# On the Use of Delay Equations in Engineering Applications

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*Abstract:* This paper is a review of applications of delay differential equations to different areas of engineering science. Starting with a general overview of delay models, we present some recent results on the use of retarded, advanced and neutral delay differential equations. An emerging area for modeling with the help of delay equations is real-time dynamic substructuring, or hybrid testing. We introduce the main idea of this technique together with the latest advances. Special emphasis is given to the development of the theory and applications of partial delay differential equations. The review concludes with a summary of some open problems and questions concerning the analysis of spatially extended delayed systems.

*Keywords:* Delay differential equations, stability, applications.

## 1. INTRODUCTION

Recently there has been an increasing interest in dynamical systems involving time delays with applications ranging from biology and population dynamics to physics and engineering, and from economics to medicine. So what makes time-delayed systems important? The answer is simple: time delays are intrinsic in many real systems, and therefore must be properly accounted for when developing models. Delay is a common feature of many real processes, and with a growing demand for more precise predictions, control and performance there is a greater need for models to behave as close to real systems as possible. A monograph by Kolmanovskii and Myshkis (1999) gives an extensive range of modeling examples from biology, chemistry, mechanics, physics, ecology, physiology and viscoelasticity. In all of those examples the nature of the processes dictates the use of delay equations as the only appropriate means of modeling.

In engineering, time delays often arise in feedback loops involving sensors and actuators, and they are also invariably present in a novel structural testing technique called real-time dynamic substructuring (RTDS), or hybrid testing (Kyrychko et al., 2006). With the rapid development of communication technologies, the transmission of measured signals to a remote control center is becoming simpler. However, the major problem facing engineers is

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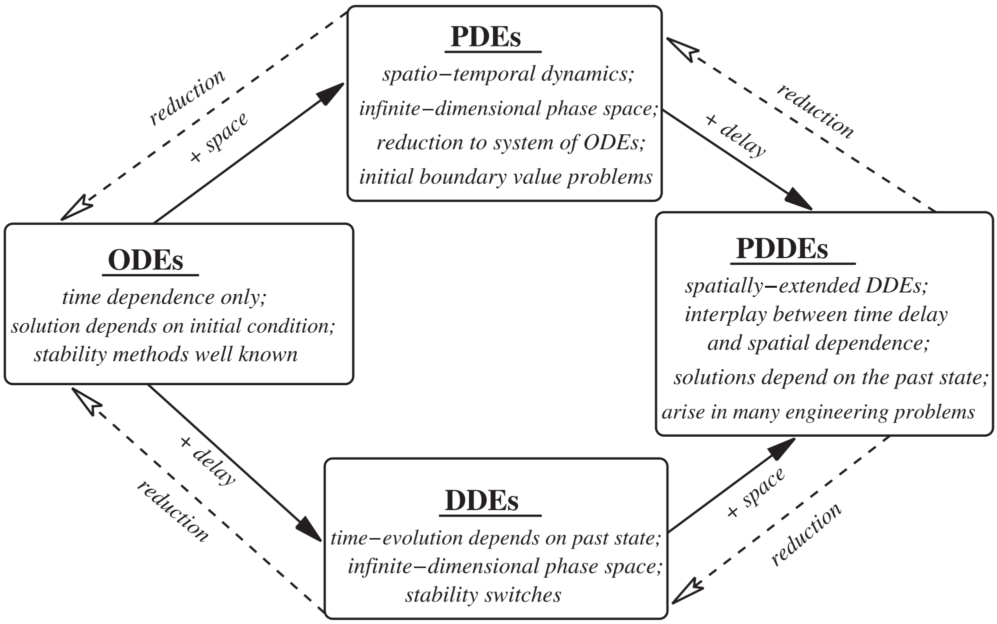


Figure 1. Schematic description of relation between ODEs, PDEs and DDEs.

an inevitable time delay between the measurement and the signal received by the controller, and this time delay should already be taken into consideration at the design stage (Chaudhuri et al., 2004) to avoid the risk of experimental instability and possible damage.

Neural networks provide a perfect real-life example where the time delay is an intrinsic part of the system and also one of the key factors that determines the dynamics. In this particular case, time delay occurs in the interaction between neurons, and is induced by the finite switching speed of amplifiers and the communication time of neurons. It has been shown that inclusion of the time delays into models of such systems leads to a complicated dynamics and even chaos (Cao and Lu, 2006). Recent studies on synchronization in coupled systems with a time delay in the interactions have shown that delays can induce oscillations but they can also enhance synchrony between coupled elements. This is very important as synchronization plays a crucial role in, for example, information processing in the brain (Dhamala et al., 2004). It is well known that time delays cause different types of oscillations, but it can also be shown that they cause an amplitude death (i.e. no oscillations in a coupled system while each subsystem oscillates when isolated) within certain regions of the parameter space. Amplitude death in coupled oscillators is sometimes associated with a Hopf bifurcation and oscillation amplitudes are damped out to reach a steady state (Reddy et al., 1998; Strogatz, 1998).

Figure 1 is a schematic diagram of how ordinary differential equations (ODEs) and partial differential equations (PDEs) can be modified to account for delays thus leading to delay differential equations (DDEs) and partial DDEs (PDDEs). It is clear that adding spatial dependence to an ODE will lead to a PDE, and the reduction of a PDE to a finite or infinite mode truncation gives a finite or an infinite system of ODEs. However, sometimes reduction of PDEs using a travelling wave type solutions yields a system of DDEs. As time evolution of DDEs takes into account past states, their phase space is infinite-dimensional, as in PDEs, but there is no spatial dependence just as in ODEs. DDEs can be divided into different classes, depending on how and where the time delay arises in the equations, on the number of delays, and whether the delays depend on the state or time variables.

One of the prominent examples of time delays arising in engineering is that of a high-speed milling where, through the variation of the chip thickness, a varying cutting force acts on the tool and this difference in the force depends on the past vibrations of the tool (Szalai et al., 2004; Gradisek et al., 2005; Stépán et al., 2005b). Another issue in milling processes is the stabilization of a time-delayed system with a parametric excitation which can be achieved via spindle speed fluctuation (Jemielniak and Widota, 1984; Segalman and Butcher, 2000; Stépán et al., 2005a; Long et al., 2007). The problem of stable control for teleoperation of mobile robots can be achieved by a time-delayed compensation placed on both the local and remote sites of the teleoperation system (Slawiñski et al., 2006).

In laser dynamics, models with time delay play an important role as a delayed feedback typically occurs due to unwanted external reflections. On the one hand, time delays tend to destabilize lasers leading to the inclusion of expensive optical isolators, but, on the other hand, chaotic outputs caused by delays can be used, for example, as carriers for encrypted optical communication schemes van Wiggeren and Roy, 1998; Erzgräber et al., 2006.

Another interesting and exciting area for time delay modeling emerges in traffic dynamics where the delay is introduced to account for the finite reaction time of drivers. Numerical bifurcation analysis of such models helps to deepen the understanding of the complicated dynamics of car-following models on the highways (Orosz et al., 2004, 2005).

Furthermore, it is becoming clear that time delays can be used to control unstable motion, where the unstable periodic orbits embedded in a chaotic attractor are stabilized by constructing a control force from the difference of the current state and the state one period in the past. This method is known as time-delay auto-synchronization and, recently, it was shown that it can also be successfully applied to stabilize unstable fixed points (H¨ovel and Schöll, 2005; Yanchuk et al., 2006). In the area of synchronization it has been shown that feeding back a time-delayed mean field into the ensemble of globally coupled systems can both enhance and suppress collective oscillations (Rosenblum and Pikovsky, 2004).

Introducing a global time-delayed feedback to control patterns in active media leads to the changes in the properties of existing wave patterns and to the formation of new kinds of patterns (Mikhailov and Showalter, 2006).

A recent overview of results on time-delayed systems and control approaches involving delays as well as some open problems relating to the constructive use of delay outputs and the digital implementation of distributed delays was performed by Richard (2003).

The outline of this paper is as follows. In the next section we introduce three types of delay equations and illustrate their use by presenting the appropriate situations in which they arise. Section 3 discusses various techniques that can be used to perform stability analyses in delayed systems. Section 4 is devoted to the PDEs with time delays and recent results on this topic. The paper concludes with a discussion of future developments.

## 2. DELAY EQUATIONS: RETARDED, ADVANCED AND NEUTRAL

DDEs can be divided into three groups: retarded, advanced and neutral, and each of those groups represents a class of infinite-dimensional systems. The most common is the retarded differential equation with constant time delay which can be written in the form

$$\dot{x}(t) = f(t, x(t), x(t-\tau), \alpha), \quad t > 0, \quad x \in \mathbb{R}^n, \tag{1}$$

where $\tau \geq 0$, $\alpha$ is a vector of parameters, and the initial conditions are defined as

$$x(s) = \phi(s) \quad \text{for all} \quad s \in [-\tau, 0]. \tag{2}$$

It it worth noting that state and solution spaces of DDEs are infinite-dimensional, hence, the state space of a DDE is an infinite-dimensional Banach space. For further details on more elaborate delay equations and their analysis the reader is referred to Hale and Verduyn Lunel (1993), Kolmanovskii and Myshkis (1999) and Stépán (1989).

There are many areas of applications where equations of this type play a crucial role. In all of those systems a recurrent feature is the dependence of the dynamics on the current state of the system and also on its state some fixed time $\tau$ in the past. Below we present several examples of such systems which are actively studied in the context of engineering applications.

The first example is high-speed milling, which is a very common cutting process in industry. Cutting is achieved by a rotating tool, whose edges remove material by inducing mainly shear stress in a workpiece. If $x$ denotes the displacement of the cutting tool, then the typical equation of motion can be written in the form of the following delayed differential equation (Szalai et al., 2004)

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \frac{g(t)}{m} F_c(h(t)), \tag{3}$$

with

$$g(t) = \begin{cases} 0 & \text{if there exists } j \in \mathbb{Z} : \; t_j \leq t < t_{j+1}^-, \\ 1 & \text{if there exists } j \in \mathbb{Z} : \; t_{j+1}^- \leq t < t_{j+1}, \end{cases} \tag{4}$$

where $F_c$ is a cutting force and it is often modeled as a power law $F_c(h(t)) = Kw(h(t))^{3/4}$, $t_j$ are the time instants when the tool starts the free vibration, $t_{j+1}^-$ are the time instants when the tool enters the workpiece and finishes cutting at $t_{j+1}$. The other parameters in equation 3 are as follows: $\omega_n$ is the undamped natural frequency, $\zeta$ is the relative damping factor, $K$ is an experimentally determined parameter and $w$ is the constant chip width. The chip thickness $h(t)$ is represented via the delayed and current tool tip position as

$$h(t) = h_0 + x(t-\tau) - x(t), \tag{5}$$

with $h_0 = v_0\tau$ being the feed for a cutting period. It was shown (Szalai et al., 2004) that as the time delay is varied, the stability can be lost via either Neimark–Sacker or a period-doubling bifurcation, and both of these bifurcations are subcritical. In a parameter region, where both period-doubled and the quasi-periodic solutions are unstable, the system exhibits chaotic oscillations, and this has been established by analytical means and confirmed in numerical simulations. In (Gradisek et al., 2005) the authors have applied zeroth-order approximation and semi-discretization methods to predict the stability boundary of the milling process. Periodic chatter-free, quasi-periodic and periodic chatter regimes predicted by the semi-discretization method were confirmed experimentally. A stability analysis of interrupted turning processes presented by Szalai and Stépán (2006) revealed that unstable parameter domains are formed from the unions of lobes and lenses. Similar problems arise in the context of nonlinear modeling of chatter in drilling processes; see, for example, Stone and Askari (2002) and Stone and Campbell (2004).

Another interesting mechanical system that can be described by DDEs is a moving conveyor belt loaded with two oscillating connected masses. Taking into account a time lag in the inertial and quasi-elastic restoring forces, a mathematical model for the force of sliding friction in connected pairs is:

$$\begin{aligned} & m_i\ddot{x}_i(t) + \varepsilon\Phi_i[\dot{x}_i(t-\tau_i) - V_0] \\ + & \sum_{j=1}^{i+1}(-1)^{j-i}k_j\left\{[x_j(t) - x_{j-1}(t)] + \varepsilon F_j[x_j(t) - x_{j-1}(t)]\right\} = 0, \end{aligned} \tag{6}$$

where $i = 1, 2$, $\tau_1 = x_0 = k_3 = 0$ and $\tau_2 = \tau$. The parameters are $m_i$ is the mass of the $i$th oscillator, $x_i(t)$ is its position, $V_0$ is a constant velocity of the belt, $k_i$ is the stiffness coefficient of each connecting link, $\varepsilon\Phi_i$ describes the force of sliding friction and functions $\varepsilon F_j$ account for energy losses in the material (Zhirnov, 1977).

An active area of research on bifurcation analysis of DDEs is being pursued in laser dynamics. One of the possible models for two identical, mutually delay-coupled semiconductor lasers is given by (Erzgräber et al., 2005)

$$\begin{aligned} E_1 &= (1+i\alpha)N_1E_1 + \kappa e^{-iC_p}E_2(t-\tau) - i\Delta E_1, \\ E_2 &= (1+i\alpha)N_2E_2 + \kappa e^{-iC_p}E_1(t-\tau) - i\Delta E_2, \\ T\dot{N}_1 &= P - N_1 - (1+2N_1)|E_1|^2, \\ T\dot{N}_2 &= P - N_2 - (1+2N_2)|E_2|^2, \end{aligned} \tag{7}$$

where $E_{1,2}$ are the optical fields, and $N_{1,2}$ are normalized inversions. Time delay corresponds to the finite propagation time of light between the two spatially separated lasers. While the delay was kept fixed in this study, it still has a profound effect on possible dynamic regimes. In particular, the time delay provides a much richer structure of *compound laser modes* (CLMs; continuous wave solutions of the two coupled lasers) than the structure of external cavity modes for lasers with conventional optical feedback. Many of those CLMs are unstable, and the system can exhibit exceedingly complicated dynamics consisting of a succession of irregular and temporary visits to such states. Time delay plays a very important role here as it determines the number and structure of unstable CLMs that form the backbone of understanding the dynamics of the coupled laser system.

An efficient transfer of energy from a self-excited oscillator to a resonant load is possible when the wave returns from the load to the oscillator with a favorable phase, provided that the phase shifter is optimally tuned. In many cases this efficiency is impeded by the fact that the frequency band of the resonance load is much narrower than the band of free self-excited oscillations. Mathematically, this can be formulated as an optimization problem of the form:

$$\begin{aligned} \ddot{y}(t) + y(t) &= -\nu y(t) + \mu(1 - y^2(t))\dot{y}(t) + m_1 x(t - \tau), \\ \ddot{x}(t) + x(t) &= -\gamma \dot{x}(t) + m_2 y(t - \tau), \end{aligned} \tag{8}$$

where $y$ and $x$ correspond to the oscillator and load cavity, $m_1$ and $m_2$ are the coupling coefficients, $\gamma \ll 1$ is the linear damping coefficient, $\mu > 0$ is the nonlinear damping coefficient, and $\tau$ is the time delay to be optimized (Ishchenko et al., 2006). This system can also be used to qualitatively describe the interaction of self-excited oscillators with oscillators of any kind (e.g. electrodynamic, acoustic, etc.).

An interesting area of applications of delay equations with discrete time delay in engineering is digital control. This refers to control of mechanical systems by means of artificial dissipation. Since computers sample velocity at discrete time instances, this leads to a time delay in the control force. The introduction of digital sampling into the control force leads to the irregular oscillations and chaotic motion in the system. Replacing the continuous time one-dimensional system by a map, Haller and Stépán (1996) have proved the existence of a hyperbolic strange attractor. Kovács and Stépán (2003) studied the control of the interaction between robotic actuator and a workpiece in the context of rehabilitation machines. Digital effects, such as sampling times, can have undesired (destabilizing) effects on the behavior of such machines. A model for the control of robot with periodic sampling can be written in the form (Kovács and Stépán, 2003):

$$\begin{aligned} \dot{x}_1(t) &= -k_p s_2(x_1(t_j - \tau) - x_2(t_j - \tau)) + \dot{x}_2(t_j - \tau), \quad t \in [t_j, t_j + \tau), \\ m_2 \ddot{x}_2(t) &= s_2(x_1(t) - x_2(t)) + F(t), \end{aligned} \tag{9}$$

where $x_1$ and $x_2$ are the displacements of two connected springs, $s_2$ is the stiffness of connected springs, $k_p$ is the proportional gain and $F(t)$ is the external force. The signal is sampled at times $t_j = j\tau$, $j = 0, 1, 2, \ldots$ and $\tau$ is the sampling time. Using this model the authors showed that a relatively low sampling time of industrial robots leads to the necessity of a careful choice of mechanical and control parameters to avoid the instability caused by discrete time sampling.

A recent study on travelling wave solutions for lattice differential equations shows that they can be defined by advanced DDEs (the reader is referred to Abel et al. (2005) and references therein). Another example is given by two identical charged particles moving symmetrically about the origin. The motion of each particle is influenced by the electromagnetic field of the other particle at some earlier time $t - r$ and also some time later $t + q$.

The delay in this situation is itself time-dependent, and the whole system is modeled with advanced DDEs (see Hoag and Driver (1990) for further details). Advanced DDEs can be defined as

$$\dot{y}(t) = f(t, y(t), y(t+r), y(t-r), \alpha), \quad t > 0, \quad y \in \mathbb{R}^n. \tag{10}$$

In this case, the dynamics depends not only on the current state of the system and that some time ago, but also on some future states which have yet to be determined. This class of equations is much less studied, and at the moment there is little mathematical theory to analyze such equations.

Finally, DDEs of neutral type are somewhat different from the two classes described above as they involve not only the past state of the system but also the delayed highest derivative, and can be represented as

$$\dot{x}(t) = f(t, x(t), x(t-\tau), \dot{x}(t-\tau), \alpha), \quad t > 0, \quad x \in \mathbb{R}^n. \tag{11}$$

The initial value problem for neutral DDEs can be defined as before, i.e. $x(s) = \phi(s)$ for all $s \in [-\tau, 0]$. The solution of equation 11 subject to the initial condition is a continuous function $x(t)$ defined on $[-\tau, \infty]$ which is continuously differentiable except at the points $t = n\tau, n = 0, 1, 2, \ldots$ and $x(t)$ satisfies equation 11 everywhere except these points. In general, the solution $x(t)$ will have a discontinuous derivative at $t = n\tau$, $n = 0, 1, 2, \ldots$. However, the jumps will be absent if the initial function $\phi$ satisfies the necessary and sufficient condition

$$\dot{\phi}(0) = f(0, \phi(0), \dot{\phi}(-\tau)), \tag{12}$$

then the solution $x(t)$ of (11) will have a continuous derivative for all $t \geq -\tau$ (see Akhmerov et al. (2005) and Hale and Verduyn Lunel (1993)). This can be easily explained with a simple example. Consider the scalar equation (Kolmanovskii and Myshkis, 1999)

$$\dot{x}(t) = \dot{x}(t-1), \quad t \geq 0, \quad x(t) = \phi(t), \quad -1 \leq t \leq 0, \tag{13}$$

where $\phi$ is a given initial function. The solution of this equation is

$$x(t) = \phi(t-m) + m[\phi(0) - \phi(-1)], \quad m-1 \leq t \leq m, \quad m = 1, 2, \ldots. \tag{14}$$

If $\dot{\phi}(-1) \neq \dot{\phi}(0)$, then $\dot{x}(t)$ has jumps at $t = 0, 1, \ldots$. If $\dot{\phi}(-1) = \dot{\phi}(0)$ and $\phi$ is twice-differentiable with $\ddot{\phi}(-1) \neq \ddot{\phi}(0)$, then $\dot{x}(t)$ is continuous but $\ddot{x}(t)$ has jumps at $t = 0, 1, \ldots$, and so on. This property makes neutral DDEs resemble hyperbolic PDEs.

Delay equations of neutral type often arise in the modeling of coupled oscillatory systems with noninstant connection between the parts. Sometimes to simplify the model, it is assumed that time delay is negligible, however, in more realistic models time delay has to be considered. An illustrative example of neutral DDEs arising from a reduction of an initial boundary value problem for hyperbolic equations has been considered by Fliess et al. (1995). Their model describes the torsional behavior of a flexible rod with a torque applied at one end and a mass attached to the other end. Equations of motion together with the boundary and initial conditions are as follows,

$$\begin{aligned} \sigma^2 u_{tt}(t,x) &= u_{xx}(t,x), \\ u_x(t,0) &= -b(t), \quad u_x(t,L) = -J u_{tt}(t,L), \\ u(0,x) &= u_0(x), \quad u_t(0,x) = u_1(x). \end{aligned} \tag{15}$$

Here $u(x,t)$ is the angular displacement of the rod from an unexcited position, $L$ is the length of a rod, $y(t) = u(t,L)$, $J$ is the moment of inertia of a mass, $\sigma$ is the inverse of the wave propagation speed, and $b(t)$ is the control torque. The original initial boundary value problem can be reduced to a second-order neutral delay differential equation for the control output $y$ (see Fliess et al. (1995) for details):

$$\ddot{y}(t) + \ddot{y}(t-2\tau) + \dot{y}(t) - \dot{y}(t-2\tau) = v(t-\tau), \tag{16}$$

where $\tau = \sigma L$, $v(t) = (2/\sigma\kappa)b(t)$, $\kappa = \sigma/J$.

This approach can be generalized for a class of initial boundary value problems for PDEs where representation of a general solution is known. For example, consider a wave equation

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \quad 0 \le x \le 1, \quad t \ge 0, \tag{17}$$

with the following boundary conditions:

$$H_0(u, u_t, u_x)|_{x=0} = 0, \quad H_1(u, u_t, u_x)|_{x=1} = 0. \tag{18}$$

Looking for a solution to equation 17 in the form:

$$u(x,t) = \mu(t+x) + \nu(t-x). \tag{19}$$

and substituting this form of solution into the boundary conditions leads to a system of DDEs (with a possibility of reduction to a system of purely difference equations for some types of boundary conditions):

$$\begin{aligned} G_0(\mu(z), \nu(z), \mu\prime(z), \nu\prime(z)) &= 0, \quad z \ge 0, \\ G_0(\mu(z+2), \nu(z), \mu\prime(z+2), \nu\prime(z)) &= 0, \quad z \ge -1, \end{aligned} \tag{20}$$

where $G_i(y_1, y_2, y_3, y_4) = H_i(y_1+y_2, y_3+y_4, y_3-y_4)$, $i = 0, 1$. Now, the problem is reduced to the analysis of unimodal maps with the nonlinearity determined by initial conditions. In the case of quadratic nonlinearity, the bifurcation scenario was shown to be exactly the same as for standard quadratic maps, including a period-doubling route to chaos (Lozi et al., 2004).

An emerging area where the use of delay equations and, in particular, delay equations of neutral type becomes increasingly important is a structural testing technique called RTDS or hybrid testing. This novel method for testing complex structures or mechanical machinery

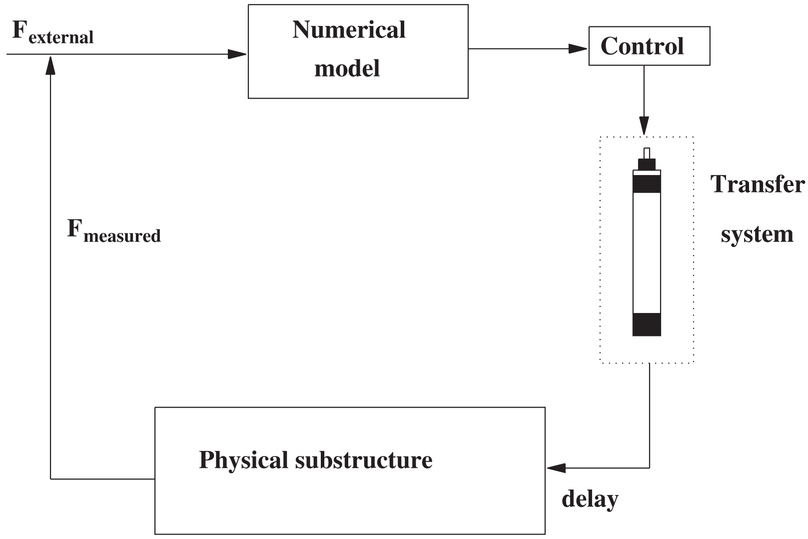


Figure 2. Schematic description of the hybrid testing.

combines the use of shaking tables, actuators and computational efforts. Figure 2 gives a schematic representation of the hybrid testing with the indication where the time delay is most likely to occur.

A main mechanism for destabilization of such an experiment is the time lags, or delays which arise mainly due to the noninstantaneous nature of the transfer systems (Horiuchi and Konno, 2001; Horiuchi et al., 2004).

Preliminary studies on the modeling of real-time substructuring with DDEs have indicated that time delays play a crucial role in determining the stability of the experimental set-up (Wallace et al., 2005; Kyrychko et al., 2006, 2007).

Kyrychko et al. (2006) have considered a coupled system consisting of a mass–spring–damper (MSD) connected to a pendulum. The pendulum was taken to be a physical structure, and the MSD was modeled numerically. An actuator was taken to a transfer system. Mathematical representation of the system was done with the help of neutral DDEs. For small angles and zero external forcing, the system can be written in a simplified form as

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m\ddot{y}(t-\tau) = 0, \tag{21}$$

where $M$, $C$ and $K$ are the mass, stiffness and damping of a MSD, and $m$ is the mass of a pendulum. Time delay is denoted by $\tau$. Theoretical investigations of the system stability were compared with the experimental results. The agreement was excellent, thus proving that DDEs approach to hybrid testing helps to gain insights to the system's behavior even before the actual experiment is performed.

Despite the successes of modeling RTDS using DDEs with constant time delay, in practice it is often the case that delay in the system is itself time-varying. It varies around a constant value and depends on the frequency of external excitation. To this end, a very interesting study was performed by Michiels et al. (2004). The authors analyzed the stability of a time-delayed system

$$\dot{x}(t) = Ax(t) + Bx(t - \tau(t)), \tag{22}$$

with appropriate initial conditions and time delay varying around a nominal value $\tau_0$ in a deterministic way:

$$\tau(t) = \tau_0 + \delta f(\Omega t), \tag{23}$$

where $f$ is a periodic function with mean zero and $\delta$ and $\Omega$ are parameters determining the amplitude and frequency of variation. They showed on three examples that delay variation always leads to larger stability regions in parameter space. This suggests that approaching hybrid testing by modeling it with time-dependent systems can lead to improved stability results, and even suggest possible regimes of varying frequency and amplitude of external excitations to enhance the overall stability of the experimental set-up.

## 3. STABILITY

Stability is an extremely important aspect in the dynamics of delayed systems since time delays can induce unwanted oscillations and cause instabilities. A simple example of the effects of time delay on the stability of the system can be seen in the following simple equation:

$$\ddot{z}(t) + z(t - \tau) = 0, \tag{24}$$

which has a periodic solution for $\tau = 0$, and no periodic solutions for any values of $\tau > 0$, except for $\tau = 2k\pi$, $k = 0, 1, 2, \ldots$. Another illustrative example is that of a delayed logistic model

$$\dot{x} = rx(t)\left[1 - \frac{x(t-\tau)}{K}\right]. \tag{25}$$

The steady state of a full carrying capacity $x = K$ is globally asymptotically stable for $\tau < \pi/2r$, and then it loses stability via a Hopf bifurcation, giving rise to a stable periodic orbit.

One of the standard means of studying the stability of fixed points of DDEs is the analysis of characteristic equations, which arise from the linearization. The major difference between standard ODEs and DDEs is in that characteristic equations of DDEs are transcendental. One of the consequences of this is the fact that characteristic equations of DDEs usually have infinitely many roots, while in the case of ODEs the number of characteristic roots coincides with the dimension of the system. Second, since the characteristic equations for delay equations are transcendental, this usually prevents one from finding a closed-form expression for the eigenvalues of linearization. However, in the case when the dimension of the system is comparatively low (one or two), it is possible to develop a complete characterization of stability, as well as obtain conditions for Hopf bifurcation (Kuang, 1993). As the dimension of the system increases, the analysis becomes ever more involved (see Ruan and Wei (2001) for the case of a third-order delayed system). In a particular case when the DDEs are of retarded type, one can express the roots of the characteristic equation in terms of the so-called Lambert $W$-function (Amann et al., 2007; Yi et al., 2007). However, this approach suffers from several limitations; first of all, it is inapplicable to neutral equations; and, second, while the Lambert function is well known, when used for studying practical problems, its use requires a substantial amount of computation.

Linear stability for DDEs implies that all of the roots of the corresponding characteristic equation have negative real part. Unlike standard ODEs, DDEs are known to possess the so-called stability switches. This essentially means that when the time delay in the system is increased from zero, a steady state may go from a stable regime to an instability via a Hopf bifurcation, and then return to the stable regime again. It can be shown that eventually, for some critical delay time, the steady state will become unstable never regaining its stability. In the case of neutral DDEs it is possible to have the following two situations. First is an asymptotically critical case, when all roots of characteristic equation have negative real part but there exists a chain of roots unboundedly approaching the imaginary axis. If all of the roots are simple, there is no exponential stability of solutions as there will be solutions behaving as $t^{-\alpha}$ as $t \to \infty$ ($\alpha > 0$). The second possibility is a supercritical case when there exists a chain of purely imaginary roots.

Stability results as obtained from the analysis of characteristic equations and their roots provide information about the behavior of the system in the vicinity of its fixed points. In order to obtain a bigger picture and understand the global aspects of the dynamics, one can use an extension of the Lyapunov stability theory for delay equations, with the Lyapunov function depending on the delayed variable. The Lyapunov–Krasovskii stability theorem states that a system is asymptotically stable if there exists a Lyapunov functional $V(t, x_t)$ such that

$$w(\|x(t)\|) \leq V(t, x(t), x(t-\tau)) \leq v(\|x_t\|_c), \tag{26}$$

and

$$\dot{V}(t, x_t) \leq -w(\|x(t)\|), \tag{27}$$

where $u$, $v$ and $w$ are strictly increasing functions with $u(0) = v(0) = w(0) = 0$, $\lim_{s\to\infty} u(s) = \infty$, $x_t(s) = x(t+s)$ for all $-\tau \leq s \leq 0$, and $\|f\|_c = \max_{-\tau \leq s \leq 0} \|f(s)\|$. The implication of this theorem is that there exists a region in a phase space (dependent on time delay) such that for all initial conditions in this region, the time evolution of the system will remain in this region. Further details regarding this matter can be found in Gu and Niculecu (2003).

Quite often, the dynamics of the system under investigation becomes too involved to be studied using analytical tools. In order to understand the full range of behaviors in this case, one has to resort to numerical simulations. At the moment software exists to perform time integration of time delayed systems, one example being the dde23 routine for Matlab (Shampine and Thompson, 2001). This routine provides the means to integrate systems of different dimensions with an arbitrary number of fixed discrete delays. To complete the picture, it is often desirable to perform the bifurcation analysis without the need to integrate the system in a particular regime of the parameter space. For this purpose one can use the popular DDE-BIFTOOL suite in Matlab (Engelborghs et al., 2001), which provides the capabilities to track the dynamics of the eigenvalues, identify and continue steady-state fold and Hopf bifurcations. DDE-BIFTOOL is capable of dealing with both fixed and also distributed and state-dependent time delays, and it can also compute homo/heteroclinic orbits. One of the limitations of this bifurcation package (the dde23 integrator suffers from the same drawback) is its inability to deal with neutral DDEs, and at present the work is being carried out to remove this limitation. A package PDDE-CONT (Szalai et al., 2006) extends the continuation capabilities of DDE-BIFTOOL to bifurcations and continuation of periodic orbits in periodically forced DDEs.

Different issues such as control of error, difficulties of solving DDEs with vanishing time delay, discontinuity of solutions, arising from numerical analysis of DDEs were discussed by Baker et al. (1995). Olgac and Sipahi (2002, 2003) have proposed a direct method for studying the stability of linear time-invariant delayed systems. Their method is based on a root clustering approach and the Rekasius transformation, which reduces the problem to finding the roots of an equivalent rational polynomial.

A recent review on the use of pseudo-spectral differentiation techniques to analyze the stability of equations with time delay is presented by Breda et al. (2005). The authors discuss an infinitesimal generator approach and a solution operator approach and give examples of stability charts for DDEs with constant and time-dependent coefficients as well as for neutral DDEs and a mixed type of functional differential equations. Baker and Paul (2006) show how discontinuities arise in solutions of neutral DDEs, and also present some methods to overcome the problem of nonsmoothness in numerical investigations.

## 4. PARTIAL DELAY DIFFERENTIAL EQUATIONS

An area of research into delay equations with engineering applications is concerned with spatially extended DDEs, i.e. PDDEs. Of particular importance and interest both to theory and applications is the interplay between spatial dependence and time delay which can give rise to complicated dynamical behavior and spatio-temporal patterns. Owing to the complexity of standard PDEs without delay, the combination of delays and spatial dependence makes the analysis of PDDEs quite challenging. However, many engineering processes can also be correctly modeled only by using PDDEs. For example, in order to describe viscoelastic effects in some materials with noninstantaneous effects of relaxation stresses, PDDEs are used. Consider a transversal motion of a linear homogeneous elastic bar with an axial load $P$. The viscoelastic effect relaxes the bending stress in the bar with a relaxation function $r \geq 0$. The equation of motion can be written in the form:

$$\begin{aligned} \rho \frac{\partial^2 u(x,t)}{\partial t^2} &= -\frac{\partial^2}{\partial x^2}[Pu(x,t) + M(x,t)], \\ M(x,t) &= EI \frac{\partial^2}{\partial x^2}\left[u(x,t) - \int_{t_0}^{t} r(\tau, t-\tau) u(x,\tau)\, d\tau\right], \\ & \qquad t \geq t_0, \quad 0 \leq x \leq l, \end{aligned} \tag{28}$$

where $\rho$ is density of the bar material, $I$ is a bending moment, $l$ is the length of the bar. A detailed description of the model is given by Kolmanovskii and Myshkis (1999).

A problem of global attractors for damped semilinear viscoelastic beams with memory was addressed by Marzocchi and Vuk (2003). The authors concentrated on a model describing a transversal vibrations of a viscoelastic bar:

$$u_{tt} + \alpha_0 u_{xxxx} + \int_0^\infty \alpha'(s) u_{xxxx}(t-s)\, ds + \delta u_{xxxxt} - \left(\beta + \int_0^1 u_x^2(\xi,t)\, d\xi\right) u_{xx} = 0, \quad (29)$$

with $\alpha_0, \delta > 0$, $\alpha' \leq 0$ and $\beta \in \mathbb{R}$. It was shown that a strong damping term helps to achieve compactness properties of the attractor, and memory is required to have an exponential delay.

Control strategies for PDDEs have been shown to exhibit unstable regimes when small delays are introduced. Datko (1988) analyzed beam and wave equations with time delays in the boundary conditions:

$$u_{tt} + u_{xxxx} = 0, \quad 0 < x < 1, \quad t > 0 \quad (30)$$

with

$$\begin{aligned} u(0,t) &= u_x(0,t) = u_{xxx}(1,t) = 0, \\ u_{xx}(1,t) &= -u_{xt}(1, t-\epsilon), \end{aligned} \quad (31)$$

where $\epsilon \geq 0$ is the small time delay. It was demonstrated that the Neumann-type boundary conditions, which stabilize the system, are not robust with respect to small delays, and the resulting system has periodic and even exponentially increasing solutions.

White and Moloney (1999) have used a full system of PDEs in order to capture the relevant physics of laser interaction. Their models includes counter-propagating travelling waves coupled to the total carrier density. The communication channel introduces time delay into the boundary conditions. The authors have demonstrated that pointwise application of a complex scalar signal is sufficient to synchronize the spatio-temporal evolution of two PDEs, and synchronization is robust to noise and strong transient perturbations. They have also discovered that added dimensionality leads to a new quasi-synchronous state, where some spatial modes are synchronized while other modes are desynchronized.

Oscillatory properties of solutions of nonlinear impulsive hyperbolic PDEs with several delays are presented by Liu et al. (2004). Some sufficient conditions for oscillation for similar type equations with Robin boundary conditions are derived by Cui et al. (2005).

A lot of research on reaction–diffusion models with the time delay has been performed with applications to population dynamics and ecology (see, for example, Kyrychko et al. (2005, 2006), Gourley and Wu (2006) and references therein).

The questions concerning travelling wave solutions for delayed reaction–diffusion-type models have been addressed by Wu and Zou (2001). Consider, for example, a delayed Belousov–Zhabotinsky system of the form:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1 - u(x,t) - rv(x, t-\tau)],$$

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - bu(x,t)v(x,t), \tag{32}$$

where $a$, $b$ and $\tau$ are constants (Zimmerman et al., 1984). Wu and Zou (2001) used a monotone iteration technique together with the upper-lower solutions to show the existence of travelling fronts in the above system. Interestingly enough, they showed that while the existence of travelling front solutions in this system is independent of delay, the minimal velocity of such fronts does depend on the delay. Moreover, a general indication of that paper was that if a scalar reaction–diffusion system has solutions in the form of travelling fronts, then so does its delayed analog, provided that the delay is sufficiently small. Stability analysis for a general class of reaction–diffusion models was extensively performed by Wu (2006) in the framework of functional differential equations.

## 5. DISCUSSION

In this paper we have attempted to summarize a range of most recent results in the analysis of time-delayed systems with applications to engineering science. We have given some general remarks on the formulation of initial value problem in the case of delayed, advanced and neutral equations. Stability of delayed systems is also addressed and we have listed the main theoretical and numerical techniques available for analyzing the stability of DDEs up to date. Special attention is given to the neutral DDEs because they arise in many engineering applications and their dynamics and bifurcation analysis are far from being completely understood.

A lot of research was/is being performed on the synchronization of coupled ODEs or DDEs without spatial extension. The complex (and, hence, more realistic) mathematical models of different engineering processes can sometimes be viewed as a coupled system of PDDEs (possibly even of neutral type). There are some early indications that spatial extension coupled with time delays can lead to a completely different properties and dynamics as opposed to ODEs or DDEs because of the quasi-synchronization phenomenon (White and Moloney, 1999). Reduction of PDDEs to DDEs always leads to a question on the number of modes needed to capture the essential dynamics of the system. This problem becomes even more complicated because of the effect that delay brings into the model. Travelling wave reduction for the PDDEs will lead to a DDE with two time delays, hence becoming a problem of DDEs with multiple delays.

Another interesting aspect needed to be addressed is the control of DDEs or even PDDEs. The current investigations concentrate on using time-delay to stabilize unstable motion of a system without delay. However, it is interesting and important to understand the control issues in the systems which have intrinsic time delays from the beginning. This again opens a direction of analysis of multi-delayed equations with emphasis on the interplay between inherent and introduced time delays. In many applications delay in the system depends on some other factors, such as time or state. A substantial analysis on the functional differential equations with state-dependent delays was performed by Hartung et al. (2006). Starting with some modeling examples, the authors give results on theoretical and numerical investigations of such systems using the methods of functional analysis. Pattern formation and Turing instability of spatially extended time-delayed systems is also one of the under-developed area which is very promising and exciting with lots of applications in, for example, liquid crystals.

In most real-world applications the use of deterministic equations seems to be a simplification while the presence of random fluctuations can seriously affect the performance of any model. In many cases to represent the dynamics of the process under investigation correctly, stochastic DDEs have to be used to account for noise. Noise in such equations can be white or colored and the two main approaches, such as Itô and Stratonovich calculus, should be used (Bocharov and Rihan, 2000) showed how the stationary probability for linear stochastic DDEs (and in some cases nonlinear) can be derived by using the corresponding delayed Fokker–Planck equations. Planck equations. Kuske (2005) studied stochastic DDEs with multiplicative noise and showed how one can use multi-scale analysis to compute the Lyapunov spectrum in such systems. Coherence resonance in DDEs with noise has been demonstrated on the example of machine tool vibrations by Buckwar et al. (2006).

The influence of noise on the synchronization of PDDEs is an important topic not only with respect to applications but also as a general framework in the study of delayed spatially extended systems. To summarize, a lot of research effort is being put into the area of DDEs but even more questions still remain to be answered and that is why delay equations are such an exciting and challenging area to work in.

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