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Stability Switches in a Neutral Delay Differential Equation with Application to Real-Time Dynamic Substructuring

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Abstract. In this paper delay differential equations approach is used to model a real-time dynamic substructuring experiment. Real-time dynamic substructuring involves dividing the structure under testing into two or more parts. One part is physically constructed in the laboratory and the remaining parts are being replaced by their numerical models. The numerical and physical parts are connected via an actuator. One of the main difficulties of this testing technique is the presence of delay in a closed loop system. We apply real-time dynamic substructuring to a nonlinear system consisting of a pendulum attached to a mass-spring-damper. We will show how a delay can have (de)stabilising effect on the behaviour of the whole system. Theoretical results agree very well with experimental data.

Introduction

There are several seismic testing techniques for studying the response of the complex engineering systems under the earthquake excitation. One of them is the real-time dynamic substructuring (see, for example [1, 2]). This testing approach comprises dividing the structure into two, sometimes more parts. One part of the whole structure is then placed in the laboratory, it is called a "substructure", and the remainder of the system is replaced by its numerical counterpart. The parts are connected together through a transfer system, usually an actuator, and this hybrid experiment is run in real-time. Use of actuators and the fact that the whole experiment is conducted in real time introduces inevitable time delays into the system. The system investigated in this paper is an autoparametric pendulum attached to a mass-spring-damper (MSD). The MSD is taken to be the numerical part of the experiment. The pendulum is a physical structure, i.e. it is constructed and placed in the laboratory, and it swings in the vertical plane. The numerical and physical parts are connected together via an electrically driven actuator. The obvious question arising from the application of the real-time dynamic substructuring is whether the hybrid system behaves in the same manner as the real (un-substructured) system would. As it was mentioned earlier, the use of an actuator induces a presence of time lags in the system. This can lead to the destabilization of the experiment, and consequently, to spurious results.

To study the effect time delays have on the system, we propose a system of delay differential equations as a mathematical model of the experiment. We analytically show how, depending on the value of the time delay, the system can lose and regain its stability. The idea of modelling with delay differential equations (DDEs) was proposed in [3] to study a single mass-spring oscillator. The authors have shown that, even for this simple linear system, the delay plays a crucial role in determining the stability of the system and accuracy of the results.

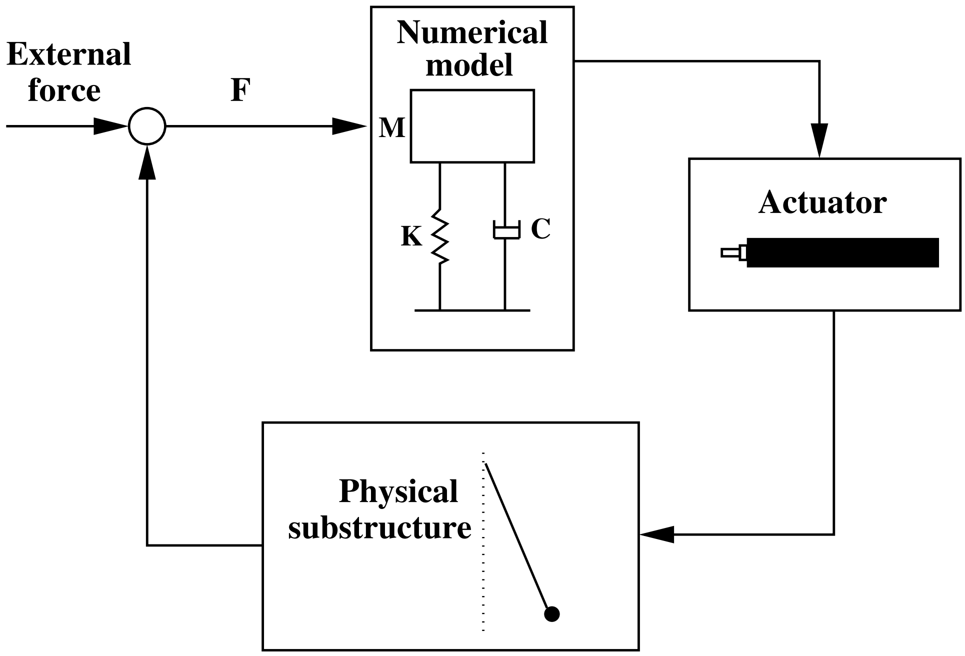

Fig. 1: Real-time substructuring loop: numerical model-transfer system-substructure.

The paper is organized as follows: Section 2 introduces a system of delay differential equations describing an MSD coupled to a pendulum. Section 3 is devoted to the stability analysis of the pendulum-MSD system for small angles. Section 4 gives a comparison between analytical and experimental results. The paper concludes with the summary of results and future plans presented in Section 5.

Mathematical description of the system

We consider the system which consists of a mass M mounted on a linear spring, which is carrying a pendulum. The pendulum of mass m_{pend} is attached to a hinged massless rod of length l. The angle θ denotes an angular deflection of the pendulum from the downward vertical position. We also take a linear viscous damping. The closed loop system is depicted in Figure 1. Since the transfer system produces a delay, the feedback force will also be delayed. Let the delay time be denoted by τ. To account for the delay in the displacement signal, the force in the system has to be described by the delayed state of the numerical model for the MSD

$$\begin{aligned}
&M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m_{\text{p}}\ddot{y}(t-\tau) \\
&+m_{\text{pend}}l[\ddot{\theta}(t-\tau)\sin\theta(t-\tau) + \dot{\theta}^2(t-\tau)\cos\theta(t-\tau)] = F_{\text{ext}}, \\
&m_{\text{pend}}l^2\ddot{\theta}(t-\tau) + k_{\text{pend}}\dot{\theta}(t-\tau) + m_{\text{pend}}g\ell\sin\theta(t-\tau) \\
&+m_{\text{pend}}l\ddot{y}(t-\tau)\sin\theta(t-\tau) = 0,
\end{aligned} \tag{1}$$

where F_{ext} is the external excitation applied in the y direction, C and K are the damping and stiffness coefficients respectively, and dot indicates the derivative with respect to time t. We will refer to the $y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$ as the position, velocity and acceleration of MSD at time t.

Since it is important to study the interaction of the pendulum with the MSD, which is simulated, we set $F_{\text{ext}} = 0$, and this converts system (1) into an autonomous system of DDEs.

Neutral delay differential equation: stability

We start by considering the case when the angle θ is close to zero ($\theta \ll 1$). The system (1) then decouples, and the second equation corresponds to decaying oscillations of a mathematical pendulum. We concentrate on the first equation, which describes the vertical motion of the pendulum-MSD system. This equation now has the form

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m_{\text{pend}}\ddot{y}(t-\tau) = 0. \tag{2}$$

Since time delay is present in the highest derivative in the equation, the equation is called a *neutral* delay differential equation. It can be written in non-dimensionalized form as

$$\ddot{z} + 2\zeta\dot{z} + z + p\ddot{z}(t-\tau) = 0, \tag{3}$$

where dot means differentiation with respect to t, and the parameters are rescaled as follows

$$\hat{t} = \omega_n t,\ \hat{\tau} = \omega_n \tau,\ \omega_n = \sqrt{\frac{K}{M}},\ p = \frac{m_{\text{pend}}}{M},\ \zeta = \frac{C}{2\sqrt{MK}},$$

and the hats are omitted for brevity. This equation has a trivial steady state $z = \dot{z} = 0$ with the corresponding characteristic equation

$$\lambda^2 + 2\zeta\lambda + 1 + p\lambda^2 e^{-\lambda\tau} = 0. \tag{4}$$

When $\tau = 0$, the characteristic equation has solutions with negative real parts only, so the steady state $z = 0$ is locally asymptotically stable. In the case $|p| > 1$, this steady state is always unstable for any positive delay τ. Therefore, we assume $|p| < 1$ in the subsequent analysis. The purely imaginary eigenvalues occur when $\lambda = \pm iv$ for $v \neq 0$, i.e. from the equation (4)

$$-v^2 + 2i\zeta v + 1 - pv^2 e^{-iv\tau} = 0.$$

Separating the last expression into the real and imaginary parts, and solving the resulting system for v and τ gives

$$v_\pm^2 = \frac{1}{(1-p^2)}\left[(1-2\zeta^2) \pm \sqrt{(1-2\zeta^2)^2 - (1-p^2)}\right], \tag{5}$$

and

$$\tau = \frac{1}{v_\pm}\left[\text{Arctan}\,\frac{2\zeta v_\pm}{{v_\pm}^2 - 1} \pm \pi n\right], \tag{6}$$

where $n = 0, 1, 2, ...$ and Arctan corresponds to the principal value of the arctan.

Theorem 1 [4]. *The solution $z = \dot{z} = 0$ of the system (3) is asymptotically stable for $\zeta > \frac{1}{\sqrt{2}}$ and $|p| < 1$ independently of delay time $\tau > 0$. Furthermore, for $\zeta < 1/\sqrt{2}$, the trivial solution of the system (3) is locally asymptotically stable for all positive delay time τ with*

$$p < 2\zeta\sqrt{1-\zeta^2}, \tag{7}$$

and in the region

$$2\zeta\sqrt{1-\zeta^2} < p < 1,$$

for the values of delay

$$\begin{aligned} 0 < \quad \tau \quad &< \frac{1}{v_+}\left[2\pi - \text{Arccos}\,\frac{1-v_+^2}{pv_+^2}\right] \\ \frac{1}{v_-}\left[2\pi n - \text{Arccos}\,\frac{1-v_-^2}{pv_-^2}\right] < \quad \tau \quad &< \frac{1}{v_+}\left[(2n+2)\pi - \text{Arccos}\,\frac{1-v_+^2}{pv_+^2}\right], \end{aligned}$$

where $n = 1, 2, ...$

In terms of original parameters, the condition of the first statement of the Theorem 1 requires that the square of the damping should be greater than twice the mass of the MSD multiplied by its stiffness, i.e. $C^2 > 2MK$, and also the ratio of the mass of the pendulum to the mass of the mass-spring-damper should be less than 1. This latter condition perfectly justifies the choice of pendulum to be the substructure, while the mass of the MSD can be easily adjusted in numerical model. For the proof of this Theorem, see [4].

Let us introduce a sequence

$$\{p_j : p_j > p_{j+1}, p_0 = 1, j = 1, ...\},$$

where $p_n, n = 1, ...$ solve the following equation

$$\tau_-(n) = \tau_+(n+1). \tag{8}$$

Here we have used the notation

$$\tau_+(n) = \frac{1}{v_+}\left[2\pi n - \text{Arccos}\,\frac{2\zeta v_+}{{v_+}^2 - 1}\right],\ \tau_-(n) = \frac{1}{v_-}\left[2\pi n - \text{Arccos}\,\frac{2\zeta v_-}{{v_-}^2 - 1}\right].$$

The detailed result on the stability switches is formulated below.

Theorem 2 [4]. *If* $p_1 \le p < 1$, *there is one stability switch at* $\tau_+(0)$. *The trivial equilibrium is stable for* $0 \le \tau \le \tau_+(0)$ *and unstable for* $\tau > \tau_+(0)$. *If* $p_{k+1} \le p < p_k$ *for* $k = 1, ...,$ *then there are exactly* $(2k+1)$ *stability switches, and the trivial equilibrium is stable for*

$$(0, \tau_+(0)) \cup (\tau_-(1), \tau_+(1)) \cup ... \cup (\tau_-(k), \tau_+(k)),$$

and unstable for

$$(\tau_+(0), \tau_-(1)) \cup (\tau_+(1), \tau_-(2)) \cup ... \cup (\tau_+(k), \infty).$$

This Theorem gives the stability and instability regions of the system in terms of the delay time τ for a given value of p (i.e. for a given ratio of mass of the pendulum to the mass of the MSD). Except for the case when mass ratio p is smaller than the lower stability border $p_c = 2\zeta\sqrt{1-\zeta^2}$, there will always be large enough delay τ after which the system will be unstable. Each time we cross stability border into an unstable region, the delayed action of the pendulum on the MSD leads to a destabilization of the numerical model. At the points $(p, \tau) = (p_n, \tau_n)$ the system undergoes codimension two Hopf bifurcation. This means that there is a pair of complex conjugate eigenvalues crossing the imaginary axis from left to right, and there is another pair crossing from right to left. Therefore, at these points the system has two frequencies simultaneously present. Possible resonances in this case are studied in [5].

Theoretical versus experimental stability results

In the last section, we have derived analytical expressions for the stability boundary depending on the parameters of the system. In order to compare theoretical results with the experimental observations, we perform a series of real experiments on the coupled MSD-pendulum system. A dSpace DS1104 RD Controller Board is employed to implement a real-time performance, with Matlab/Simulink used to represent a numerical model (mass-spring-damper). We assume that the actuator delay is the main source of time delay in the system (the exact delay in the experiment is 18ms). The experimental values of the system parameters are $m = 0.9$kg, $C = 20$kg/s, $K = 5000$N/m. Figure 2 illustrates the comparison between experimental and theoretical stability border. First, we fix the value of $M = 4$kg and vary the time delay, which allows one to find regions of stable and unstable behaviour, as shown in Figure 2(a). Solid line is a theoretical stability boundary, with areas above it being stable regions, and areas below it being the regions of instability. Diamonds show experimentally unstable points, which lie exactly on the theoretical boundary curve. Stars represent experimental stability points and these are also well agreed with the analytical findings. In order to follow the stability boundary we allow M to vary (Figure 2(b)). Again solid line is the theoretical stability border, and triangles are experimentally stable points. These figures show an excellent agreement between theoretical findings and experimental observations.

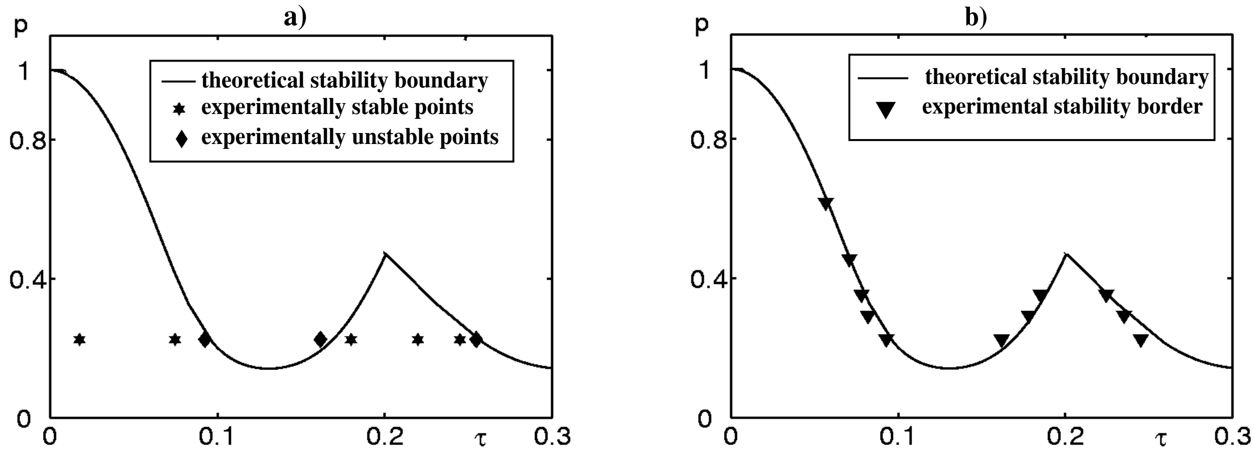

Fig. 2: Experimental and theoretical stability borders.

Conclusions

This paper is devoted to the stability analysis of the real-time dynamic substructuring testing technique applied to the pendulum coupled to a mass-spring-damper system. Mathematically, the system was modelled using delay differential equations (DDEs) approach. This allowed us to include the effects of the time delay present in the experiment.

We started by deriving the equations of motion represented by the system of two neutral delay differential equations. This system was studied for the case of small angles $\theta \ll 1$, which gives a single neutral DDE for the vertical motion. We performed the stability analysis of this equation and found that depending on the value of the time delay, the system undergoes successive changes in its stability. This means that an initially stable system becomes unstable as the delay time increases. For even larger time delays the system regains its stability. If the delay time is larger still, the stability is lost again. Stability analysis has also shown that if the ratio of the masses of the pendulum and the mass-spring-damper is larger than 1, the experiment will be always unstable.

A series of experimental tests was performed, and they confirm our analytical findings. The experimental stability points were compared with analytically found stability border, and the agreement is excellent.

Further details regarding numerical simulations of the present system and the case of viscous damping present in the connections, can be found in [4].

References

[1] A. Blakeborough, M.S. Williams, A.P. Darby and D.M. Williams, The development of real-time substrucute testing, *Proc. R. Soc. A* **359**, 1869-1891 (2001).

[2] M.S. Williams, A. Blakeborough, Laboratory testing of structures under dynamic loads: an introductory review, *Proc. Roy. Soc. Lond.* A **359**, 1651-1669 (2001).

[3] M.I. Wallace, J. Sieber, S.A. Neild, D.J. Wagg and B. Krauskopf, Stability analysis of real-time dynamic substructuring using delay differential equations models, *Earthq. Eng. Struct. Dyn.* **34**, 1817-1832 (2005).

[4] Y.N. Kyrychko, K.B. Byuss, A. Gonzalez-Buelga, S.J. Hogan and D.J. Wagg, Real-time dynamic substructuring in a coupled oscillator-pendulum system, *Proc. R. Soc. A* **462**, 1271-1294 (2006).

[5] S.A. Campbell, Resonant codimension two bifurcation in a neutral functional differential equation, *Nonl. Anal. TMA* **30**, 4577-4594 (1997).