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JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 176 (2005) 433–443

www.elsevier.com/locate/cam

Persistence of travelling wave solutions of a fourth order diffusion system

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Received 11 February 2004; received in revised form 18 May 2004

Abstract

In this paper the extended Burgers–Huxley equation with the fourth-order derivative is considered. First, the convergence to the uniform steady state is proved, which means the solution of the equation with positive initial data will remain positive for time t sufficiently large. Then, the persistence of the travelling wave solution for the extended equation on the unbounded domain is investigated. We have proved that this solution will persist under small perturbation of the equation.

Keywords: Burgers–Huxley equation; Convergence; Perturbation theory; Travelling wave solution

1. Background

Almost all branches of mathematics and physics are associated with problems involving nonlinear partial differential equations. These equations model diverse real-life phenomena in biology, chemistry, physics, etc., and understanding the behaviour of their solutions provides an important insight in the dynamics of the underlying problem. A fundamental equation used in modelling of diffusion processes is the KPP-Fisher equation [18], which admits travelling front solutions connecting the two steady states. An area of recent active interest is the improvement of this model by including temporal delay, long-range

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doi:10.1016/j.cam.2004.07.028

diffusion and higher order nonlinearities [18,19]. One of possible generalisations of the Fisher equation is the so-called Burgers–Huxley equation which has the following form

$$u_t = u_{xx} - \alpha u u_x - \beta u(u-1)(u-\gamma), \tag{1}$$

where the real parameters α, β are positive and γ can be of either sign. This equation includes as particular cases several known evolution equations: when $\beta = 0$ it reduces to the Burgers equation; when $\alpha = 0$ it is the FitzHugh–Nagumo equation [12,13], and when $\alpha = 0$ and $\gamma = -1$ it is the Newell–Whitehead equation [16]. Symmetries and integrability of this equation have been addressed by Estévez and Gordoa [8] (see also [7,9]). The following analytical expression for the travelling wave solutions connecting the two steady states $u=0$ and $u=1$ of Eq. (1) has been recently found with the help of symbolic computations and relevant nonlinear transformations [10,21]:

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\beta}{r-\alpha}(x-ct)\right], \tag{2}$$

where $r = \sqrt{\alpha^2 + 8\beta}$, and the wave speed is defined as

$$c = \frac{(\alpha - r)(2\gamma - 1) + 2\alpha}{4}. \tag{3}$$

In this paper we consider the extended Burgers–Huxley equation

$$u_t = -\delta u_{xxxx} + u_{xx} - \alpha u u_x - \beta u(u-1)(u-\gamma) \tag{4}$$

with the parameters $\alpha > 0$, $\beta > 0$, $\gamma < 0$, $\delta > 0$, where the fourth-order derivative term is added to account for long-range effects as they appear, for example, in the studies of population dynamics [18,4]. The questions to be addressed are as follows. First, we consider the Eq. (4) on a finite domain with periodic boundary conditions and prove the convergence result, namely, that under certain restrictions on the initial data $u(x,0)$, the solutions of (4) tend to 1 uniformly in x. Then we use the geometric singular perturbation theory to prove the persistence of the travelling wave solutions (2) of the Eq. (1) in the presence of a small fourth-order derivative term ($\delta \ll 1$). These travelling waves are qualitatively similar to those of the Burgers–Huxley equation.

2. Nonlinear stability of the uniform steady state $u = 1$.

In this section we employ the technique used by Bartuccelli et al. [3] to prove the convergence result for the Eq. (4) in the following setting:

$$\begin{aligned} &u_t = -\delta u_{xxxx} + u_{xx} - \alpha u u_x - \beta u(u-1)(u-\gamma), \ 0 \leqslant x \leqslant L, \ t > 0, \\ &\text{initial condition } u(x,0) = u_0(x), \\ &\text{periodic boundary conditions at } x = 0, L. \end{aligned} \tag{5}$$

We centre Eq. (5) on the uniform steady state $u \equiv 1$ by introducing a function $v(x,t)$ as

$$u(x,t) = 1 + v(x,t) \tag{6}$$

and the following time-dependent functionals

$$J_N := \left\| \frac{\partial^N v}{\partial x^N} \right\|_2^2 = \int_0^L \left(\frac{\partial^N v}{\partial x^N} \right)^2 \mathrm{d}x.$$

Substituting (6) in (5) we obtain an equation for the function v in the form

$$v_t = -\delta v_{xxxx} + v_{xx} - \alpha v_x - \alpha v v_x - \beta(1-\gamma)v - \beta(2-\gamma)v^2 - \beta v^3. \tag{7}$$

If one can show that $\|v(\cdot,t)\|_\infty \to 0$ as $t \to \infty$ then this implies a uniform convergence of solutions of the Eq. (5) to the non-trivial steady state $u = 1$. Thus, we start the analysis by investigating the evolution equation for the L^2-norm of v, namely, J_0. Differentiating J_0 with respect to time and inserting the RHS of (7) after some computations give

$$\frac{1}{2}\dot{J}_0 = -\delta J_2 - J_1 - \beta(1-\gamma)J_0 - \beta(2-\gamma)\int_0^L v^3\,\mathrm{d}x - \beta\int_0^L v^4\,\mathrm{d}x, \tag{8}$$

where the α-terms vanish under the periodic boundary conditions. By using the fact that

$$-\beta(2-\gamma)\int_0^L v^3\,\mathrm{d}x \leqslant \beta(2-\gamma)\|v\|_\infty J_0$$

and

$$-\beta\int_0^L v^4\,\mathrm{d}x \leqslant -\beta L^{-1} J_0^2,$$

the Eq. (8) turns into

$$\tfrac{1}{2}\dot{J}_0 \leqslant -\delta J_2 - J_1 - \beta(1-\gamma)J_0 + \beta(2-\gamma)\|v\|_\infty J_0 - \beta L^{-1} J_0^2. \tag{9}$$

Now, the last term to be estimated is $\|v\|_\infty$. This can be achieved using the following recent interpolation inequality with the sharp and explicit constant [14]

$$\|v\|_\infty \leqslant c J_2^{1/8} J_0^{3/8} + L^{-1/2} J_0^{1/2},\ c = \left(\frac{4}{27}\right)^{1/8}. \tag{10}$$

Applying this inequality to the fourth term in (9) we arrive at

$$\begin{aligned}\tfrac{1}{2}\dot{J}_0 \leqslant & -\delta J_2 - J_1 - \beta(1-\gamma)J_0 + \beta c(2-\gamma)J_2^{1/8}J_0^{11/8} \\ & + \beta(2-\gamma)L^{-1/2}J_0^{3/2} - \beta L^{-1}J_0^2.\end{aligned} \tag{11}$$

Next, we employ Young's inequality to split the term $\beta c(2-\gamma)J_2^{1/8}J_0^{11/8}$ into two as follows:

$$\beta c(2-\gamma)J_2^{1/8}J_0^{11/8} \leqslant \frac{1}{8}\delta J_2 + \frac{7}{8}\frac{(\beta c(2-\gamma))^{8/7}}{\delta^{1/7}} J_0^{11/7}.$$

Substituting this expression in (11) one obtains

$$\begin{aligned}\frac{1}{2}\dot{J}_0 \leqslant & -\frac{7\delta}{8}J_2 - J_1 - \beta(1-\gamma)J_0 + \frac{7}{8}\frac{(\beta c(2-\gamma))^{8/7}}{\delta^{1/7}}J_0^{11/7} \\ & + \beta(2-\gamma)L^{-1/2}J_0^{3/2} - \beta L^{-1}J_0^2.\end{aligned}$$

By neglecting the first two negative-definite terms one gets the following inequality for J_0

$$\frac{1}{2}\dot{J}_0 \leqslant -\beta(1-\gamma)J_0 + \frac{7}{8}\frac{(\beta c(2-\gamma))^{8/7}}{\delta^{1/7}}J_0^{11/7} + \beta(2-\gamma)L^{-1/2}J_0^{3/2} - \beta L^{-1}J_0^2 := f(J_0). \tag{12}$$

From assumption $\beta > 0$, $\gamma < 0$ it follows that $\beta(1-\gamma) > 0$, and we may conclude that for J_0 small, and for J_0 large the function $f(J_0)$ is negative. It is easy to check that

$$f(L) = \frac{7}{8}\frac{(\beta c(2-\gamma))^{8/7}}{\delta^{1/7}}L^{11/7} > 0,$$

hence $f(J_0)$ is positive in some intermediate range. Therefore, if $J_0(0) < J^*$, where J^* is the smallest positive root of $f(J_0) = 0$, then $J_0 \to 0$ as $t \to \infty$.

Next, we differentiate the Eq. (7) with respect to x and obtain the following evolution equation for J_1:

$$\frac{1}{2}\dot{J}_1 = -\delta\int_0^L v_x v_{xxxxx}\,\mathrm{d}x + \int_0^L v_x v_{xxx}\,\mathrm{d}x - \alpha\int_0^L v_x v_{xx}\,\mathrm{d}x - \alpha\int_0^L v v_x v_{xx}\,\mathrm{d}x - \alpha\int_0^L v_x^3\,\mathrm{d}x - \beta(1-\gamma)\int_0^L v_x^2\,\mathrm{d}x - 2\beta(2-\gamma)\int_0^L v v_x^2\,\mathrm{d}x - 3\beta\int_0^L v^2 v_x^2\,\mathrm{d}x.$$

After some computations we obtain

$$\frac{1}{2}\dot{J}_1 \leqslant -\frac{7\delta}{8}J_3 - J_2 + \frac{7}{8\delta^{1/7}}\left(\frac{\alpha c}{2}\right)^{8/7}J_1^{11/7} - \beta(1-\gamma)J_1 + \frac{\beta(2-\gamma)^2}{3}J_1.$$

Finally, combining the last two terms gives

$$\frac{1}{2}\dot{J}_1 \leqslant -\frac{7\delta}{8}J_3 - J_2 + \frac{7}{8\delta^{1/7}}\left(\frac{\alpha c}{2}\right)^{8/7}J_1^{11/7} + \frac{\beta(\gamma^2-\gamma+1)}{3}J_1.$$

By omitting the second negative-definite term and using the fact that $-J_3 \leqslant -J_1^3/J_0^2$ we arrive at

$$\frac{1}{2}\dot{J}_1 \leqslant -\frac{7\delta}{8}\frac{J_1^3}{J_0^2} + \frac{7}{8\delta^{1/7}}\left(\frac{\alpha c}{2}\right)^{8/7}J_1^{11/7} + \frac{\beta(\gamma^2-\gamma+1)}{3}J_1. \tag{13}$$

As it was previously proved, $J_0 \to 0$ as $t \to \infty$, and consequently, we may conclude that

$$J_1(t) \leqslant \text{const}, \; t \geqslant 0.$$

With the help of the interpolation inequality for $\|v\|_\infty$ in the form

$$\|v\|_\infty \leqslant J_1^{1/4}J_0^{1/4} + L^{-1/2}J_0^{1/2}$$

and employing the above-mentioned results we have that $\|v\|_\infty \to 0$ as $t \to \infty$, and accordingly $\lim_{t\to\infty}u(x,t) = 1$ uniformly in x.

Below, we summarise our findings in the following theorem which represents a condition on the initial data that is sufficient for convergence.

Theorem 1. *Suppose that* $\alpha, \beta, \delta > 0$ *and* $\gamma < 0$. *If the initial data satisfy*

$$\int_0^L (u(x,0)-1)^2 \, \mathrm{d}x < J^*,$$

where J^* *is the smallest positive root of*

$$f(J_0) = -\beta(1-\gamma)J_0 + \frac{7}{8}\frac{(\beta c(2-\gamma))^{8/7}}{\delta^{1/7}} J_0^{11/7} + \beta(2-\gamma)L^{-1/2}J_0^{3/2} - \beta L^{-1} J_0^2 = 0$$

with $c = \left(\frac{4}{27}\right)^{1/8}$, *then the solution* $u(x,t)$ *of* (5) *satisfies*

$$\lim_{t\to\infty} u(x,t) = 1,$$

uniformly in $x \in [0, L]$.

3. Travelling waves

In this section, the geometric singular perturbation theory and Fenichel's invariant manifold theory [11,15] are used to prove the persistence of the travelling wave solutions for the Eq. (4) on an infinite domain. Similar techniques have been used to prove persistence for the delayed Fisher equation in Ashwin et al. [2] and also for the fourth-order diffusion equation in Akveld and Hulshof [1].

It is known that the Burgers–Huxley equation (1) admits travelling wave solutions of the form (2) connecting the two steady states $u = 0$ and $u = 1$. We intend to show that for the extended Burgers–Huxley equation with the small perturbation parameter multiplying the fourth-order derivative term these travelling wave solutions persist. Let $\delta = \varepsilon^2$ with $\varepsilon \ll 1$, then (4) becomes

$$u_t = -\varepsilon^2 u_{xxxx} + u_{xx} - \alpha u u_x - \beta u(u-1)(u-\gamma), \quad x \in (-\infty, \infty). \tag{14}$$

Looking for the travelling wave solutions of the form

$$u(x,t) = U(z), \quad \text{where } z = x - ct$$

and inserting this into (14) we obtain

$$-\varepsilon^2 U'''' + U'' - \alpha U U' + cU' - \beta U(U-1)(U-\gamma) = 0. \tag{15}$$

By defining $U' = v$, $v' = y$ and $\varepsilon y' = w$ one can rewrite (15) as the following system of ODEs

$$\begin{aligned}
U' &= v,\\
v' &= y,\\
y' &= \frac{1}{\varepsilon} w,\\
w' &= \frac{1}{\varepsilon}(y - \alpha U v + cv - \beta U(U-1)(U-\gamma)),
\end{aligned} \tag{16}$$

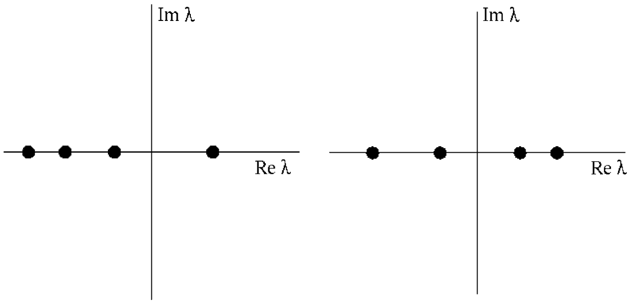

Fig. 1. Qualitative positions of the eigenvalues λ. Left: Spectrum of A_0. Right: Spectrum of A_1.

or, equivalently,

$$Y_z = F(Y), \quad F(Y) = \begin{pmatrix} v \\ y \\ \frac{1}{\varepsilon} w \\ \frac{1}{\varepsilon}(y - \alpha U v + cv - \beta U(U-1)(U-\gamma)) \end{pmatrix}, \quad Y = \begin{pmatrix} U \\ v \\ y \\ w \end{pmatrix}.$$

The equilibrium steady states for this system are $\Sigma^0 = (U, v, y, w) = (0, 0, 0, 0)$ and $\Sigma^1 = (U, v, y, w) = (1, 0, 0, 0)$. The linearisation near the steady state Σ^0 has the following properties. Let

$$A_0 \stackrel{\text{def}}{=} DF(\Sigma^0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\varepsilon \\ -\beta\gamma/\varepsilon & c/\varepsilon & 1/\varepsilon & 0 \end{pmatrix}$$

and the characteristic equation for A_0 is

$$\varepsilon^2 \lambda^4 - \lambda^2 - c\lambda + \beta\gamma = 0. \tag{17}$$

For $\varepsilon = 0$, this equation has two roots in the left complex half-plane. Recalling that $\gamma < 0$, one has to require $c \geqslant 2\sqrt{-\beta\gamma}$ to ensure that both of these roots are real. Violation of this condition would result in oscillations of U about the origin, which should be excluded since we restrict ourselves to the case of $U \geqslant 0$. For $\varepsilon > 0$, the qualitative positions of the eigenvalues λ are pictured in Fig. 1 (left). Similarly, the linearisation near the steady state Σ^1 is

$$A_1 \stackrel{\text{def}}{=} DF(\Sigma^1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\varepsilon \\ \beta(\gamma-1)/\varepsilon & (c-\alpha)/\varepsilon & 1/\varepsilon & 0 \end{pmatrix}$$

with the corresponding characteristic equation for A_1

$$\varepsilon^2 \lambda^4 - \lambda^2 + (\alpha - c)\lambda - \beta(\gamma - 1) = 0. \tag{18}$$

Positions of λ in this case are displayed in Fig. 1 (right). From Fig. 1 one can gain that for ε sufficiently small there are the following situations. Let $c > 0$ be defined by (3). Then, for $\alpha > 0$, $\beta > 0$, $\gamma < 0$ the Eq. (17) has four real roots: three negative and one positive. Likewise, for the same range of parameters, the eigenvalues at the steady state Σ^1 determined by equation (18) are also all real: two positive and two negative. Therefore, the sum of the dimensions of stable and unstable manifolds is five, while the phase space has the dimension four. For this reason these manifolds might intersect along one-dimensional curve in $\mathbb{R}^4$. Below we shall prove the existence of a connection between $\Sigma^0 = (0, 0, 0, 0)$ and $\Sigma^1 = (1, 0, 0, 0)$. We rewrite the system (16) in the following way:

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}z} &= v,\\
\frac{\mathrm{d}v}{\mathrm{d}z} &= y,\\
\varepsilon\frac{\mathrm{d}y}{\mathrm{d}z} &= w,\\
\varepsilon\frac{\mathrm{d}w}{\mathrm{d}z} &= y - \alpha U v + cv - \beta U(U-1)(U-\gamma)
\end{aligned} \tag{19}$$

and with $\zeta = z/\varepsilon$, it becomes

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}\zeta} &= \varepsilon v,\\
\frac{\mathrm{d}v}{\mathrm{d}\zeta} &= \varepsilon y,\\
\frac{\mathrm{d}y}{\mathrm{d}\zeta} &= w,\\
\frac{\mathrm{d}w}{\mathrm{d}\zeta} &= y - \alpha U v + cv - \beta U(U-1)(U-\gamma).
\end{aligned} \tag{20}$$

We call this system the "fast system" associated with (19). If in (19) $\varepsilon = 0$, then U and v are governed by

$$\frac{\mathrm{d}^2U}{\mathrm{d}z^2} + c\frac{\mathrm{d}U}{\mathrm{d}z} - \alpha U\frac{\mathrm{d}U}{\mathrm{d}z} - \beta U(U-1)(U-\gamma) = 0, \quad v = \frac{\mathrm{d}U}{\mathrm{d}z},$$

while y and w must lie on the set

$$M_0 := \{(U, v, y, w) \in \mathbb{R}^4 : w = 0 \quad \text{and} \quad y - \alpha U v + cv - \beta U(U-1)(U-\gamma) = 0\},$$

which is a two-dimensional submanifold of $\mathbb{R}^4$.

We claim that for ε sufficiently small there exists a two-dimensional sub-manifold M_ε of $\mathbb{R}^4$ which is within $\mathcal{O}(\varepsilon)$ of M_0 and which is invariant for the flow (19). By Fenichel's invariant manifold theory such a perturbed invariant manifold M_ε will exist if M_0 is "normally hyperbolic".

Definition (*Fenichel [11]*). The manifold M_0 is said to be normally hyperbolic if the linearisation of the *fast system*, restricted to M_0, has exactly dim M_0 eigenvalues on the imaginary axis, with the remainder of the system hyperbolic.

The linearisation of the fast system (20), restricted to M_0 (i.e. $\varepsilon = 0$) has the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ s - \alpha v & c - \alpha U & 1 & 0 \end{pmatrix} \tag{21}$$

with

$$s = 2\beta\gamma U - 3\beta U^2 + 2\beta U - \beta\gamma.$$

This matrix has eigenvalues $0, 0, -1, 1$. Thus, M_0 is normally hyperbolic, and the perturbed manifold M_ε exists.

Next, we determine the dynamics on M_ε. In order to do it let us write

$$\begin{aligned} M_\varepsilon = \{(U, v, y, w) \in \mathbb{R}^4 : w = g(U, v, \varepsilon),\ y = h(U, v, \varepsilon) \\ + \alpha U v - cv + \beta U(U-1)(U-\gamma)\}, \end{aligned} \tag{22}$$

where the functions g and h (to be found) satisfy

$$g(U, v, 0) = h(U, v, 0) = 0.$$

Substitution of (22) into (19) gives that $g(U, v, \varepsilon)$ and $h(U, v, \varepsilon)$ satisfy the following system:

$$\begin{aligned} &\varepsilon \Big[v \frac{\partial h}{\partial U} + \frac{\partial h}{\partial v}(h + \alpha U v - cv + \beta U(U-1)(U-\gamma)) - ch + c^2 v - \alpha c U v \\ &\quad - \beta c U(U-1)(U-\gamma) - 3\beta U^2 v + 2\beta\gamma U v + 2\beta U v - \beta\gamma v \Big] = g, \end{aligned}$$

$$\varepsilon \left[v \frac{\partial g}{\partial U} + \frac{\partial g}{\partial v}(h + \alpha U v - cv + \beta U(U-1)(U-\gamma)) \right] = h.$$

Now, we expand g and h in Taylor series in the variable ε:

$$g(U, v, \varepsilon) = g(U, v, 0) + \varepsilon g_\varepsilon(U, v, 0) + \tfrac{1}{2}\varepsilon^2 g_{\varepsilon\varepsilon}(U, v, 0) + \cdots,$$

$$h(U, v, \varepsilon) = h(U, v, 0) + \varepsilon h_\varepsilon(U, v, 0) + \tfrac{1}{2}\varepsilon^2 h_{\varepsilon\varepsilon}(U, v, 0) + \cdots.$$

Powers of ε^0 give

$$g(U, v, 0) = h(U, v, 0) = 0,$$

as expected. At the next order of ε we obtain

$$\begin{aligned} &g_\varepsilon(U, v, 0) = c^2 v - \alpha c U v - c\beta U(U-1)(U-\gamma) - \beta v[3U^2 - 2\gamma U - 2U + \gamma], \\ &h_\varepsilon(U, v, 0) = 0 \end{aligned}$$

and powers of ε^2 give

$$g_{\varepsilon\varepsilon}(U, v, 0) = 0,$$

$$\begin{aligned} h_{\varepsilon\varepsilon}(U, v, 0) = v[&c(-\alpha v - 3\beta U^2 + 2\beta U(\gamma + 1) - \beta\gamma) - 6\beta U + 2\beta\gamma + 2\beta] \\ &+ [-cv + \alpha U v + \beta U(U-1)(U-\gamma)][c^2 - \alpha c U \\ &- 3\beta U^2 + 2\beta\gamma U + 2\beta U - \beta\gamma]. \end{aligned}$$

Thus,

$$h(U, v, \varepsilon) = \varepsilon^2 h_1(U, v, \varepsilon),$$

where

$$h_1(U, v, \varepsilon) = \tfrac{1}{2} h_{\varepsilon\varepsilon}(U, v, 0) + \mathcal{O}(\varepsilon),$$

and the system (19) becomes

$$\begin{aligned} \frac{dU}{dz} &= v, \\ \frac{dv}{dz} &= \alpha U v - cv + \beta U(U-1)(U-\gamma) + \varepsilon^2 h_1(U, v, \varepsilon). \end{aligned} \tag{23}$$

These equations determine the dynamics on the “slow” manifold M_ε.

4. The flow on the manifold M_ε

When $\varepsilon=0$, system (23) reduces to a system of coupled first-order ODEs for the travelling front solution of the Burgers–Huxley equation (1). This system has the following equilibria of interest: $(U, v) = (0, 0)$ and $(U, v) = (1, 0)$. Let (U_0, v_0) be the solution of (23) when $\varepsilon = 0$, then in the (U, v) phase plane this solution is a connection between (1, 0) and (0, 0). We now employ the Fredholm theory to show that for $\varepsilon > 0$ sufficiently small there exists a heteroclinic connection between the critical points (1, 0) and (0, 0) of (23). This connection corresponds to a travelling wave solution of (14).

To seek such a connection, set

$$U = U_0 + \varepsilon^2 \tilde{U}, \quad v = v_0 + \varepsilon^2 \tilde{v}$$

and substitute into (23). To the lowest order in ε the system governing $(\tilde{U}, \tilde{v})$ is

$$\frac{d}{dz}\begin{pmatrix} \tilde{U} \\ \tilde{v} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 3\beta U_0^2 - 2\beta(\gamma+1)U_0 + \beta\gamma + \alpha v_0 & \alpha U_0 - c \end{pmatrix}\begin{pmatrix} \tilde{U} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 0 \\ h_1(U_0, v_0, 0) \end{pmatrix} \tag{24}$$

and we want to prove this system has a solution satisfying

$$\tilde{U}, \tilde{v} \to 0 \text{ as } z \to \pm\infty.$$

By Fredholm theory, the system (24) has a square-integrable solution iff the following compatibility condition holds

$$\int_{-\infty}^{\infty} \left(\mathbf{x}(z), \begin{pmatrix} 0 \\ h_1(U_0(z), v_0(z), 0) \end{pmatrix} \right) dz = 0,$$

for all functions $\mathbf{x}(z)$ in the kernel of the adjoint of the operator defined by the left-hand side of (24). The adjoint system for (24) has the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 0 & -3\beta U_0^2 + 2\beta(\gamma+1)U_0 - \beta\gamma - \alpha v_0 \\ -1 & c - \alpha U_0 \end{pmatrix} \mathbf{x}. \tag{25}$$

As $z \to \infty$ we have $U_0 \to 0$, $v_0 \to 0$, and the matrix in (25) is then a constant matrix with eigenvalues λ satisfying

$$\lambda^2 - c\lambda - \beta\gamma = 0. \tag{26}$$

From (26) we can see that both eigenvalues are positive or have a positive real part (since $\gamma < 0$, $\beta > 0$), and as $z \to \infty$ any solution of (26), other then the zero solution, must grow exponentially. The only solution in L^2 is therefore the zero solution $\mathbf{x}(z) = \mathbf{0}$, and consequently the Fredholm orthogonality condition holds. Thus, we have proved the existence of the desired connection on the manifold M_ε. These results are summarised in the following theorem.

Theorem 2. *For $c > 0$ defined in* (3), *there exists ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$, Eq.* (14) *admits a travelling wave solution $u(x,t) = U(z)$ satisfying $U(-\infty) = 1$ and $U(\infty) = 0$, where $z = x - ct$.*

5. Conclusions

Starting with the Burgers–Huxley equation (1) we extended it by adding a fourth-order derivative term and addressed the following two questions: under what conditions on the initial data will the solutions of the perturbed equation converge to the uniform steady state $u = 1$, and if the coefficient near the fourth-order derivative term is sufficiently small, what would happen with the travelling wave solutions (2).

The equations similar to the Burgers–Huxley equation (1) with fourth-order derivative have been considered in the context of population biology [18,4], the theory of phase transitions [6], the studies of the second order materials [5,17], etc. Positivity of solutions for such equations is always particularly important since little is known about the sign of the fourth-order derivative term in the evolution and the maximum principle [20] does not apply. We establish the eventual positivity of the solutions for all t sufficiently large by proving the uniform convergence of the solutions to the positive steady state $u = 1$ under certain restrictions on the L^2-norm of the initial data. These results are accumulated in Theorem 1.

Since travelling wave solutions are always important for the above-mentioned equations, therefore it is natural to ask a question about persistence of the travelling wave solution (2) in the extended equation (4). Considering equation (14) with the help of the invariant manifold theory and geometric singular perturbation theory we have proved (for $\varepsilon \ll 1$) the persistence of the solutions (2).

In general, the techniques used in this paper can be employed for various other equation, for example, the generalised Fisher equation [10]

$$u_t = -\varepsilon^2 u_{xxxx} + u_{xx} + pu(1-u^r)(q+u^r),$$

or the generalised Burgers–Huxley equation [8,21]

$$w_t = w_{xx} - \alpha w^m w_x + \beta w(1-w^m)(w^m - \gamma).$$

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