Choice via Grouping Procedures

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Abstract

In this paper, we consider a natural procedure of decision-making, called a “Grouping Choice Method,” which leads to a kind of bounded rational choices. In this procedure, a decision-maker (DM) first divides the set of available alternatives into some groups, and in each group, she chooses the best element (“winner”) for her preference relation. Then, among the winners in the first round, she selects the best one as her final choice. We characterize Grouping Choice Methods in three different ways. First, we show that a choice function is a Grouping Choice Method if and only if it is a Rational Shortlist Method (Manzini and Mariotti, 2007) in which the first rationale is transitive. Second, Grouping Choice Methods are axiomatically characterized by means of a new axiom called Elimination, in addition to two well-known axioms, Expansion and Weak WARP (Manzini and Mariotti, 2007). Third, Grouping Choice Methods are also characterized by a weak version of Path Independence.
1 Introduction

To construct models to explain (seemingly) irrational choices of individuals or societies is one of the central themes in economic theory recently. In this paper, we consider a natural procedure of decision-making, called “Grouping Choice Methods,” which leads to a kind of bounded rational choices. In this procedure, a decision-maker (DM) first divides the set of available alternatives into some groups, and in each group, she chooses the best element (“winner”) for her preference relation. Then, among the winners in the first round, she selects the best one as her final choice.

Such choice behaviors are often observed in the real life. For example, suppose that a family would like to buy a house. Three houses \{x, y, z\} are available, of which \(x\) and \(y\) are located in town A, and \(z\) in town B. They first choose the best house in each town, and then makes a final choice from the “winners” in the first round. Suppose that they prefer \(x\) to \(y\), \(y\) to \(z\), and \(z\) to \(x\).\(^1\) Now, when \(y\) and \(z\) are available, each of them is the only house in each town. Hence, \(y\) is chosen from \{\(y, z\)\} because they prefer \(y\) to \(z\). On the other hand, when all three houses are available, they first choose \(x\) as the best house in town A since they prefer \(x\) to \(y\). Because \(z\) is the only and hence the best house in town B, the set of winners in the first round is \{\(x, z\)\}. Then they choose \(z\) because they prefer \(z\) to \(x\). Thus, \(z\) is selected from \{\(x, y, z\)\}. Notice that the family’s preference relation is cyclic in this example, and yet a final choice can be determined by this procedure of choice with grouping. However, these choices are inconsistent with Samuelson’s (1938) Weak Axiom of Revealed Preferences (WARP), which requires that if \(y\) is chosen when \(z\) is available, then \(z\) should not be chosen whenever \(y\) is available.

In this paper, we formalize and analyze decision-making procedures as described above. First, we define a grouping rule as a correspondence that specifies groups for each set of available alternatives. We assume that each DM is endowed with a single preference relation. Given a grouping rule and a pref-

\(^1\)Note that a family’s preference relation may become cyclic because it is a collective preference relation (if they decide by majority voting, for instance).
ference relation, and for each set of available alternatives, a Grouping Choice Method first takes a maximal element in each group in the set for the preference relation, and then selects a maximal element among these first-round maximums.

We characterize Grouping Choice Methods in three different ways. First, we show that a choice function is a Grouping Choice Method if and only if it is a Rational Shortlist Method (Manzini and Mariotti, 2007) in which the first rationale is transitive. In Rational Shortlist Methods, a DM is endowed with two preference relations, called “rationales,” and for each set of available alternatives, she sequentially applied the two rationales to make selection.

Second, we axiomatically characterize Grouping Choice Methods. Manzini and Mariotti (2007) showed that Rational Shortlist Methods are characterized by a weak version of WARP and a standard choice-consistency property under expansion of the set of alternatives, simply called “Expansion.” Their Weak WARP requires that if an alternative $x$ is chosen in binary comparison with $y$ as well as in a set $S$ containing both $x$ and $y$, then $y$ should not be chosen in any “intermediate” set $T$ between $\{x, y\}$ and $S$ (that is, $\{x, y\} \subseteq T \subseteq S$). Because the class of Grouping Choice Methods is a restricted class of Rational Shortlist Methods, Grouping Choice Methods also satisfy Weak WARP and Expansion. In addition to the above two axioms, we introduce a new axiom called “Elimination.” This property means that if an alternative $y$ is never chosen in the presence of another alternative $x$, then (i) whenever $y$ is chosen in a menu (without $x$) and then $x$ becomes newly available, $x$ should be chosen in the new menu, or (ii) whenever $y$ and $x$ are present, eliminating $y$ from the menu dose not affect the choice. We show that these three axioms fully characterize Grouping Choice Methods.\(^2\)

Third, we consider a weak version of Path Independence, which we call Grouping Path Independence.\(^3\) Path Independence means that final outcomes

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\(^2\)Au and Kawai (2011) characterized Rational Shortlist Methods in which both of the two rationales are transitive by a different set of axioms.

\(^3\)The original version of Path Independence was introduced by Plott (1973).
should be independent of the “paths” to lead to them. To describe our version, let a grouping rule and a set $S$ of available alternatives be given. Suppose that a choice rule first selects an alternative in each group in $S$ specified by the grouping rule. Then, the rule makes a final choice among these alternatives selected from the groups. Suppose further that grouping in $S$ is changed either by merging some groups or by splitting some groups. Then, apply the choice rule as the same way as above, but under new grouping in $S$. Grouping Path Independence requires that this type of change in grouping should not affect the final choice. We show that, given a grouping rule $G$, a choice function satisfies Grouping Path Independence for $G$ if and only if it is a Grouping Choice Method with $G$ and some preference relation. This means that a choice function satisfying Grouping Path Independence can be rationalized by a preference relation.

In the literature on individual or social decision-making, many authors have proposed and studied models to explain choice behaviors which are inconsistent with single preference maximization over the sets of feasible alternatives. Among them, sequential applications of multiple criteria are often considered in both individual and social choices.\footnote{\text{Such contributions include Suzumura (1983), Aizerman (1985), Aizerman and Aleskerov (1995), Roelofsma and Read (2000), Kalai, Rubinstein, and Spiegler (2002), Tadenuma (2002, 2005), Manzini and Mariotti (2007, 2012a), Houy (2007), Houy and Tadenuma (2009), Au and Kawai (2011), and Apesteguia and Ballester (2013).}} It is interesting that choices by sequential maximization of two rationales where the first one is transitive may be alternatively described as decision-making by single preference maximization with a grouping procedure. The two distinct procedures may explain the same set of choice outcomes.

Manzini and Mariotti (2012b) consider another decision-making procedure which looks similar to ours. In their procedure, a DM first “categorize” alternatives. Here “categorization” is the same as “grouping” in our procedure. In their model, however, a DM is endowed with two distinct preference relations, one over the “categories” (subsets of the set of alternatives) and the other over
alternatives. Then, she first eliminates all alternatives in categories dominated by another category, and then chooses an alternative that is maximal among the remaining ones. In contrast to their model, a DM is endowed with a single preference relation over the alternatives in our model, just like the standard choice theory. The point of departure from the classical theory lies only in introducing a grouping process before maximization.

In social choice contexts, agenda setting is crucial for determining final outcomes, especially when a social preference relation contains cycles as in the case of majority voting. Here agenda setting is the same as grouping in our model. Hence, our results may shed some light on bounded rationality of social choice under various agenda settings.

We introduce basic notation and definitions, and define Grouping Choice Methods in Section 2. Sections 3, 4, and 5 present three characterizations of Grouping Choice Methods, respectively. The final section contains some concluding remarks. All proofs are relegated in the Appendix.

2 Grouping Choice Methods

First, we introduce basic notation and definitions throughout the paper. Let \( X \) be a finite set of alternatives, and \( \mathcal{X} \) the set of all nonempty subsets of \( X \). A choice function is a function \( C : \mathcal{X} \rightarrow X \) such that for every \( S \in \mathcal{X} \), \( C(S) \in S \). A binary relation (or rationale) on \( X \) is a set \( P \subseteq X \times X \). For simplicity, \((x, y) \in P\) is written as \( x \mathrel{P} y \). A binary relation \( P \) is asymmetric if \( x \mathrel{P} y \) implies not \( y \mathrel{P} x \). Let \( \mathcal{P} \) be the set of all asymmetric binary relations on \( X \).

We say that \( x \in X \) and \( y \in X \) with \( x \neq y \) are comparable in \( P \in \mathcal{P} \) if \( x \mathrel{P} y \) or \( y \mathrel{P} x \) holds. An asymmetric binary relation \( P \in \mathcal{P} \) is complete if for all \( x, y \in X \) with \( x \neq y \), \( x \) and \( y \) are comparable in \( P \). It is transitive if for all \( x, y, z \in X \), \( x \mathrel{P} y \) and \( y \mathrel{P} z \) implies \( x \mathrel{P} z \). It contains a cycle if there exist an integer \( n \) with \( n \geq 3 \) and \( n \) alternatives \( x_1, \ldots, x_n \in X \) such that \( x_i \mathrel{P} x_{i+1} \) for all \( i \in \{1, \ldots, n-1\} \) and \( x_n \mathrel{P} x_1 \). It is acyclic if it contains no cycle.
For each \( P \in \mathcal{P} \) and each \( S \in \mathcal{X} \), let \( M(S; P) \subseteq S \) denotes the set of maximal elements in \( S \) for \( P \):
\[
M(S; P) = \{ x \in S \mid \nexists y \in S \text{ such that } y \preceq P x \}.
\]

For each \( S \in \mathcal{X} \), let \( |S| \) denote the number of elements in \( S \).

Next, we introduce a new procedure of decision making, which we call a “Grouping Choice Method.” In this procedure, a DM first divides the set of feasible alternatives into some groups, and from each group, she selects an element (“winner”). Then, she chooses an element among the winners in the first round. In order to formally define the new procedure, we first introduce “grouping rules.”

**Definition 1.** A **grouping rule** is a correspondence \( G \) that associates with every \( S \in \mathcal{X} \) a family \( G(S) \) of subsets of \( S \). Let \( \mathcal{G} \) be the class of all grouping rules that satisfy the following conditions.

1. **(G1)** For every \( S \in \mathcal{X} \), \( \cup_{S_k \in G(S)} S_k = S \),
2. **(G2)** For every \( S \in \mathcal{X} \), there exist no \( S_i, S_j \in G(S) \) with \( S_i \neq S_j \) and \( S_i \subseteq S_j \).
3. **(G3)** For all \( S, T \in \mathcal{X} \), if there exists \( S_i \in G(S) \) such that \( \{x, y\} \subseteq S_i \) and \( \{x, y\} \subseteq T \), then there exists \( T_j \in G(T) \) such that \( \{x, y\} \subseteq T_j \).

Each member of \( G(S) \) is called a **group** in \( S \).

Condition (G1) means that every element in \( S \) belongs to some group in \( G(S) \). Condition (G2) says that no group is a strict subset of another group. Condition (G3) is consistency in grouping. To motivate the condition, consider again the situation for a family to buy a house. Let \( X = \{x, y, z, w\} \) be the set of all houses where \( x, y \) are located in town A while \( z, w \) are in town B. At first, all houses are available, and they divide \( X \) into \( \{x, y\} \) and \( \{z, w\} \) by location. But then, \( w \) is sold so that \( S = \{x, y, z\} \) becomes the new set of available houses. Then, if they still divide \( S \) according to location, \( \{x, y\} \) should be a group in \( S \). That is, \( x \) and \( y \) are always in a group as long as both are available. Condition (G3) requires this kind of consistency in grouping procedures.
Now we are ready to define our new decision procedure.

**Definition 2.** A choice function $C$ is a **Grouping Choice Method** if and only if there exist a grouping rule $G \in \mathcal{G}$ and a binary relation $P \in \mathcal{P}$ such that for every $S \in \mathcal{X}$, $C(S) = M(\bigcup_{S_k \in G(S)} M(S_k; P); P)$.

### 3 Grouping Choice Methods and Sequential Applications of Multiple Criteria

Sequential applications of multiple criteria in individual or social choices have been studied by many authors as cited in the Introduction. In this section, we clarify the relationship of Grouping Choice Methods with the models of sequential applications of multiple criteria.

Manzini and Mariotti (2007) defined and analyzed Rational Shortlist Methods. In the Methods, a DM is endowed with a pair of preference relations, called “rationale,” and for each set of available alternatives, she first takes all maximal elements for the first rationale in the set, and then among these elements, she selects the maximum for the second rationale. An obvious difference of Grouping Choice Methods from Rational Shortlist Methods is that a DM is endowed with single preference relation in the former whereas with two preference relations in the latter. Despite this difference, there is a strong connection between the two methods of choice. Our first theorem shows that a choice function is a Grouping Choice Method if and only if it is a Rational Shortlist Method in which the first rationale is transitive. It is interesting that the same set of choice outcomes may be explained in either of the two models of decision-making: the model of sequential maximization with a pair of rationales and that of maximization of a single preference relation with a grouping procedure.

**Theorem 1.** A choice function is a Grouping Choice Method if and only if it is a Rational Shortlist Method in which the first rationale is transitive.
It is worth noting the relationship of two rationales in a Rational Shortlist Method with the single preference relation in the corresponding Grouping Choice Method. Let a pair of rationales \((P_1, P_2)\) be given. Construct the single preference \(P_{12}\) as follows. For all \(x, y \in X\),

\[
x P_{12} y \iff x P_1 y \text{ or } [\text{not}(y P_1 x) \text{ and } x P_2 y].
\]

The preference relation \(P_{12}\) was defined and studied in Tadenuma (2002) and Houy and Tadenuma (2009), and called the \textit{lexicographic composition} of \((P_1, P_2)\). In this composition, an alternative \(x\) is superior to another alternative \(y\) if and only if (1) \(x\) is superior to \(y\) by the first criterion \(P_1\) or (2) \(x\) is superior to \(y\) by the second criterion \(P_2\) when \(x\) and \(y\) are not comparable by \(P_1\). Houy and Tadenuma (2009) scrutinize differences between the two ways of decision-making with a given pair of preference relations: one is a Rational Shortlist Method, and the other is to construct the (single) lexicographic composition and then maximize it. Despite the differences, the lexicographic composition plays a key role to connect Rational Shortlist Methods with Grouping Choice Methods, as seen below.

Let a pair of preference relations \((P_1, P_2)\) be given. Suppose that there exist a grouping rule \(G\) and a preference relation \(P\) such that \(M((S; P_1); P_2) = M(\bigcup_{S_k \in G(S)} M(S_k; P); P)\) for every \(S \in X\). Then, we have \(M((\{x, y\}; P_1); P_2) = M((\{x, y\}; P)\) for all \(x, y \in X\). This means that \(x P y\) if and only if \([x P_1 y]\) or \([\text{not}(y P_1 x) \text{ and } x P_2 y]\). Hence, \(P = P_{12}\). Thus, as a corollary of Theorem 1, we have the following.

\textbf{Corollary 1.} If a choice function is a Rational Shortlist Method with a pair of rationales \((P_1, P_2)\) in which the first rationale is transitive, then it is a Grouping Choice Method with the lexicographic composition of \((P_1, P_2)\) and some grouping rule.
4 Axiomatic Characterization of Grouping Choice Methods

In this section, we define three natural properties of choice functions. Then, we show that Grouping Choice Methods satisfy all these properties, and conversely, every choice function satisfying the three properties is a Grouping Choice Method.

The first property is standard choice-consistency under expansion of available alternatives. It says that if an alternative is chosen in each of two sets of available alternatives, then it should be chosen in the union of the two sets.

**Expansion:** For all \( S, T \in \mathcal{X} \), if \( x = C(S) = C(T) \), then \( x = C(S \cup T) \).

The next property is a weaker version of Samuelson’s WARP, which was introduced by Manzini and Mariotti (2007). This axiom requires that if an alternative \( x \) is chosen in \( \{x, y\} \) and a set \( S \) containing both \( x \) and \( y \), then \( y \) should not be chosen in any “intermediate” set between \( \{x, y\} \) and \( S \).

**Weak WARP:** For all \( x, y \in X \) and all \( S, T \in \mathcal{X} \), if \( \{x, y\} \subseteq T \subseteq S \) and \( x = C(\{x, y\}) = C(S) \), then \( y \neq C(T) \).

The third property says that if an alternative \( y \) is never chosen in the presence of another alternative \( x \), then (i) whenever \( y \) is chosen in a menu (in the absence of \( x \)) and then \( x \) becomes newly available, \( x \) should be chosen in the new menu (that is, \( x \) “eliminates” the initial winner \( y \)), or (ii) whenever both \( x \) and \( y \) are present, eliminating \( y \) from the menu does not affect the choice.

**Elimination:** For all \( x, y \in X \), if \( y \neq C(S) \) for every \( S \in \mathcal{X} \) with \( x \in S \), then

1. for every \( S \in \mathcal{X} \) with \( x \notin S \) and \( y = C(S) \), \( x = C(S \cup \{x\}) \), or
2. for every \( S \in \mathcal{X} \) with \( x \in S \) and \( y \in S \), \( C(S) = C(S \setminus \{y\}) \).

Our next theorem shows that the above three properties characterize Grouping Choice Methods.
Theorem 2. A choice function satisfies Expansion, Weak WARP, and Elimination if and only if it is a Grouping Choice Method.

5 Grouping Path Independence

To present our final characterization of Grouping Choice Method, we need to introduce some additional definitions. The idea behind the following definitions is simple. Given a grouping in some set $S$, we consider two ways to change the grouping. In one way, we merge two groups into one and iterate this operation to obtain a new grouping. In the other way, we split a group into two groups and iterate it.

Let $S \in \mathcal{X}$. Let $S_1$ and $S_2$ be two families of subsets of $S$. We say that

1. $S_2$ is obtained by merging from $S_1$ if (i) $T_i = S_j \cup S_k$ for some $T_i \in S_2$ and some $S_j, S_k \in S_1$ and (ii) $S_1 \setminus \{S_j, S_k\} = S_2 \setminus \{T_i\}$; and that
2. $S_2$ is obtained by splitting from $S_1$ if (i) $S_i = T_j \cup T_k$ for some $S_i \in S_1$ and some $T_j, T_k \in S_2$, and (ii) $S_1 \setminus \{S_i\} = S_2 \setminus \{T_j, T_k\}$.

Then, we say that a family $\mathcal{T}$ of subsets of $S$ is obtained by iteratively merging (resp., iteratively splitting) from a family $\mathcal{S}$ if there exists a sequence of families of subsets of $S$, $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ such that $\mathcal{S}_1 = \mathcal{S}$, $\mathcal{S}_\ell = \mathcal{T}$, and for all $h \in \{1, \ldots, \ell - 1\}$, $\mathcal{S}_{h+1}$ is obtained by merging (resp., splitting) from $\mathcal{S}_h$.

The following property requires that the above types of changes in grouping should not affect the final outcomes.

Grouping Path Independence: Let a grouping rule $G$ be given. We say that a choice function $C$ satisfies Grouping Path Independence for $G$ if the following condition holds: For every $S \in \mathcal{X}$, if $\Sigma = \{T_1, \ldots, T_m\}$ is obtained either by iteratively merging or by iteratively splitting from $G(S)$, then $C(S) = C(\cup_{T_k \in \Sigma} C(T_k))$.

We are now ready to state our third characterization of Grouping Choice Methods.

Theorem 3. Let a grouping rule $G$ be given. A choice function $C$ satisfies
Grouping Path Independence for G if and only if it is a Grouping Choice Method with G and some asymmetric binary relation P.

We note that the necessity part of the above theorem does not rely on the property (G3) of grouping rules. Hence, this part holds in the class of grouping rules that satisfy (G1) and (G2) but not necessarily (G3). This means that a choice function satisfying Grouping Path Independence can be rationalized by a preference relation in more general cases of grouping.

6 Concluding Remarks

In this paper, we introduce a new, natural procedure of decision making, called a Grouping Choice Method. We clarify the relationships between the two distinct procedures, Rational Shortlist Methods and Grouping Choice Methods. We also axiomatically characterize Grouping Choice Methods by using new properties.

An advantage of our grouping procedure lies in its simplicity. Grouping before maximization is quite common in every day decision-makings. Moreover, our method assumes only one preference relation for a DM, as in the classical theory. We do not need to imagine more complex DMs with multiple criteria. Yet the outcomes of choice are the same as those in the case where the DMs would sequentially maximize their multiple preference relations.

A key to determine properties of a Grouping Choice Method is the grouping rule. We impose three conditions (G1) to (G3) on grouping rules. However, there may be some situations to which it is not appropriate to apply (G3). Consider again the example in which a family buys a house. They divide available houses into groups by their locations. Suppose that the set of all houses is \( \{x, y, z, w\} \) where \( x \) and \( y \) are located in district \( a \) of town \( A \), \( z \) is located in another district \( b \) of town \( A \), and \( w \) is located in another town \( B \). If the set of available houses is \( \{x, z, w\} \), then she divides it into \( \{x, z\} \) and \( \{w\} \) because \( x \) and \( z \) are located in the same town and \( w \) is located in the other town. But when she faces \( \{x, y, z\} \), she might divide \( \{x, y, z\} \) into \( \{x, y\} \) and
\{z\} because \(x\) and \(y\) are located in the same district and \(z\) is located in another district. This is a violation of (G3). The condition (G3) requires that grouping should not depend on the sets of available alternatives. However, there may be cases in which grouping changes with a change of the whole menu.

If we weaken the requirements on grouping rules, we may explain a broader range of choice behaviors, including more “irrational” ones. Indeed, without (G3), we can construct an example of a grouping choice method that violates both Expansion and Weak WARP.\textsuperscript{5} It may be an interesting future topic to study what kind of bounded rationality Grouping Choice Methods keep with weaker requirements on grouping rules.

**Appendix: Proofs**

We first introduce additional definitions and notation. Let a grouping rule \(G\) be given. For all \(x, y \in X\), define \(x \leftrightarrow y\) as

\[
x \leftrightarrow y \Leftrightarrow \exists S \in \mathcal{X} : \exists S_i \in G(S) \text{ such that } \{x, y\} \subseteq S_i
\]

That is, \(x \leftrightarrow y\) means that \(x\) and \(y\) belong to the same group in some subset \(S \in \mathcal{X}\). Notice that the following relation holds since the grouping rule satisfies (G3).

\[
x \leftrightarrow y \Leftrightarrow \forall S \in \mathcal{X} : \exists S_i \in G(S) \text{ such that } \{x, y\} \subseteq S_i
\]

That is, the relation \(x \leftrightarrow y\) also means \(x\) and \(y\) belong to the same group in every subset of \(X\). We write not[\(x \leftrightarrow y\)] as \(x \not\leftrightarrow y\):

\[
x \not\leftrightarrow y \Leftrightarrow \forall S \in \mathcal{X} : \exists S_i \in G(S) \text{ such that } \{x, y\} \subseteq S_i
\]

Given a pair of asymmetric binary relations \((P_1, P_2)\), define the binary relation \(P_2^*\) as follows (Houy and Tadenuma, 2009, p.1776): for all \(x, y \in X\),

\[
x P_2^* y \Leftrightarrow x \text{ and } y \text{ are not comparable in } P_1 \text{ and } x P_2 y.
\]

\textsuperscript{5}In the above example of choosing a house, consider \(G\) such that \(G(X) = \{\{x, y\}, \{z\}, \{w\}\}\), \(G(\{x, y, z\}) = \{\{x, y\}, \{z\}\}\), \(G(\{x, y, w\}) = \{\{x, y\}, \{w\}\}\), \(G(\{x, z, w\}) = \{\{x, z\}, \{w\}\}\), and \(G(\{y, z, w\}) = \{\{y, z\}, \{w\}\}\). Then, let \(P = \{(y, x), (x, z), (w, x), (z, y), (w, y), (z, w)\}\).
Given $P \in \mathcal{P}$, define the transitive closure $T(P)$ of $P$ as follows: for all $x, y \in X$,

$$x T(P) y \iff \exists z_1, \ldots, z_k \in X \text{ with } k \geq 2 \text{ such that } x = z_1, y = z_k \text{ and } \forall i \in \{1, \ldots, k - 1\}, z_i P z_{i+1}.$$ 

To prove theorems, we need some lemmas.

**Lemma 1.** Assume that a pair of asymmetric binary relations $(P_1, P_2)$ sequentially rationalizes a choice function $C$. Then the following claims hold.

(a) $P_1$ is acyclic.

(b) For all $S \in \mathcal{X}$, $M(M(S; P_1); P_2^*) = M(M(S; P_1); P_2)$ holds. That is, $(P_1, P_2^*)$ also sequentially rationalizes $C$.

(c) For all $x, y \in X$ with $x \neq y$, $x$ and $y$ are comparable in one and only one of $P_1$ and $P_2^*$.

**Proof.** Assume that a pair of asymmetric binary relations $(P_1, P_2)$ sequentially rationalized a choice function $C$.

(a) If $P_1$ contains a cycle $x_1, \ldots, x_n \in X$, then $M(S; P_1) = \emptyset$ where $S = \{x_1, \ldots, x_n\}$, and hence $C(S) = M(M(S; P_1); P_2) = \emptyset$. This contradicts nonemptiness of $C$.

(b) Let $S \in \mathcal{X}$. For all $x, y \in M(S; P_1)$, $x$ and $y$ are not comparable in $P_1$, and hence $x P_2 y$ holds if and only if $x P_2^* y$ holds. Therefore, $M(M(S; P_1); P_2) = M(M(S; P_1); P_2^*)$.

(c) Assume that $x$ and $y$ are neither comparable in $P_1$ nor in $P_2^*$. By the above claim, $(P_1, P_2^*)$ sequentially rationalizes $C$. Then, we have $\{x, y\} = M(M(\{x, y\}; P_1); P_2^*) = C(S)$. This contradicts $C(S)$ is a singleton for every $S \in \mathcal{X}$. Hence, $x$ and $y$ must be comparable in $P_1$ or $P_2^*$. By the definition of $P_2^*$, $x$ and $y$ are comparable in only one of $P_1$ and $P_2^*$.

$\square$
Lemma 2. Assume that a pair of asymmetric binary relations \((P_1, P_2)\) sequentially rationalizes a choice function \(C\). Let \(S \in \mathcal{X}\) and \(x = C(S)\). Then, the following claims hold.

(d) There exists no \(y \in S\) such that \(y P_1 x\).

(e) For every \(y \in S\), if \(y P_2^* x\), then there exist \(k\) alternatives \(z_1, \ldots, z_k \in S \setminus \{x\}\) with \(k \geq 1\) such that

(i) \(z_1 P_1 y\),

(ii) \(z_{i+1} P_1 z_i\) and \(z_i P_2^* x\) for all \(i = 1, \ldots, k - 1\) if \(k \geq 2\), and

(iii) \(x P_1 z_k\) or \(x P_2^* z_1\).

Proof. Since \(x = C(S) = M(M(S; P_1); P_2)\), we have \(x \in M(S; P_1)\). Hence, Claim (d) follows.

Assume that \(y \in S\) and \(y P_2^* x\). By claim (b) in Lemma 1, \(x = M(M(S; P_1); P_2^*)\). If there exists no \(z_1 \in S\) with \(z_1 P_1 y\), then \(y \in M(S; P_1)\) holds, and \(x \notin M(M(S; P_1); P_2^*)\), which is a contradiction. Hence, there exists \(z_1 \in S\) with \(z_1 P_1 y\). If \(z_1 = x\), then \(x P_1 y\), which contradicts \(y P_2^* x\). Thus, \(z_1 \neq x\). If \(x P_1 z_1\) or \(x P_2^* z_1\), we are done.

Assume that neither \(x P_1 z_1\) nor \(x P_2^* z_1\) holds. By claim (d), \(z_1 P_1 x\) does not hold. Then, by claim (c) in Lemma 1, we have \(z_1 P_2^* x\). If there exists no \(z_2 \in S\) with \(z_2 P_1 z_1\), then \(z_1 \notin M(S; P_1)\) and \(x \notin M(M(S; P_1); P_2^*)\), which is a contradiction. Thus, there exists \(z_2 \in S\) with \(z_2 P_1 z_1\). It follows from \(z_2 P_1 z_1\) and \(z_1 P_2^* x\) that \(x \neq z_2\). If \(x P_1 z_2\) or \(x P_2^* z_2\), we are done. If not, then by the same argument as above, there exists \(z_3 \in S\) such that \(z_3 P_1 z_2\). Iterating this procedure, we have a sequence \(z_1, z_2, \ldots\) such that \(z_{i+1} P_1 z_i\) and \(z_i P_2^* x\) for all \(i = 1, 2, \ldots\). Because \(P_1\) is acyclic by claim (a) in Lemma 1, it must be the case that \(z_i \neq y\) for all \(i = 1, 2, \ldots\) and \(z_i \neq z_j\) for all \(i, j\) with \(i \neq j\). Moreover, since \(S\) is finite, this procedure must terminate. Let the \(k\)th iteration terminate the procedure. Then, we have \(x P_1 z_k\) or \(x P_2^* z_k\).

Lemma 3. Assume that a pair of asymmetric binary relations \((P_1, P_2)\) sequentially rationalizes a choice function \(C\) and satisfies the following property:

Property T: \(\forall x, y \in X; x T(P_1) y \Rightarrow x P_1 y\) or \(x P_2^* y\).
Then, \((T(P_1), P^*_2)\) also sequentially rationalizes \(C\).

**Proof.** Assume that \((P_1, P_2)\) sequentially rationalizes \(C\) and satisfies Property T. Let \(S \in \mathcal{X}\) and \(x = C(S)\). By claim (b) in Lemma 1, \(x = M(M(S; P_1); P^*_2)\).

Now we prove \(x = M(M(S; T(P_1)); P^*_2)\).

First we show \(x \in M(S; T(P_1))\). Suppose, on the contrary, that there exists \(y \in S\) with \(y T(P_1) x\). By claim (d) in Lemma 2, \(y P_1 x\) does not hold. Then, by Property T, we have \(y P^*_2 x\). Consequently, \([x P_1 z \text{ or } x P^*_2 z]\) and \([z P_1 x \text{ or } z P^*_2 x]\) are incompatible because (i) both \(P_1\) and \(P^*_2\) are asymmetric, and (ii) by the definition of \(P^*_2\), \(x P_1 z\) and \(z P^*_2 x\) are incompatible, and so as \(x P^*_2 z\) and \(z P_1 x\). Hence, we have \(x \in M(S; T(P_1))\).

Since \(P_1 \subseteq T(P_1)\), we have \(M(S; T(P_1)) \subseteq M(S; P_1)\). It follows from \(x = M(M(S; T(P_1)); P^*_2)\) that \(x \in M(M(S; T(P_1)); P^*_2)\). Notice that \(P^*_2\) is complete in \(M(S; P_1)\) by claim (c) in Lemma 1. Hence, it is also complete in \(M(S; T(P_1))\). Thus, we have \(x = M(M(S; T(P_1)); P^*_2)\).

**Lemma 4.** Assume that a choice function \(C\) is a Grouping Choice Method with an asymmetric binary relation \(P\) and a grouping rule \(G\). Then, the following claims hold:

(f) \(P\) is complete.

(g) For every \(S \in \mathcal{X}\), if \(x \leftrightarrow y\) for all \(x, y \in S\), then \(M(S; P) \neq \emptyset\).

(h) For every \(S \in \mathcal{X}\) and all \(x, y \in S\), if \(x \leftrightarrow y\) and \(y P x\), then \(x \neq C(S)\).

(i): For every \(S \in \mathcal{X}\) and all \(x, y \in S\), if \(y P x\) and \(x = C(S)\), then there exists \(z \in S \setminus \{x, y\}\) such that \(x P z\), \(z P y\), and \(y \leftrightarrow z\).

**Proof.**

Claim (f). For all \(x, y \in X\), \(C(\{x, y\})\) is a single element in \(\{x, y\}\). Hence, we have \(x P y\) or \(y P x\).
Claim (g). Suppose, on the contrary, that \( S \in \mathcal{X} \) and for all \( x, y \in S \), \( x \leftrightarrow y \) but \( M(S; P) = \emptyset \). Since \( S \) is finite and \( P \) is asymmetric and complete, \( P \) contains a cycle in \( A \), that is, there exist \( x_1, x_2, \ldots, x_n \in S \) with \( n \geq 3 \) such that \( x_i P x_{i+1} \) for all \( i \in \{1, \ldots, n - 1\} \) and \( x_n P x_1 \). Take minimal \( j \in \{3, \ldots, n\} \) such that \( x_j P x_1 \). Because \( P \) is complete, we have \( x_1 P x_{j-1}, x_{j-1} P x_j, \) and \( x_j P x_1 \). Let \( T = \{x_1, x_{j-1}, x_j\} \). By the initial supposition, we have \( v \leftrightarrow w \) for all \( v, w \in T \). Then it must be the case that \( T \in G(T) \) or \( \{x_1, x_{j-1}\}, \{x_{j-1}, x_j\}, \{x_1, x_j\} \subseteq G(T) \). If \( T \in G(T) \), it follows from (G2) in the definition of Grouping Rules, \( \{T\} = G(T) \). Thus, we have \( C(T) = M(T; P) = \emptyset \), which contradicts non-emptiness of \( C \). If \( \{x_1, x_{j-1}\}, \{x_{j-1}, x_j\}, \{x_1, x_j\} \subseteq G(T) \), then we also have \( C(T) = M(\cup_{T_k \in G(T)} M(T_k; P); P) = M(T; P) = \emptyset \), which is a contradiction. Therefore, it must be the case that \( M(S; P) \neq \emptyset \).

Claim (h). Assume that \( S \in \mathcal{X}, x, y \in S, x \leftrightarrow y, \) and \( y P x \). Let \( S_i \in G(S) \) be a group in \( S \) such that \( x, y \in S_i \). Then, \( x \notin M(S_i; P) \). Because \( G \) satisfies (G3), we have \( v \leftrightarrow w \) for all \( v, w \in S_i \). By Claim (g), \( M(S_i; P) \neq \emptyset \).

Let \( z \in M(S_i; P) \). Since \( P \) is complete, we have \( z P x \). Thus, \( x \notin M(\cup_{S_k \in G(S)} M(S_k; P); P) = C(S) \).

Claim (i). Assume that \( S \in \mathcal{X}, x, y \in S, y P x, \) and \( x = C(S) \). If \( y \notin \cup_{S_k \in G(S)} M(S_k; P) \), then \( y P x \) and \( x = C(S) \) are incompatible. Hence, \( y \notin \cup_{S_k \in G(S)} M(S_k; P) \). Let \( S_i \in G(S) \) be a group with \( y \in S_i \). By Claim (g), \( M(S_i; P) \neq \emptyset \). Then, there exists \( z \in M(S_i; P) \). By completeness of \( P \), we have \( z P y \). Then, \( z \neq x \) since \( y P x \). Because \( G \) satisfies (G3), we have \( y \leftrightarrow z \). Moreover, from \( x = C(S) = M(\cup_{S_k \in G(S)} M(S_k; P); P) \) and \( z \in \cup_{S_k \in G(S)} M(S_k; P) \), we have \( x P z \).

\[ \Box \]

**Proof of Theorems 1 and 2**

We prove Theorems 1 and 2 together in the following three parts:

Part 1: to show that every Grouping Choice Method satisfies Expansion, Weak
WARP, and Elimination.

Part 2: to show that if a choice function satisfies Expansion, Weak WARP, and Elimination, then it is a Rational Shortlist Method in which the first rationale is transitive.

Part 3: to show that if a choice function is a Rational Shortlist Method in which the first rationale is transitive, then it is a Grouping Choice Method.

Part 1: We show that every Grouping Choice Method satisfies Expansion, Weak WARP, and Elimination.

Let $C$ be a Grouping Choice Function with an asymmetric binary relation $P$ and a grouping rule $G$.

Expansion:
Suppose, on the contrary, that $x = C(S) = C(T)$ but $y = C(S \cup T) \neq x$. Since $C$ satisfies Weak WARP, $y \neq C(\{x, y\})$. Hence, $x = C(\{x, y\})$ and $x P y$. Then, by Claim (i) in Lemma 4, there exists $z \in S \cup T$ with $z \leftrightarrow x$ and $z P x$. Without loss of generality, assume $z \in S$. It follows from Claim (h) in Lemma 4 that $x \neq C(S)$, which is a contradiction. Thus, $x = C(S \cup T)$ must hold.

Weak WARP:
Assume $T \in \mathcal{X}$ and $x = C(\{x, y\}) = C(T)$. Let $S \in \mathcal{X}$ be a set such that $\{x, y\} \subseteq S \subseteq T$. By $x = C(\{x, y\})$, we have $x P y$. It follows from $x = C(T)$ and Claim (h) in Lemma 4 that there exists no $z \in T$ such that $x \leftrightarrow z$ and $z P x$. Because $S \subseteq T$, there exists no $z \in S$ with $x \leftrightarrow z$ and $z P x$. Then, Claim (i) in Lemma 4 and $x P y$ together imply $y \neq C(S)$.

Elimination:
Let $x, y \in X$. Assume that $y \neq C(A)$ for all $A \in \mathcal{X}$ with $x \in A$. Suppose, on the contrary, that there exist $S, T \in \mathcal{X}$ such that $y = C(S)$, $x \neq C(S \cup \{x\})$, and $x \in T$, $C(T) \neq C(T \cup \{y\})$. Let $v = C(S \cup \{x\})$ and $w = C(T)$.

In the following, we have seven steps to derive a contradiction.
Step 1: We show that $v \neq x, y$ and $w \neq x, y$.

By the assumption and the supposition, $v \neq x, y$ and $w \neq x, y$. Moreover, if $x = w = C(T)$, then we have $x = C(T) = C(T \cup \{y\})$ because $x = C(\{x, y\})$ by the assumption and $C$ satisfies Expansion. Therefore, we have $w \neq x$.

Step 2: We show $x P y$, $x \leftrightarrow y$, $y P v$, $v \not\leftrightarrow y$, and $v P x$.

Since $v \neq x$, we have $v \in S$. Because $y = C(S)$ and $C$ satisfies Weak WARP, it must be the case that $y = C(\{v, y\})$. Hence, $y P v$.

It follows from $v = C(S \cup \{x\})$ and Claim (i) in Lemma 4 that there exists $a \in S \cup \{x\}$ such that $v P a$, $a P y$ and $a \leftrightarrow y$. If $a \in S$, then it contradicts $y = C(S)$ by Claim (h) in Lemma 4. Therefore, we have $a = x$. Thus, $v P x$, $x P y$ and $x \leftrightarrow y$.

If $v \leftrightarrow y$, then by Claim (h) in Lemma 4, we have $v \neq C(S \cup \{x\})$, which is a contradiction. Thus, we have $v \not\leftrightarrow y$.

Step 3: We show $y P w$ and $w \leftrightarrow y$.

Let $z = C(T \cup \{y\})$. By the initial supposition, $z \neq w = C(T)$. From the initial assumption and $x \in T$, we have $z \neq y$, and hence $z \in T$. If $z P w$, then $C(\{z, w\}) = z$, which contradicts the fact that $C$ satisfies Weak WARP. Thus, $w P z$. It follows from $z = C(T \cup \{y\})$ and Claim (i) in Lemma 4 that there exists $b \in T \cup \{y\}$ such that $z P b$, $b P w$ and $b \leftrightarrow w$. If $b \in T$, then it contradicts $w = C(T)$ by Claim (h) in Lemma 4. Therefore, we have $b = y$. Thus, $y P w$, and $w \leftrightarrow y$.

Step 4: We show $w \not\leftrightarrow x$.

Assume not: let $w \leftrightarrow x$. Then, by this assumption and the above steps, we have $w \leftrightarrow x$, $w \leftrightarrow y$, and $x \leftrightarrow y$. It follows from Claim (g) in Lemma 4 that $M(\{w, x, y\}; P) \neq \emptyset$. From Step 2 and Step 3, we have $x P y$ and $y P w$ which imply $x = M(\{w, x, y\}; P)$. Then, by asymmetry of $P$, we have $x P w$. From the combination of $x P w$ and $w \leftrightarrow x$, Claim (h) in Lemma 4 states that $w \neq C(T)$. It is a contradiction. Hence, we have $w \not\leftrightarrow x$.

Step 5: We show $v \neq w$, $x P w$, $w P v$, $v \leftrightarrow w$, and $x = C(\{v, w, x, y\})$. 

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By Steps 2 and 3, we have $v \not\leftrightarrow y$ and $w \leftrightarrow y$. Hence, $v \neq w$.

Consider $C(\{v, w, x, y\})$. By Step 2, $x P y$ and $x \leftrightarrow y$. It follows from Claim (h) in Lemma 4 that $y \neq C(\{v, w, x, y\})$. Similarly, since $y P w$ and $w \leftrightarrow y$ by Step 3, we have $w \neq C(\{v, w, x, y\})$.

Since $w \leftrightarrow y$ by Step 3, there exists $A_i \in G(\{v, w, x, y\})$ with $\{w, y\} \subseteq A_i$. Because $v \not\leftrightarrow y$ and $w \not\leftrightarrow x$ by Steps 2 and 4, we have $\{w, y\} = A_i$. From $y P w$ in Step 3, we have $y = M(A_i; P)$, and hence $y \in \cup_{A_k \in G(\{v, w, x, y\})} M(A_k; P)$. Then, since $y P v$ by Step 2, $v \neq M(\cup_{A_k \in G(\{v, w, x, y\})} M(A_k; P); P)$.

Thus, it must be the case that $x = C(\{v, w, x, y\})$. It follows from $v P x$ in Step 2 and Claim (i) in Lemma 4 that there exists $d \in \{w, y\}$ such that $x P d$, $d P v$ and $d \leftrightarrow v$. However, since $v \not\leftrightarrow y$ by Step 2, we have $d \neq y$. Hence, $x P w$, $w P v$ and $v \leftrightarrow w$.

Step 6: We show that there exists $t \in T \setminus \{x, w\}$ such that $t \neq v, y$, $w P t$, $t P x$, and $t \leftrightarrow x$.

Since $w = C(T)$ and $x P w$ by Step 5, it follows from Claim (h) in Lemma 4 that there exists $t \in T \setminus \{x, w\}$ such that $w P t$, $t P x$, and $t \leftrightarrow x$. Because $x = C(\{v, w, x, y\})$ by Step 5, it must be the case that $t \neq v, y$.

Step 7: We show that none of $t, v, w, x$, and $y$ is equal to $C(\{t, v, w, x, y\})$, which is a contradiction.

By Step 2, $x P y$ and $x \leftrightarrow y$. It follows from Claim (h) in Lemma 4 that $y \neq C(\{t, v, w, x, y\})$. By a similar argument, we can show that none of $v, w, x$, and $y$ is equal to $C(\{t, v, w, x, y\})$.

Since $v \leftrightarrow y$ by Step 5, there exists $B_j \in G(\{t, v, w, x, y\})$ with $\{v, w\} \subseteq B_j$. Suppose $t \in B_j$. It follows from $w \leftrightarrow t$, $w P t$ by Step 6, and Claim (h) in Lemma 4 that $t \neq C(\{t, v, w, x, y\})$.

Next, suppose $t \notin B_j$. Because $v \not\leftrightarrow y$ and $w \not\leftrightarrow x$ by Steps 2 and 4, we have $x, y \notin B_j$. Then, $B_j = \{v, w\}$. Since $w P v$ by Step 5, we have $w = M(B_j; P)$, and hence $w \in \cup_{B_k \in G(\{u, v, w, x, y\})} M(B_k; P)$. Because $w P t$ in Step 6, we have $t \neq M(\cup_{B_k \in G(\{u, v, w, x, y\})} M(B_k; P); P) = C(\{t, v, w, x, y\})$. 19
Part 2: We show that if a choice function satisfies Expansion, Weak WARP, and Elimination, then it is a Rational Shortlist Method in which the first rational is transitive.

Assume that a choice function $C$ satisfies Expansion, Weak WARP, and Elimination. In the following, we construct an asymmetric and transitive first rationale and an asymmetric second rationale that sequentially rationalize $C$.

Define $P_1$ and $P_2$ as follows. For all $a, b \in X$ with $a \neq b$,

$$a P_1 b \iff \exists d \in X \setminus \{a, b\} \text{ such that } b = C(\{b, d\}) \text{ and } d = C(\{a, b, d\}).$$

$$a P_2 b \iff a = C(\{a, b\}).$$

In the following, we have three steps.

Step 1: We check that $P_1$ and $P_2$ are asymmetric.

By definition, $P_2$ is asymmetric.

Suppose, on the contrary, that $P_1$ is not asymmetric. Then, there exist $a, b \in X$ such that $a P_1 b$ and $b P_1 a$. By definition, there exist $d, e \in X \setminus \{a, b\}$ such that $b = C(\{b, d\})$, $d = C(\{a, b, d\})$, $a = C(\{a, e\})$, and $e = C(\{a, b, e\})$. If $a = C(\{a, b\})$, then by $a = C(\{a, e\})$ and Expansion, we have $a = C(\{a, b, e\})$, which contradicts $e = C(\{a, b\})$. Similarly, $b = C(\{a, b\})$ leads to a contradiction. Thus, $P_1$ must be asymmetric.

Step 2: We show that $(P_1, P_2)$ sequentially rationalizes $C$.

Let $S \in X$ and $x = C(S)$.

For all $a, b \in X$ with $a \neq b$, either $a = C(\{a, b\})$ or $b = C(\{a, b\})$ holds. Hence, $P_2$ is complete, which implies that $M(M(S; P_1); P_2)$ has at most one element. Thus, we only need to show $x \in M(M(S; P_1); P_2)$. It suffices to check that (1) there exists no $y \in S$ with $y P_1 x$; and (2) for all $y \in S$ with $y P_2 x$, there exists $z \in S$ such that $z P_1 y$.

(1) Suppose, on the contrary, that there exists $y \in S$ with $y P_1 x$. By the definition of $P_1$, there exists $z \in X \setminus \{x, y\}$ such that $x = C(\{x, z\})$ and $z = C(\{x, y, z\})$. By Expansion, $x = C(S \cup \{z\})$. It follows from $\{x, z\} \subseteq$
\{x, y, z\} \subseteq S \cup \{z\}, x = C(\{x, z\}) = C(S \cup \{z\})$, and Weak WARP that $z \neq C(\{x, y, z\})$, which is a contradiction.

(2) Assume \(y \in S\) and \(y P_2 x\). By the definition of \(P_2\), we have \(y = C(\{x, y\})\). Because \(C\) satisfies Expansion and Weak WARP, by Manzini and Mariotti (2007, Theorem 1), there exists a pair of asymmetric binary relations \((Q_1, Q_2)\) that sequentially rationalizes \(C\). Note that \((Q_1, Q_2)\) is not necessarily equal to \((P_1, P_2)\).

Since \(y = C(\{x, y\})\), we have \(y Q_1 x\) or \(y Q_2 x\). Because \(x = C(S)\), it follows from Claim (e) in Lemma 2 that there exist \(k\) alternatives \(z_1, \ldots, z_k \in S \setminus \{x\}\) such that \([x Q_1 z_k\) or \(x Q_2 z_k]\), \(z_1 Q_1 y\), and \(z_{i+1} Q_1 z_i\) and \(z_i Q_2 x\) for all \(i = 1, \ldots, k - 1\) if \(k \geq 2\).

There are two cases: (i) \(k \geq 2\) and (ii) \(k = 1\). First, assume \(k \geq 2\). If \(z_k Q_1 y\), the case is reduced to that of \(k = 1\). Hence, we assume that \(z_k Q_1 y\) does not hold.

Let \(z_h \in \{z_1, \ldots, z_{k-1}\}\) be the alternative such that \(z_k Q_1 z_h\) and for no \(j < h\), \(z_h Q_1 z_j\). Then, for all \(T \in \mathcal{X}\) with \(z_k \in T\), \(z_h \not\in C(T)\).

Consider \(C(\{x, z_h\})\) and \(C(\{x, z_h, z_k\})\). By \(z_h Q_2 x\), we have \(z_h = C(\{x, z_h\})\). Since \(x Q_1 z_k\) or \(x Q_2 z_h\) holds, \(x = C(\{x, z_h\})\). Then, it follows from \(x = C(\{x, z_h\}) = C(S)\) and Weak WARP that \(z_k \neq C(\{x, z_k, z_h\})\).

Next we prove \(x \neq C(\{x, z_k, z_{h-1}\})\) and \(x = C(\{x, z_k, z_{h-1}\})\), where if \(h = 1\), let \(z_{h-1} = y\). If \(x = C(\{x, z_k, z_{h-1}\})\), then by \(z_{h-1} Q_2 x\), we have \(z_k Q_1 z_{h-1}\), which contradicts the definition of \(z_h\). If \(z_{h-1} = y\), it contradicts the assumption that \(z_k Q_1 y\) does not hold. Thus, \(x \neq C(\{x, z_k, z_{h-1}\})\).

Because \(z_k Q_1 z_h\) and \(z_k Q_1 z_{h-1}\), we have \(z_h, z_{h-1} \neq C(\{x, z_k, z_h, z_{h-1}\})\).

It follows from \(x = C(\{x, z_k\}) = C(S)\) and Weak WARP that \(z_k \neq C(\{x, z_k, z_h, z_{h-1}\})\), Therefore, \(x = C(\{x, z_k, z_h, z_{h-1}\})\).

We have shown (A) \(z_h = C(\{x, z_h\})\) and \(z_k \neq C(\{x, z_h \cup \{z_k\})\), and (B) \(x \neq C(\{x, z_k, z_{h-1}\})\) and \(x = C(\{x, z_k, z_{h-1}\}) \cup \{z_h\})\). However, because \(z_h \not\in C(T)\) for all \(T \in \mathcal{X}\) with \(z_k \in T\), it follows from Elimination that at least one of the claims (A) and (B) cannot hold. This is a contradiction.

\footnote{Recall that for all \(a, b \in X\), \(a Q_2 b\) if and only if \(a\) and \(b\) are not comparable in \(Q_1\) and \(a Q_2 b\).}
Hence, it must be the case that \( k = 1 \). That is, \( [x Q z_1] \) or \( [x Q^*_2 z_1] \) and \( z_1 Q_1 y \). By Claim (d) in Lemma 2, we have \( y \neq C(\{x, z_1, y\}) \). It follows from 
\[
F(x) = C(\{x, z_1\}) = C(S) \quad \text{and Weak WARP that} \quad z_1 \neq C(\{x, z_1, y\}).
\]
Therefore, 
\[
x = C(\{x, z_1, y\}).
\]
Recall \( y = C(\{x, y\}) \). Then, by the definition of \( P_1 \), we obtain \( z_1 P_1 y \).

**Step 3**: We show that \((T(P_1), P_2^*)\) sequentially rationalizes \( C \).

We have shown that \((P_1, P_2)\) sequentially rationalizes \( C \). By Lemma 3, if \((P_1, P_2^*)\) satisfies Property T:

\[
\forall a, b \in X : a T(P_1) b \Rightarrow a P_1 b \text{ or } a P_2^* b,
\]

then \((T(P_1), P_2^*)\) also sequentially rationalizes \( C \), which means that \( C \) is a Rational Shortlist Method in which the first rationale is transitive. Therefore, it remains to show that \((P_1, P_2^*)\) satisfies Property T.

Suppose, on the contrary, that there exist \( z_1, \ldots, z_n \in X \) such that \( z_i P_1 z_{i+1} \) for all \( i \in \{1, \ldots, n-1\} \) but neither \( z_1 P_1 z_n \) nor \( z_1 P_2^* z_n \) holds. Then, \( n \geq 3 \).

Because \((P_1, P_2^*)\) sequentially rationalizes \( C \), we have \( z_n = C(\{z_1, z_n\}) \) and \( z_1 = C(\{z_1, z_2, \ldots, z_n\}) \). Then, there exists \( j \in \{1, \ldots, n-2\} \) such that (A) 
\[
C(\{z_1, \ldots, z_j, z_n\}) \neq C(\{z_1, \ldots, z_j, z_n\} \cup \{z_{j+1}\}).
\]
Moreover, by the definition of \( P_1 \), there exists \( w \in X \setminus \{z_j, z_{j+1}\} \) such that (B) \( z_{j+1} = C(\{w, z_{j+1}\}) \) but \( w = C(\{w, z_{j+1}\} \cup \{z_j\}) \neq z_j \).

However, since \( z_j P_1 z_{j+1} \), it follows that \( z_{j+1} \neq C(S) \) for all \( S \in \mathcal{X} \) with \( z_j \in S \). Then, by Elimination, one of the claims (A) and (B) cannot hold, which is a contradiction. Thus, \((P_1, P_2^*)\) satisfies Property T.

**Part 3**: We show that if a choice function is a Rational Shortlist Method in which the first rationale is transitive, then it is a Grouping Choice Method.

Assume that a choice function \( C \) is sequentially rationalized by a pair of asymmetric binary relations \((P_1, P_2)\) and \( P_1 \) is transitive. Define a binary relation \( P \) as follows. For all \( x, y \in X \) with \( x \neq y \),

\[
x P y \iff x = C(\{x, y\}).
\]
Then, $P$ is asymmetric.

For every $S \in \mathcal{X}$, let $g(S)$ be the class of all subsets $S_i$ of $S$ such that for all $x, y \in S_i$ with $x \neq y$, either $x P_1 y$ or $y P_1 x$ holds. Then, we define a correspondence $G$ as follows.

$$G(S) = \{S_i \in g(S) \mid \forall S_j \in g(S) \text{ such that } S_j \neq S_i \text{ and } S_i \subseteq S_j\}.$$ 

That is, $G(S)$ is the class of “maximal” subsets (in inclusion relations) of $S$ in which every element is comparable with every other element in $P_1$.

Now we check that $G$ is a grouping rule, that is, for every $S \in \mathcal{X}$, $G(S)$ satisfies the conditions (G1), (G2), and (G3) in Definition 2.

(G1) By the definition of $G$, $\cup_{S_i \in G(S)} S_i \subseteq S$. For every $x \in S$, because $\{x\} \in g(S)$ by the definition of $g(S)$, there exists $S_i \in G(S)$ with $x \in S_i$. Therefore, we have $S \subseteq \cup_{S_i \in G(S)} S_i$.

(G2) By the definition of $G$, it satisfies (G2).

(G3) Assume that there exists $S_i \in G(S) \subseteq g(S)$ such that $\{x, y\} \subseteq S_i$, and $\{x, y\} \subseteq T$. By definition, either $x P_1 y$ or $y P_1 x$ holds. This implies $\{x, y\} \in g(T)$. Then, there exists $T_j \in G(T)$ such that $\{x, y\} \subseteq T_j$.

Next, we show that $C$ is a grouping choice method with $P$ and $G$. Let $S \in \mathcal{X}$ and $x = C(S)$. First, we prove that there exists $S_k \in G(S)$ such that $x = M(S_k; P)$. Since $G(S)$ satisfies (G1), there exists $S_i \in G(S)$ with $x \in S_i$. If $S_i = \{x\}$, then obviously, $x = M(S_k; P)$. Assume $|S_i| \geq 2$. Let $y \in S_i \setminus \{x\}$. It follows from $x = C(S)$ and Lemma 2 that $y P_1 x$ does not hold. Because $y$ is comparable with $x$ in $P_1$, it must be the case that $x P_1 y$. Then, we have $x = C(\{x, y\})$. By the definition of $P$, $x P y$. This holds for every $y \in S_i \setminus \{x\}$. Thus, we have $x = M(S_i; P)$.

Second, we show $x P y$ for every $y \in S \setminus \{x\}$ such that $y = M(S_j; P)$ for some $S_j \in G(S)$. Suppose, on the contrary, that there exists $y \in S$ such that $y = M(S_j; P)$ for some $S_j \in G(S)$ but $y P y$ does not hold. By the definition of $P$, $x \neq C(\{x, y\})$, and hence $y = C(\{x, y\})$. Then, either $y P_1 x$ or $y P_2 x$. Since $x = C(S)$, it cannot be the case that $y P_1 x$. It follows from $y P_2 x$ and Claim
(e) in Lemma 2 that there exists \( w \in S \) with \( w \mathrel{P_1} y \). Then, \( w = C(\{y, w\}) \).
Hence, we have \( w \mathrel{P} y \). Because \( y = M(S_j; P) \), it follows that \( w \notin S_j \).

If \( S_j = \{y\} \), it contradicts \( \{w, y\} \in g(S) \) and the construction of \( G \). Hence, we have \( |S_j| \geq 2 \). Let \( z \in S_j \setminus \{y\} \). Then, either \( z \mathrel{P_1} y \) or \( y \mathrel{P_1} z \) holds. If \( z \mathrel{P_1} y \), then \( z = C(\{y, z\}) \), which implies \( z \mathrel{P} y \), which contradicts \( y = M(S_j; P) \). Thus, \( y \mathrel{P_1} z \) must be the case. It follows from \( w \mathrel{P_1} y \) and transitivity of \( P_1 \) that \( w \mathrel{P_1} z \). This holds for every \( z \in S_j \setminus \{y\} \). Hence, we have \( S_j \cup \{w\} \in g(S) \), which contradicts \( S_j \in G(S) \).

**Proof of Theorem 3**

To prove the theorem, we use a standard property of choice consistency. It says that if an alternative in a set \( S \) “wins” over every other alternative in \( S \) in binary choices, then it should be chosen in \( S \).

**Condorcet Consistency:** For every \( S \in \mathcal{X} \), if there exists \( x \in S \) such that \( x = C(\{x, y\}) \) for every \( y \in S \setminus \{x\} \), then \( x = C(S) \).

The following lemma may be interesting of itself.

**Lemma 5.** If a choice function \( C \) satisfies Grouping Path Independence for a grouping rule \( G \), then it satisfies Condorcet Consistency.

*Proof.* The proof is by induction. Assume that a choice function \( C \) satisfies Grouping Path Independence for a grouping rule \( G \). Let \( S \in \mathcal{X} \). Assume that there exists \( x \in S \) such that \( x = C(\{x, y\}) \) for every \( y \in S \setminus \{x\} \). If \( |S| = 2 \), then by the above assumption, \( x = C(S) \). Assume \( x = C(S) \) holds if \( |S| \leq k - 1 \) where \( k \geq 3 \). We show that it also holds if \( |S| = k \).

We divide two cases: (i) \( G(S) = \{S\} \) and (ii) \( G(S) \neq \{S\} \). First, assume \( G(S) = \{S\} \). Consider the family of subsets of \( S \), \( \{\{x\}, S \setminus \{x\}\} \), which is obtained by iteratively splitting from \( G(S) \). Since \( C \) satisfies Grouping Path Independence for \( G \), we have \( C(S) = C(\{C(\{x\}), C(\{S \setminus \{x\}\})\}) = C(\{x, v\}) \) where \( v = C(S \setminus \{x\}) \). By the initial assumption, we have \( x = C(\{x, v\}) \). Hence, we have \( x = C(S) \).
Second, assume \( G(S) \neq \{ S \} \). Let \( G(S) = \{ S_1, \ldots, S_n \} \) where \( n \geq 2 \). Without loss of generality, assume \( x \in S_1 \). Consider the family of subsets of \( S \), \( \{ S_1, \cup_{k \in G(S), k \neq S_1} S_k \} \), which is obtained by iteratively merging from \( G(S) \). From condition (G2) in the definition of grouping rules, it cannot be the case that \( S_1 = S \). Hence, we have \( |S_1| \leq k - 1 \). By the assumption of induction, we have \( x = C(S_1) \). Now, since \( C \) satisfies Grouping Path Independence for \( G \), we have \( C(S) = C(\{ C(S_1), C(\cup_{k \in G(S), k \neq S_1} S_k) \}) = C(\{ x, y \}) \) where \( y = C(\cup_{k \in G(S), k \neq S_1} S_k) \). By the initial assumption, we have \( x = C(\{ x, y \}) \).

Therefore we have \( x = C(S) \).

We now prove Theorem 3.

Let a grouping rule \( G \) be given.

[Sufficiency]
Assume that a choice function \( C \) is a Grouping Choice Method with \( G \) and some asymmetric binary relation \( P \). We show that \( C \) satisfies Grouping Path Independence for \( G \).

Let \( S \in X \). Let \( \Sigma = \{ T_1, \ldots, T_m \} \) be obtained either by iteratively merging or by iteratively splitting from \( G(S) = \{ S_1, \ldots, S_n \} \). We need to show that \( C(\cup_{T_i \in \Sigma} C(T_i)) = C(S) \). Let \( x = C(S) = M(\cup_{j \in G(S)} M(S_j; P); P) \).

Without loss of generality, assume \( x \in T_1 \in \Sigma \). We show \( x = C(T_1) \).

Assume that \( \Sigma \) is obtained by iteratively merging from \( G(S) \). Suppose, on the contrary, that \( x \neq C(T_1) = y \). By Claim (f) in Lemma 4, \( P \) is complete.

Case 1: \( y \mathbin{P} x \).
There exists \( S_k \in G(S) \) with \( y \in S_k \subseteq T_1 \). From Claims (f) and (g) in Lemma 4, \( M(S_k; P) = \{ z \} \) for some \( z \in S_k \subseteq T_1 \). Then, \( z \neq y \) because \( x = M(\cup_{j \in G(S)} M(S_j; P); P) \) and \( y \mathbin{P} x \). Hence, we have \( z \in T_1 \), \( y \leftrightarrow z \), and \( z \mathbin{P} y \). It follows from Claim (h) in Lemma 4 that \( y \neq C(T_1) \), which is a contradiction.

Case 2: \( x \mathbin{P} y \).
We have \( x = C(\{ x, y \}) = C(S) \) and \( \{ x, y \} \subseteq T_1 \subseteq S \). It follows from Weak WARP that \( y \neq C(T_1) \), which is a contradiction.

Hence, it must be the case that \( x = C(T_1) \).
Next, assume that Σ is obtained by iteratively splitting from \(G(S)\). Then, there exists \(S_j \in G(S)\) such that \(T_1 \subseteq S_j\). It follows from \(x = C(S)\) and Claims (f) and (h) in Lemma 4 that \(x P y\) for every \(y \in S_j \setminus \{x\}\) and hence for every \(y \in T_1 \setminus \{x\}\). Thus, it must be case that \(x = C(T_1)\).

We have shown \(x \in \bigcup_{T_i \in \Sigma} C(T_i)\). To show \(x = C(\bigcup_{T_i \in \Sigma} C(T_i))\), suppose, on the contrary, \(C(\bigcup_{T_i \in \Sigma} C(T_i)) = y \neq x\). As in Case 2 above, supposing \(x P y\) leads to a contradiction. Suppose \(y P x\). As in Case 1 above, if \(\Sigma\) is obtained by iteratively merging from \(G(S)\), then we have a contradiction. Suppose that \(\Sigma\) is obtained by iteratively splitting from \(G(S)\). There exists \(S_k \in G(S)\) with \(y \in S_k\). From Claims (f) and (g) in Lemma 4, \(M(S_k; P) = \{z\}\) for some \(z \in S_k\). Because \(y P x\) and \(x = M(\bigcup_{S_j \in G(S)} M(S_j; P); P)\), it must be the case that \(z \neq y\). Hence, \(z P y\) and \(y \leftrightarrow z\). There exists \(T_h \in \Sigma\) such that \(z \in T_h \subseteq S_k\). Then, \(z P w\) for every \(w \in T_h \setminus \{z\}\). Therefore, \(z = C(T_h) \subseteq \bigcup_{T_i \in \Sigma} C(T_i)\). It follows from \(z P y, y \leftrightarrow z, z \in \bigcup_{T_i \in \Sigma} C(T_i)\), and Claim (h) in Lemma 4 that \(y \neq C(\bigcup_{T_i \in \Sigma} C(T_i))\), which is a contradiction. Thus, we have \(x = C(\bigcup_{T_i \in \Sigma} C(T_i))\).

**[Necessity]**

Assume that a choice function \(C\) satisfies Grouping Path Independence for \(G\). Define a binary relation \(P\) as follows: for all \(x, y \in X\), \(x P y \iff x = C(\{x, y\})\). Because either \(C(\{x, y\}) = x\) or \(C(\{x, y\}) = y\) holds, \(P\) is complete and asymmetric. Hence, for every \(A \in \mathcal{X}\), \(|M(A; P)| \leq 1\).

Let \(S \in \mathcal{X}\) and \(x = C(S)\). We show that \(\{x\} = M(\bigcup_{S_j \in G(S)} M(S_j; P); P)\). First, we show that \(\{x\} = M(S_k; P)\) for some \(S_k \in G(S)\). Suppose, on the contrary, \(x \notin M(S_j; P)\) for all \(S_j \in G(S)\) with \(x \in S_j\). Then, for every \(S_j \in G(S)\) with \(x \in S_j\), there exists \(y_j \in S_j\) such that \(y_j P x\). By definition, \(y_j = C(\{x, y_j\})\). Construct a family \(\Sigma\) of subsets of \(S\) as follows:

\[
\Sigma = \{\{x, y_j\}, S_j \setminus \{x, y_j\} \mid S_j \in G(S) \text{ and } x \in S_j\} \cup \{S_j \mid S_j \in G(S) \text{ and } x \notin S_j\}.
\]

Then, \(\Sigma\) is obtained by iteratively splitting from \(G(S)\). Now we have \(x \neq C(T_i)\) for all \(T_i \in \Sigma\), and hence \(x \neq C(\bigcup_{T_i \in \Sigma} C(T_i))\). However, by Grouping
Path Independence, we have \( x = C(S) = C(\cup_{T_i \in \Sigma} C(T_i)) \neq x \), which is a contradiction. Thus, we have \( \{x\} = M(S_k; P) \) for some \( S_k \in G(S) \).

Second, we show \( x P y \) for every \( y \in S \) such that \( \{y\} = M(S_j; P) \) for some \( S_j \in G(S) \). Suppose, on the contrary, that there exists \( y \in S \) such that \( \{y\} = M(S_j; P) \) and \( y P x \). Because \( P \) is complete, it follows that for every \( z \in S_j \), \( y P z \), and hence \( y = C(\{y, z\}) \). Since \( C \) satisfies Condorcet Consistency by Lemma 5, we have \( y = C(S_j) \). Consider the family of subsets of \( S \), \( \{S_j, \cup_{S_h \in \mathcal{S}(S), S_h \neq S_j} S_h\} \). This family is obtained by iteratively merging from \( G(S) \). Because \( C \) satisfies Grouping Path Independence for \( G \), we have \( C(S) = C(\{C(S_j), C(\cup_{S_h \in \mathcal{S}(S), S_h \neq S_j} S_h)\}) = C(\{y, C(\cup_{S_h \in \mathcal{S}(S), S_h \neq S_j} S_h)\}) \). However, since \( y P x \), we have \( y = C(\{y, x\}) \). It follows that \( C(\{y, C(\cup_{S_h \in \mathcal{S}(S), S_h \neq S_j} S_h)\}) \neq x \), which contradicts \( x = C(S) = C(\{y, C(\cup_{S_h \in \mathcal{S}(S), S_h \neq S_j} S_h)\}) \). Thus, \( x P y \) for every \( y \in S \) such that \( \{y\} = M(S_j; P) \) for some \( S_j \in G(S) \). Therefore, \( \{x\} = M(\cup_{S_j \in G(S)} M(S_j; P); P) \).


References


