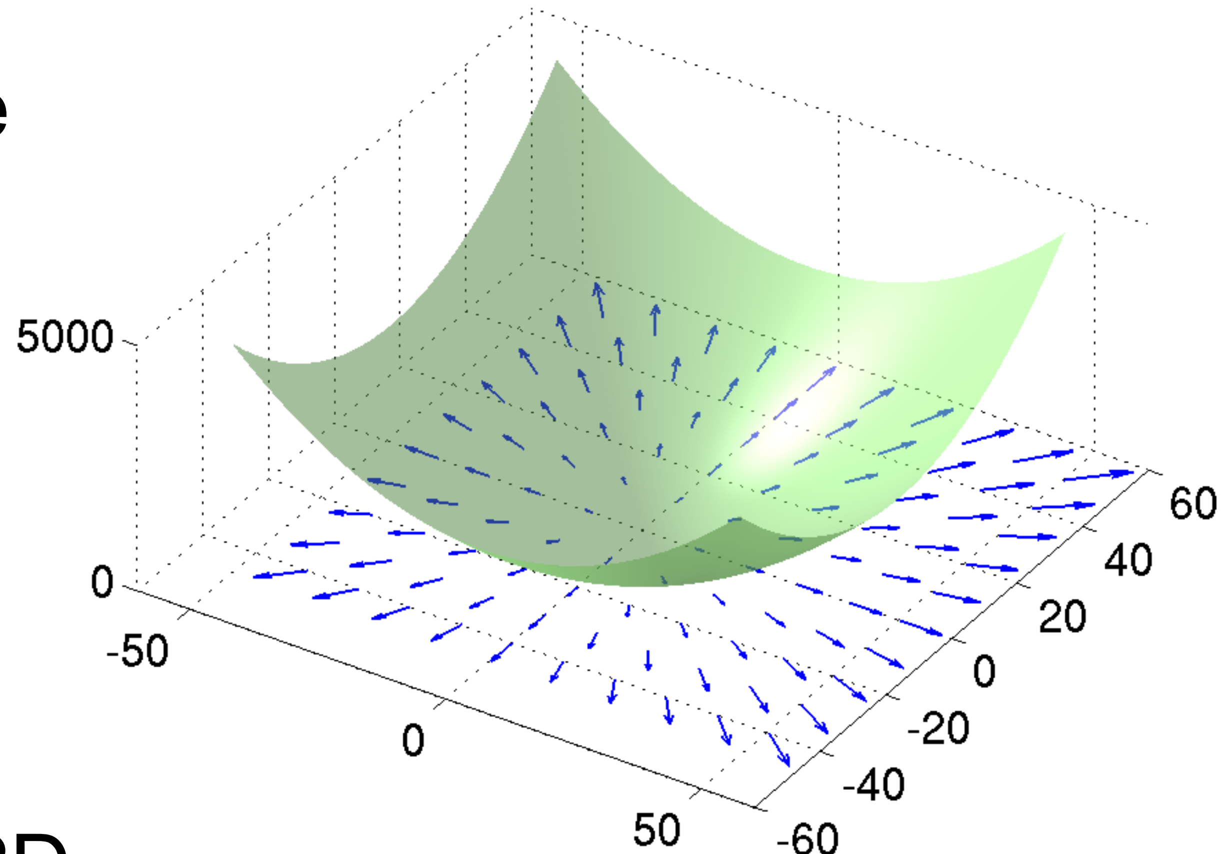
The partial derivative $\frac{\partial f}{\partial x_1}$ shows how much f changes when x_1 is changed.

The **gradient** gives the direction of the steepest slope:

$$\left(\begin{array}{c} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{array} \right) = \left(\begin{array}{c} 2\,x_1 \\ 2\,x_2 \end{array} \right)$$

The gradient

Points to the direction of steepest ascent.



Note – it is 2D vectors in this case.

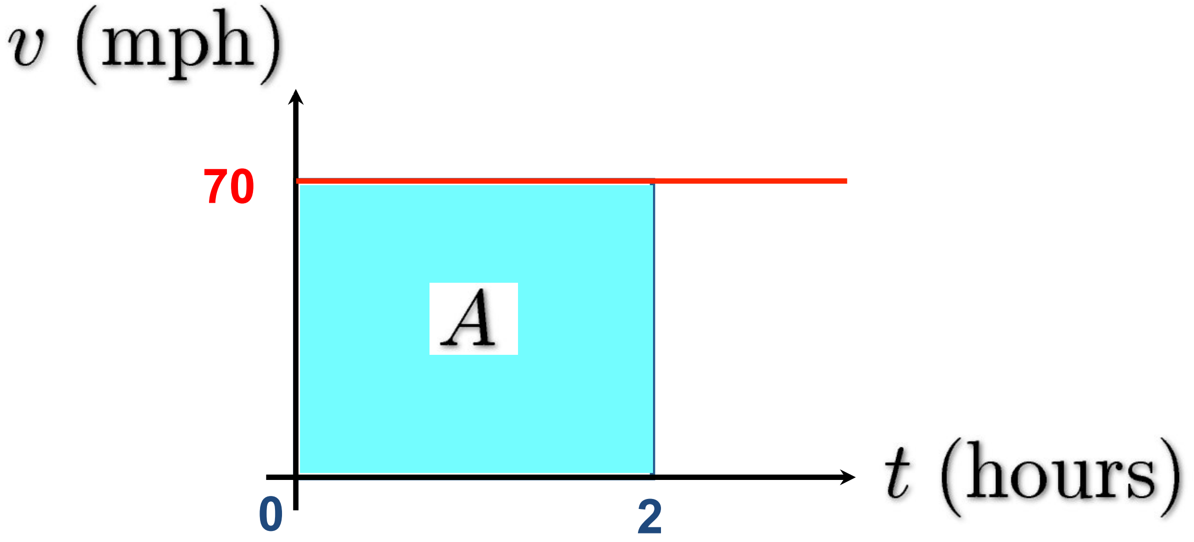
Applications

- Gradients are important for finding minima (so-called gradient descent):
  If you always go against the gradient, you go the steepest way down.
- The gradient can tell you when you are in a (local) extremum (minimum or maximum):
  In this case the gradient is 0.

INTEGRATION

Area under a graph

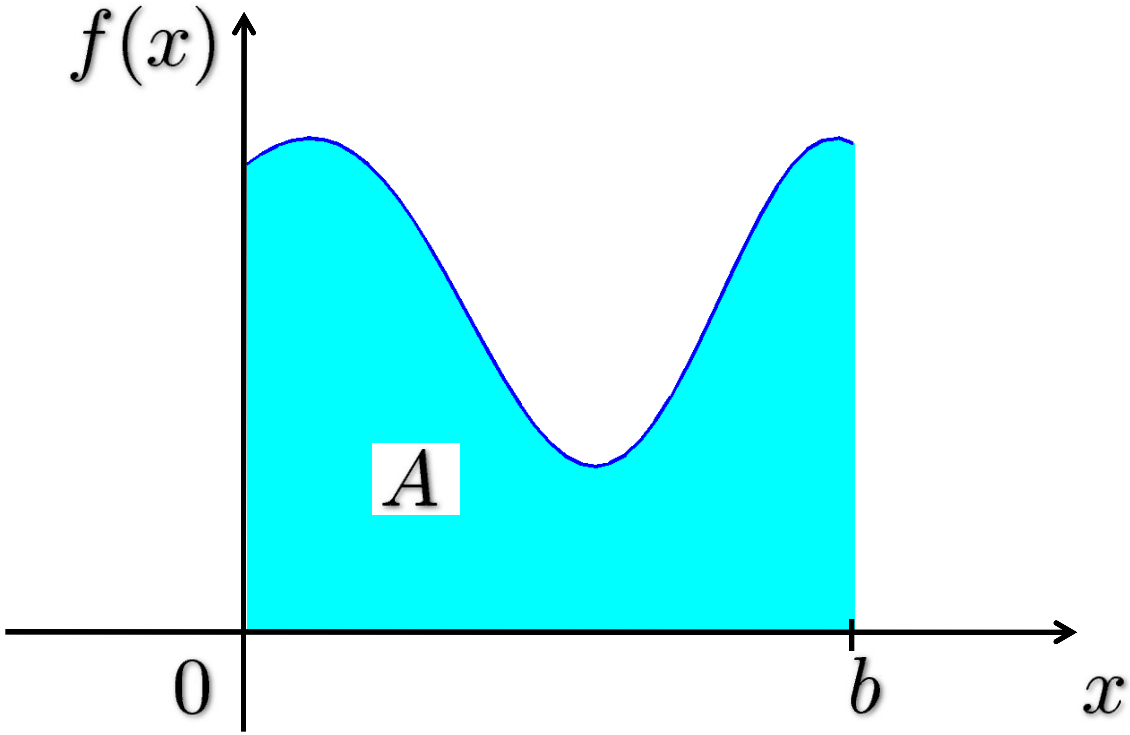
- Car travelling at 70 mph



Area = distance traveled:

$$A = v \cdot t = 70 \cdot 2 \text{ miles} = 140 \text{ miles}$$

What if we are interested in the area under this curve:

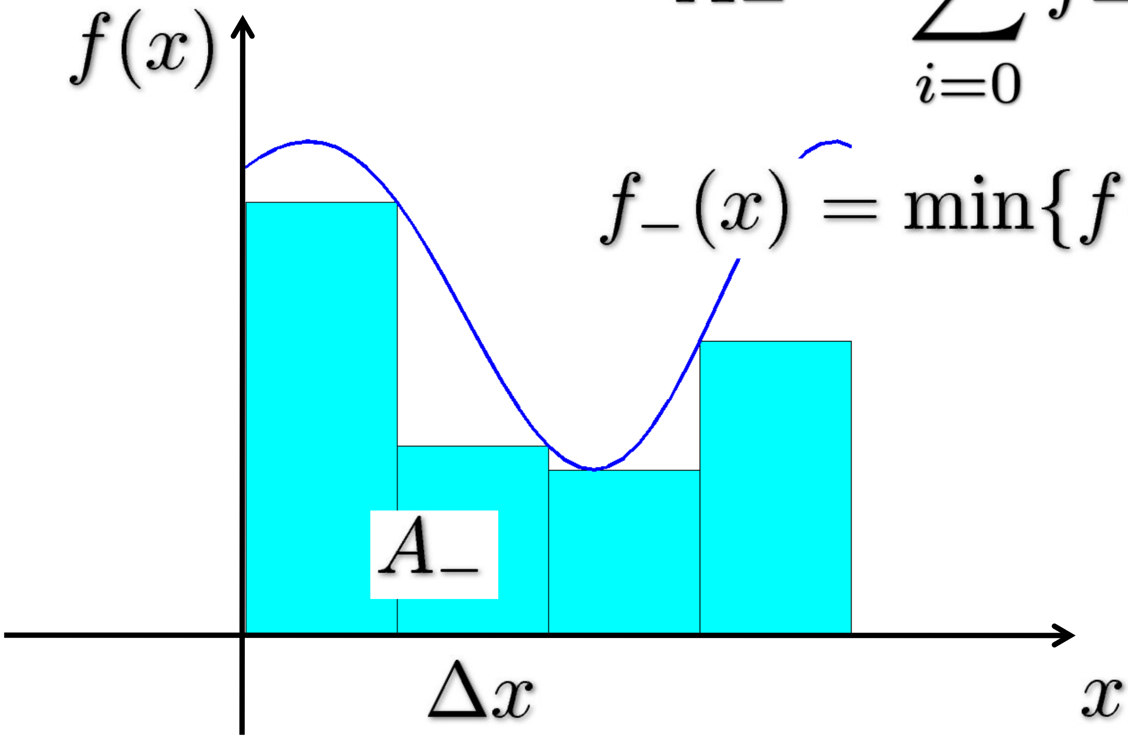


$$A = ?$$

Try something we know about

$$A_- = \sum_{i=0}^{4} f_-(i \cdot \Delta x) \cdot \Delta x$$

$$f_-(x) = \min\{f(z) : z \in [x, x + \Delta x]\}$$

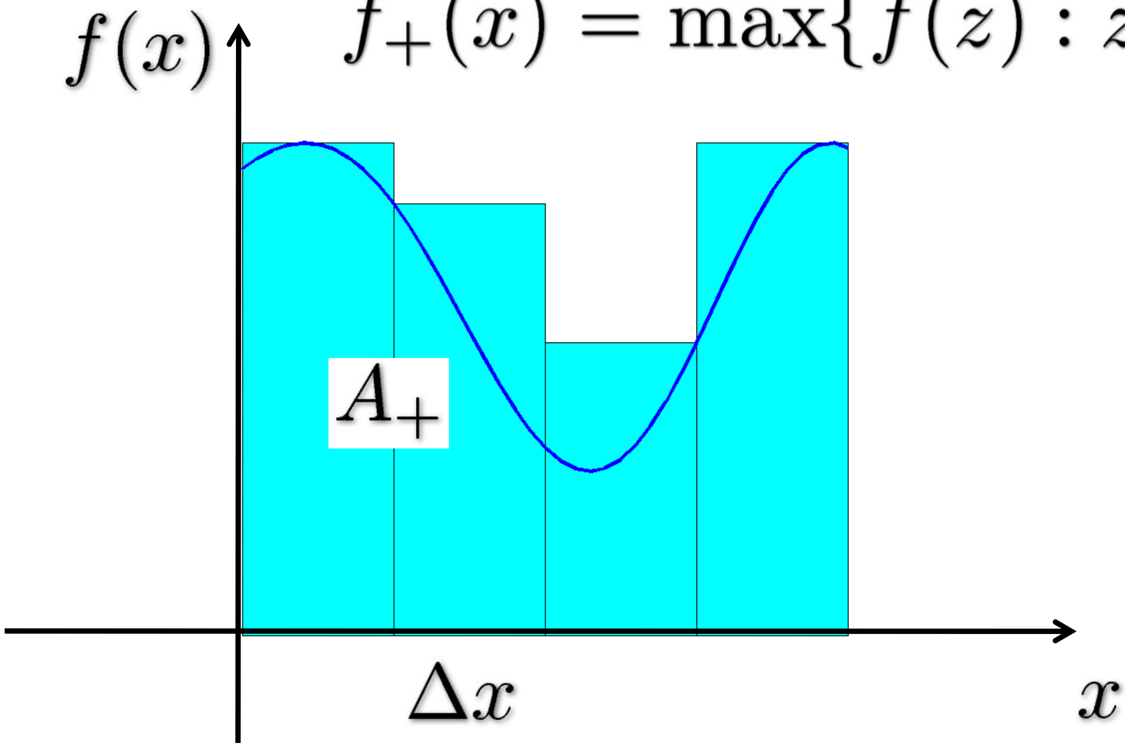


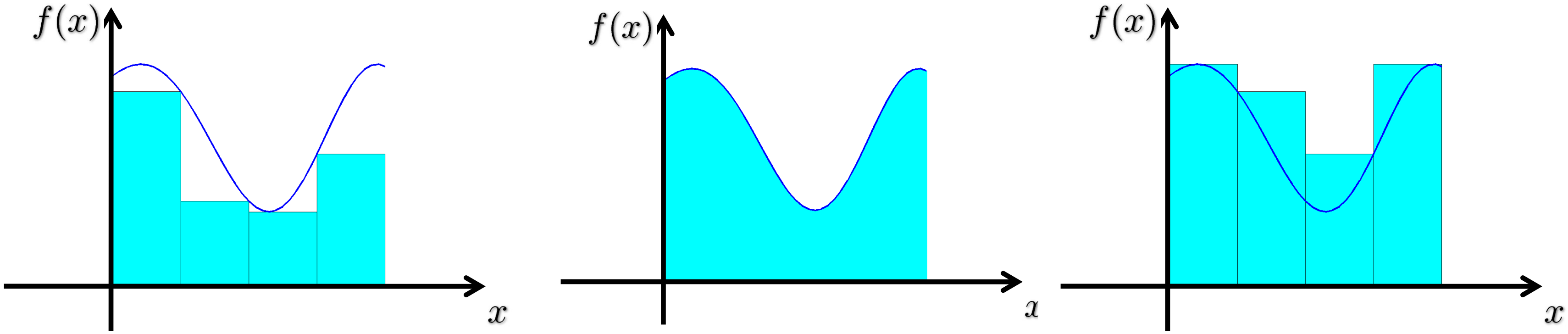
This is called “lower Riemann sum”

Upper Riemann sum

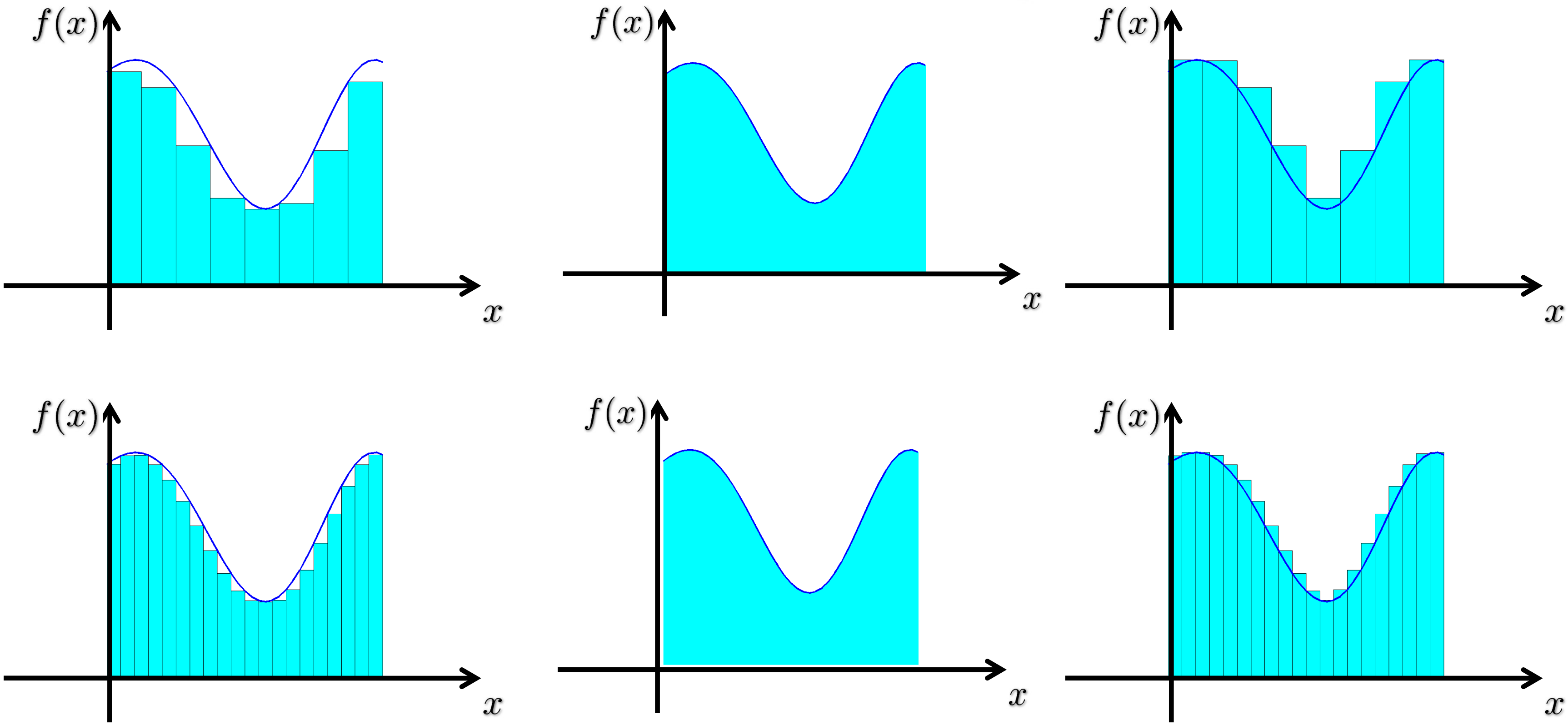
$$A_+ = \sum_{i=0}^{4} f_+(i \cdot \Delta x) \cdot \Delta x$$

$$f_+(x) = \max\{f(z) : z \in [x, x + \Delta x]\}$$





$$A_- \leq A \leq A_+$$



Riemann integral

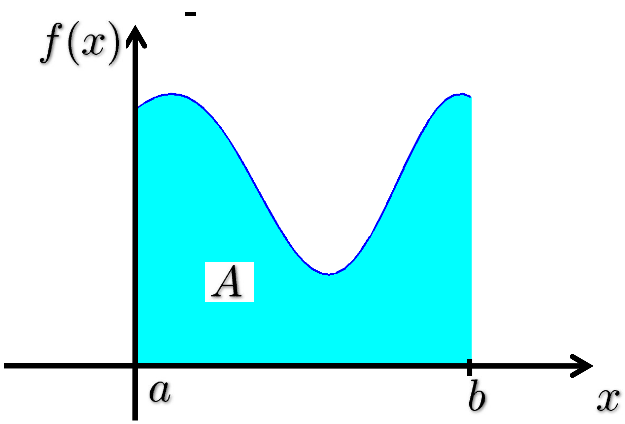
For many functions

$$\lim_{\Delta x \to 0} A_- = \lim_{\Delta x \to 0} A_+$$

The upper and lower Riemann sum become the same for small steps.

Such functions are called “Riemann integrable”, and

(Riemann) inte



$$A = \lim_{\Delta x \to 0} A_- = \lim_{\Delta x \to 0} A_+$$

is called “(Riemann) integral”

Notation:

upper limit

$$A = \int_a^b f(x)\,dx$$

integration variable
(dummy variable)

Integral sign

lower limit

Example from first principles

$$\int_0^T x\,dx = \lim_{\Delta x \to 0} \sum_{i=0}^{T/\Delta x} (i \cdot \Delta x) \cdot \Delta x$$

$$= \lim_{N\to\infty} \sum_{i=0}^{NT} \frac{i}{N} \cdot \frac{1}{N} = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=0}^{NT} i$$

$$= \lim_{N\to\infty} \frac{1}{N^2} \left(\frac{NT(NT+1)}{2} \right)$$

$$= \lim_{N\to\infty} \frac{1}{2} T^2 + \frac{T}{2N} = \frac{1}{2} T^2$$

Main theorem of differential and integral calculus

In principle, one could calculate integrals from first principles, but fortunately…

Integration is the opposite of differentiation!

Plausibility argument

Take a difference

Differentiation

$$f'(x) = \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

divide by Δx

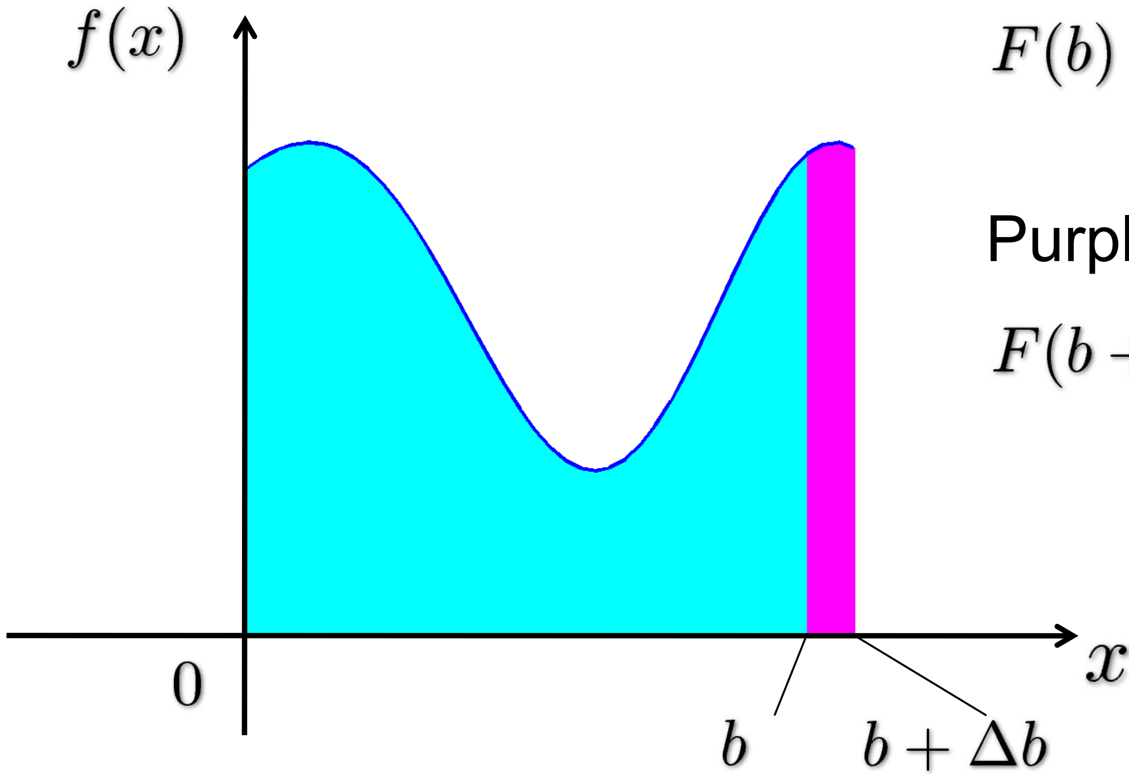
Integration

Sum it up

$$\int_0^T f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=0}^{T/\Delta x} f(i \cdot \Delta x) \cdot \Delta x$$

multiply with Δx

Differentiation inverts integration



Area from 0 to b (cyan):

$$F(b) = \int_0^b f(x)\,dx$$

Purple area from b to b+Δb:

$$F(b + \Delta b) - F(b)$$

$$\approx f(b)\Delta b$$

$$\frac{F(b + \Delta b) - F(b)}{\Delta b} \approx f(b)$$

$$\frac{d}{db}F(b) = f(b)$$

Rules of Differentiation

Rule name	Function	Derivative
Polynomials	$f(x) = x^n$	$f'(x) = n\,x^{n-1}$
Constant factor	$g(x) = a\,f(x)$	$g'(x) = a\,f'(x)$
Sum and Difference	$h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$

Become rules of integration

Rule name	~~Function~~ Integral	~~Derivative~~ Function
Polynomials	$\int_0^x f(t)dt = x^n + C$	$f(x) = nx^{n-1}$
Constant factor	$\int_0^x g(t)dt = a\int_0^x f(t)dt$	$g(x) = a\,f(x)$
Sum and Difference	$\int_0^x h(t)dt = \int_0^x f(t)dt + \int_0^x g(t)dt$	$h(x) = f(x) + g(x)$

Special functions

Function	Integral
$f(x) = x^n$	$\int_0^x t^n dt = \frac{1}{n+1} x^{n+1}$
$\exp(x) = e^x$	$\exp(x) = e^x$
$\frac{1}{x}$	$\log(x) = \ln(x)$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$

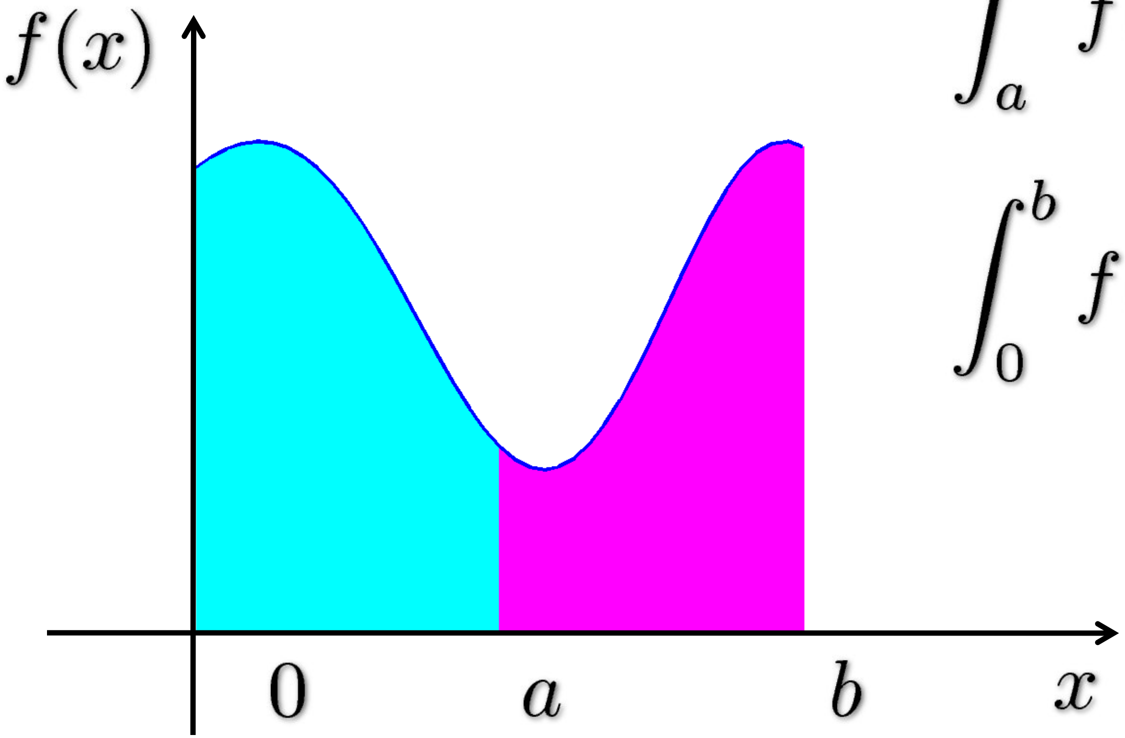
Integration constant

The “antiderivative”, “primitive integral” or “indefinite integral” is only defined up to a constant:

$$\frac{d}{dx}F(x) = f(x)$$

$$\frac{d}{dx}F(x) + C = f(x)$$

Practical tips



$$\int_a^b f(x)\,dx =$$

$$\int_0^b f(x)\,dx - \int_0^a f(x)\,dx$$

$$= F(b) - F(a)$$

Note how the integration constant does not matter here.

Practical tips II

$$\int_a^b f(x)\,dx = \big[\,F(x)\,\big]_a^b = F(b) - F(a)$$

"anti-derivative"

Example:

$$\int_2^3 x^3\,dx = \left[\,\frac{1}{4}x^4\,\right]_2^3 = \frac{1}{4}3^4 - \frac{1}{4}2^4 =$$

$$\frac{81-16}{4} = \frac{65}{4}$$

Example

$$\int_1^2 x^3 + 2x\,dx = [\frac{1}{4}x^4 + 2 \cdot \frac{1}{2}x^2]_1^2$$

$$= 4 + 4 - (\frac{1}{4} + 1) = \frac{27}{4}$$

More Examples

$$\int_1^2 x \exp(x^2)\, dx = \left[\exp(x^2)\right]_1^2 = \exp(4) - \exp(1)$$

$$\int_1^2 \frac{1}{x}\, dx = \left[\log(x)\right]_1^2 = \log(2) - \log(1) = \log(2)$$

$$\begin{aligned}\int_1^2 \sin(x)\, dx = \left[-\cos x\right]_1^2 &= -\cos(2) - (-\cos(1)) \\ &= \cos(1) - \cos(2)\end{aligned}$$