Dynamical Systems
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713414890

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Y. N. Kyrychko [a]; K. B. Blyuss [a]; P. Hövel [b]; E. Schöll [b]
[a] Department of Engineering Mathematics, University of Bristol, Bristol, BS8 1TR, UK [b] Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstr. 36, 10623 Berlin, Germany

First Published on: 21 April 2009

To cite this Article Kyrychko, Y. N., Blyuss, K. B., Hövel, P. and Schöll, E.(2009)'Asymptotic properties of the spectrum of neutral delay differential equations',Dynamical Systems,

To link to this Article: DOI: 10.1080/14689360902893285

URL: http://dx.doi.org/10.1080/14689360902893285

Dynamical Systems
2009, 1–12, iFirst

Asymptotic properties of the spectrum of neutral delay differential equations

Y.N. Kyrychko[a]*, K.B. Blyuss[a], P. Hövel[b] and E. Schöll[b]

[a]*Department of Engineering Mathematics, University of Bristol, Bristol, BS8 1TR, UK;* [b]*Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstr. 36, 10623 Berlin, Germany*

(*Received 6 September 2008; final version received 11 March 2009*)

Spectral properties and transition to instability in neutral delay differential equations are investigated in the limit of large delay. An approximation of the upper boundary of stability is found and compared to an analytically derived exact stability boundary. The approximate and exact stability borders agree quite well for the large time delay, and the inclusion of a time-delayed velocity feedback improves this agreement for small delays. Theoretical results are complemented by a numerically computed spectrum of the corresponding characteristic equations.

Keywords: neutral delay differential equations; stability; spectral properties

AMS Subject Classifications: 39A11; 37N35

1. Introduction

An important area of research in physics and engineering is control theory, and a recent monograph by Schöll and Schuster [1] gives a good overview of the developments in this field. From a dynamical systems perspective, one could consider stabilization of unstable fixed points or unstable periodic orbits [2–4]. In fact, even when these unstable periodic orbits are embedded in a chaotic attractor, they can still be stabilized by weak external forces, as it has been first proposed in a seminal paper by Ott *et al*. [5]. Since then, several other methods of controlling unstable motion have been proposed. One of them is a time-delayed feedback control proposed by Pygaras [6], which can be easily implemented in a wide range of experiments and is non-invasive, i.e. it vanishes as soon as unstable motion becomes stable [7–18]. This method utilizes a difference between a signal at the current time and the same signal at some time ago. The scheme can be improved by introducing multiple time delays into the control loop [19]. Further considerations of multiple delay control, also referred to as extended time-delay autosynchronization can be found in [20–24].

From a theoretical point of view, introduction of a time delay into the system leads to an infinite-dimensional phase space and transcendental characteristic equations [25,26]. This adds a significant difficulty to the stability and bifurcation analyses of such systems. Some analytical results on time-delayed feedback control can be found, for instance, in [27–32]. In the case of linear time-delayed systems with a non-delayed highest derivative,

*Corresponding author. Email: y.kyrychko@bristol.ac.uk

ISSN 1468–9367 print/ISSN 1468–9375 online

DOI: 10.1080/14689360902893285
http://www.informaworld.com

one can use the Lambert function to find the solutions of the corresponding characteristic equation [33,34]. However, for the neutral equations, i.e. equations which have time delays in the highest derivative term, this approach fails, and other approaches should be used. Furthermore, since neutral delay differential equations (NDDEs) often possess discontinuities in their solution, the numerical treatment and bifurcation analyses of such equations are much more involved than those of regular delay differential equations (DDEs). For instance, the existing packages for bifurcation analysis of DDEs, such as DDE-BIFTOOL [35] and PDDE-CONT [36] are currently unable to perform continuation for neutral systems.

This article is devoted to the analytical and numerical analysis of a time-delayed system of neutral type. Such models arise in a variety of contexts, such as biological and population dynamics models, see, for example, [37–39]. Balanov *et al.* [40], for instance, derived a neutral DDE as a model for torsional waves on a driven drill-string. Another example studied by Blakely and Corron [41] is a model of a chaotic transmission line oscillator, in which an NDDE was used to correctly reproduce experimental observations of fast chaotic dynamics. The system to be studied in this article was first introduced by Kyrychko *et al.* [42] in the context of hybrid testing, where it proved to be a good physical model for the description of the effects of actuator delays. It is noteworthy that in hybrid testing, quite often one encounters significant time delays due to actuator response time. Furthermore, actuator delay strongly depends on the stiffness of the system, hence it can vary considerably even in different runs of the same experiment. For this reason, the actual values attained by the actuator time delay can be quite high [43–45].

It is important to note that NDDEs are different from DDEs in that they may possess a continuous as well as a point spectrum, and their stability properties are far from being completely understood. Here, we investigate two different kinds of time-delayed feedback, both of which arise naturally in experimental settings. The first of these includes time delay in the feedback force, while the second introduces a velocity feedback. To understand the stability properties of the system, we will analyse asymptotic behaviour of the eigenvalue spectrum and identify regions of (in)stability in terms of the system's parameters. The validity of these results will be compared to the numerical solution of the corresponding characteristic equation. For the particular system under investigation it is possible to find the stability spectrum analytically, and therefore it serves as a perfect test model for which it is possible to compare exact and approximate stability boundaries. It will be shown that although the approximation may deviate quite significantly from the exact boundary for small time delays, it gives a good agreement for larger time delays. Therefore, this approximation can be used for systems described by neutral DDEs with large time delays, where it is impossible to find the stability boundary analytically.

2. Stability analysis

2.1. *Delayed force*

Consider the following NDDE [42]:

$$\ddot{z}(t) + 2\zeta\dot{z}(t) + z(t) + p\ddot{z}(t-\tau) = 0, \tag{1}$$

where dot means differentiation with respect to time t, and τ is the time delay. In the context of hybrid testing experiments on a pendulum-mass-spring-damper system, ζ stands for a rescaled damping rate, and p is the mass ratio. Introducing $v(t) = \dot{z}(t)$ and

$u(t) = v(t) + pv(t)$, this equation can be rewritten as a system of differential equations with a shift

$$\begin{aligned} \dot{z}(t) &= u(t) - pv(t-\tau), \\ \dot{u}(t) &= -2\zeta[u(t) - pv(t-\tau)] - z(t), \\ v(t) &= u(t) - pv(t-\tau). \end{aligned} \tag{2}$$

With the initial data $(z(0), u(0)) = (z_0, u_0) \in \mathbb{R} \times \mathbb{R}$ and $v(s) = \phi(s) \in C[-\tau, 0]$, this system can first be solved on $0 \le t \le \tau$ interval, then on $\tau \le t \le 2\tau$ and so on, provided the following sewing condition is satisfied: $\phi(0) = u_0 - p\phi(-\tau)$. This condition ensures that there are no discontinuities in the solutions at $t = k\tau$, $k \in \mathbb{Z}_+$. For arbitrary initial conditions the sewing condition does not hold, and leads to jumps in the derivative of the solution [46].

The Equation (1) has a single steady state $z^* = 0$. The stability of this steady state is determined by the real part of the complex roots $\lambda \in \mathbb{C}$ of the corresponding characteristic equation

$$\lambda^2 + 2\zeta\lambda + 1 + p\lambda^2 e^{-\lambda\tau} = 0. \tag{3}$$

As was already mentioned in the introduction, the existing bifurcation packages for the analysis of delay equations, such as DDE-BIFTOOL [35] and PDDE-CONT [36], currently do not provide capabilities of calculating eigenvalues for NDDEs. One of the reasons for this lies in the so-called behavioural discontinuity, a feature unique to NDDEs as compared to standard DDEs. This refers to the fact that even when all characteristic roots are stable for $\tau = 0$, for τ being small and positive infinitely many of these roots may have unbounded real parts. In other words, a small variation of the time delay leads to an infinitely large root variation [47,48].

Several methods based on the linear multi-step approach and pseudospectral differentiation have recently been put forward which provide an efficient tool for computing the characteristic spectrum of NDDEs [49–51]. We have used this method to compute the spectrum of Equation (3), which is shown in Figure 1. It can be observed that for small time delays (Figure 1(a)) the steady state is stable, as all the eigenvalues are in the left half-plane. As time delay increases, a pair of complex conjugate eigenvalues crosses the imaginary axis, as demonstrated in Figure 1(b) and (c), leading to an instability. As time delay increases still further, the unstable eigenvalues return to the left half-plane, thus restoring the stability.

In the case when the mass ratio p in Equation (1) exceeds unity, the steady state is unstable for any positive time delay τ. It is worth noting that for $|p| < 1$, the steady state may undergo stability changes/switches as the time delay is varied. To understand the dynamics of the system in the neighbourhood of these stability changes, one can use the framework of *pseudocontinuous spectrum* used by Yanchuk *et al.* [31,52] for the analysis of scaling behaviour of eigenvalues for large time delays, who followed an earlier work of Lepri *et al.* [53] on scaling of the spectra. Following this approach, one can express the asymptotic approximation of the eigenvalues for large τ as

$$\lambda = \frac{1}{\tau}\gamma + i\left(\Omega + \frac{1}{\tau}\phi\right) + \mathcal{O}\left(\frac{1}{\tau^2}\right), \tag{4}$$

where γ, Ω and ϕ are real-valued quantities, which are associated with the real and imaginary part of the eigenvalue λ, respectively. Substituting this representation into

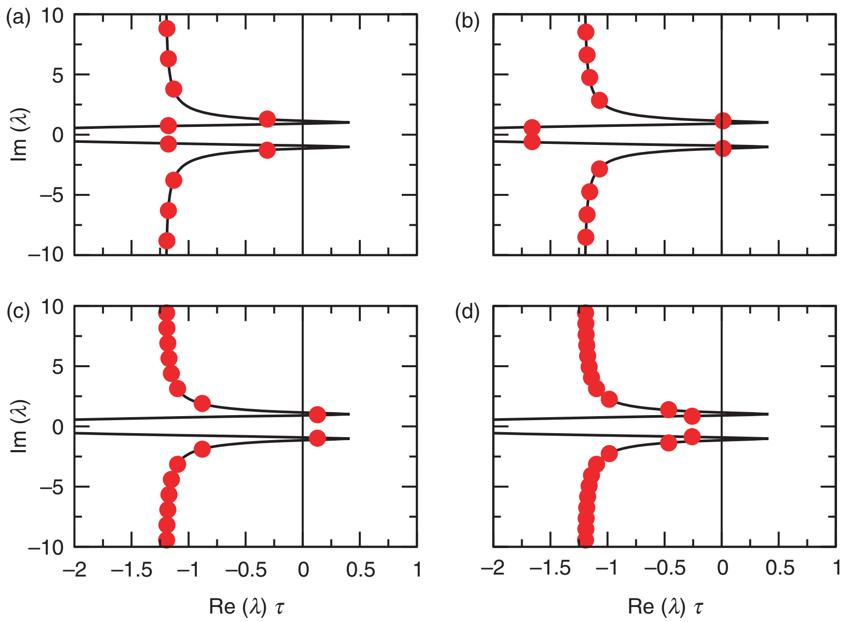

Figure 1. (Colour online) Spectrum of the characteristic Equation (3) for different time delays: (a) $\tau=2.5$, (b) $\tau=3.32$, (c) $\tau=5$ and (d) $\tau=7$. Parameter values are: $\zeta=0.1$, $p=0.3$. The solid lines show the asymptotic pseudocontinuous spectrum given by Equation (7).

the characteristic Equation (3), gives to the leading order in $\mathcal{O}(1/\tau)$:

$$1-\Omega^2+2i\zeta\Omega-p\Omega^2e^{-\gamma}e^{-i\phi}e^{-i\Omega\tau}=0. \tag{5}$$

By choosing $\Omega=\Omega^{(n)}=2\pi n/\tau$, $n=\pm1,\pm2,\pm3,\ldots$ in Equation (5), we can simplify this equation to

$$1-\Omega^2+2i\zeta\Omega-p\Omega^2e^{-\gamma}e^{-i\phi}=0. \tag{6}$$

From (4) it follows that $\mathrm{Re}(\lambda)\approx\gamma(\Omega)/\tau$ and $\mathrm{Im}(\lambda)\approx\Omega$ up to the leading order, and therefore the eigenvalues λ accumulate in the complex plane along curves $(\gamma(\Omega),\Omega)$, with the real axis scaling as $\tau\mathrm{Re}(\lambda)$. Solving Equation (6) gives an expression for the real part γ of the eigenvalue as a function of the Hopf frequency Ω

$$\gamma(\Omega)=-\frac{1}{2}\ln\frac{1}{p^2}\left[1+\frac{4\zeta^2-2}{\Omega^2}+\frac{1}{\Omega^4}\right]. \tag{7}$$

A steady state can lose its stability via a Hopf bifurcation, at which point the tip of curve $\gamma(\Omega)$ will cross the imaginary axis. If this happens, there will be an interval of frequencies $\Omega_1<\Omega<\Omega_2$, for which $\gamma(\Omega)>0$ and $\gamma(\Omega_1)=\gamma(\Omega_2)=0$. This instability can be prevented, provided the interval of unstable frequencies $\Omega_1<\Omega<\Omega_2$ lies inside the interval $\left[\Omega^{n_0},\Omega^{n_0+1}\right]$ for some n_0 [31]. Here, $\Omega_{1,2}$ are two positive roots of the equation $\gamma(\Omega)=0$, which can be found from Equation (7) as

$$\Omega_{1,2}^2=\frac{1}{1-p^2}\left[1-2\zeta^2\pm\sqrt{(1-2\zeta^2)^2-1+p^2}\right]. \tag{8}$$

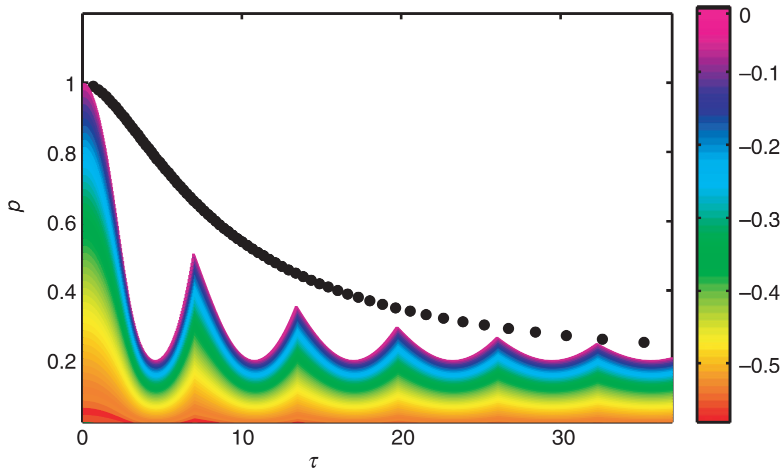

Figure 2. (Colour online) Comparison of the approximate upper bound of stability according to Equation (10) (dotted line) with an exact stability boundary for $\zeta = 0.1$ in the (τ, p) plane. The grayscale (colour code) encodes the value of the largest real part of the complex eigenvalues λ.

For further analytical progress, we expand this expression for small values of ζ, which gives

$$\Delta\Omega = \Omega_1 - \Omega_2 = \frac{1}{\sqrt{1-p}} - \frac{1}{\sqrt{1+p}} - \frac{\zeta^2}{p}\left(\frac{1}{\sqrt{1+p}} + \frac{1}{\sqrt{1-p}}\right). \tag{9}$$

Since the actual values of the frequencies are $\Omega^{(n)} = 2\pi n/\tau$ for any integer n, the distance between any two successive frequencies is $2\pi/\tau$, and hence the necessary condition for stability $\Delta\Omega < 2\pi/\tau$ can be written as

$$\frac{1}{\sqrt{1-p}} - \frac{1}{\sqrt{1+p}} - \frac{\zeta^2}{p}\left(\frac{1}{\sqrt{1+p}} + \frac{1}{\sqrt{1-p}}\right) < 2\pi/\tau. \tag{10}$$

For large enough time delay τ, p asymptotically approaches a lower bound of stability which corresponds to $\Delta\Omega = 0$. It can be obtained from Equation (8) by using

$$\begin{aligned} 0 &= (\Omega_1 - \Omega_2)(\Omega_1 + \Omega_2) = \Omega_1^2 - \Omega_2^2 \\ &= \frac{2}{1-p^2}\sqrt{(1-2\zeta^2)^2 - 1 + p^2}, \end{aligned}$$

which yields

$$p = 2\zeta\sqrt{1-\zeta^2} \approx 2\zeta. \tag{11}$$

Figure 2 shows the plot of the approximate stability boundary (10) as a function of time delay τ for a given small value of the damping ζ. The grayscale (colour code) in this figure indicates the value of the largest real part of the eigenvalues in the spectrum of the characteristic Equation (3) for each value of p and τ. As it follows from Figure 2, the analytically derived formula (10) for the maxima on the stability boundary deviates from the exact stability peaks (which correspond to codimension-two Hopf bifurcation) for small delays, but it provides a good approximation for large time delay τ.

The Appendix contains an exact analytic expression for the stability boundary in terms of system's parameters.

2.2. *Delayed viscous damping*

In order to analyse the influence of velocity feedback on the stability of NDDE, we modify Equation (1) as

$$\ddot{z}(t) + 2\zeta_1\dot{z}(t) + z(t) + p\ddot{z}(t-\tau) + 2\zeta_2\dot{z}(t-\tau) = 0. \tag{12}$$

This equation was introduced in Ref. [42], where it was shown that depending on the difference between two damping parameters ζ_1 and ζ_2, the stability domain may shrink and even split into separate stability regions in the parameter plane (the so-called death islands). The characteristic equation now modifies to

$$\lambda^2 + 2\zeta_1\lambda + 1 + p\lambda^2 e^{-\lambda\tau} + 2\zeta_2\lambda e^{-\lambda\tau} = 0. \tag{13}$$

Figure 3 shows the numerical approximation of the roots of this equation in the neighbourhood of the origin. From this figure it follows that similar to the situation without velocity feedback, the system undergoes successive stability switches as the time delay is varied.

Assuming in Equation (13) the same asymptotic behaviour (4) of the eigenvalues for large time delay (i.e. the real part of the eigenvalue scales as $1/\tau$), gives to the leading order

$$1 - \Omega^2 + 2i\zeta_1\Omega - p\Omega^2 e^{-\gamma} e^{-i\phi} + 2i\zeta_2\Omega e^{-\gamma} e^{-i\phi} = 0, \tag{14}$$

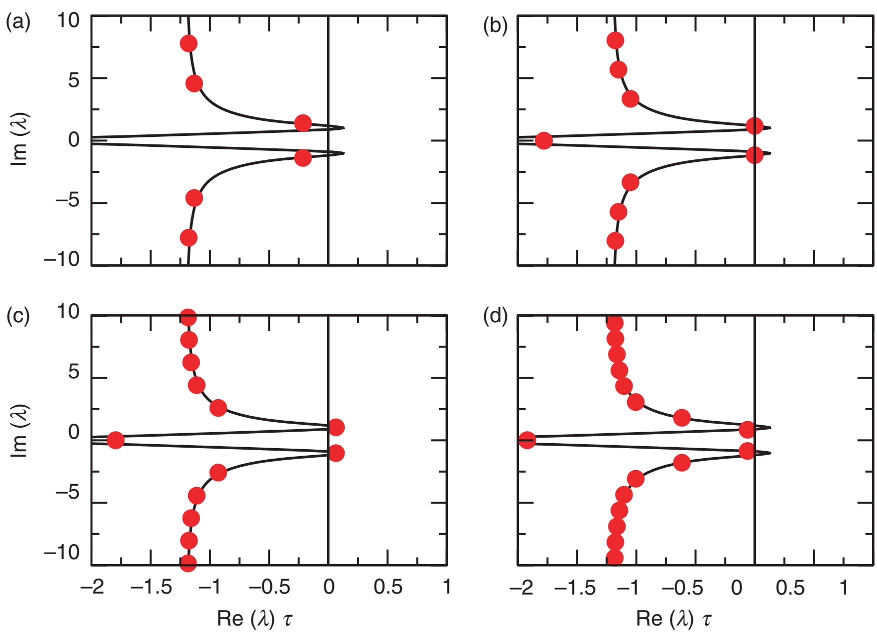

Figure 3. (Colour online) Spectrum of the characteristic Equation (13) for different time delays: (a) $\tau = 2$, (b) $\tau = 2.725$, (c) $\tau = 3.5$ and (d) $\tau = 5$. Parameter values are: $\zeta_1 = 0.25$, $\zeta_2 = 0.24$, $p = 0.3$. The solid lines show the asymptotic pseudocontinuous spectrum given by Equation (15).

with the constraint $\Omega = \Omega^{(n)} = 2\pi n/\tau$, $n = \pm 1, \pm 2, \pm 3, \ldots$. One can solve this equation for the real part of the eigenvalue γ at the Hopf bifurcation as a function of frequency Ω as

$$\gamma(\Omega) = -\frac{1}{2}\ln\frac{\left(1-\Omega^2\right)^2 + 4\zeta_1^2\Omega^2}{p^2\Omega^4 + 4\zeta_2^2\Omega^2}. \tag{15}$$

Transition to instability occurs when $\gamma(\Omega) = 0$, which gives the expression for instability frequencies

$$\Omega_{1,2}^2 = \frac{1}{1-p^2}\left[1 + 2\zeta_2^2 - 2\zeta_1^2 \pm \sqrt{\left(1 + 2\zeta_2^2 - 2\zeta_1^2\right)^2 - 1 + p^2}\right]. \tag{16}$$

In a manner similar to the analysis of the delayed force feedback, one can make further analytical progress by assuming that both damping coefficients are small: $|\zeta_1| \ll 1$, $|\zeta_2| \ll 1$. The necessary stability condition $\Delta\Omega = \Omega_1 - \Omega_2 < 2\pi/\tau$ gives the following asymptotic approximation for the maxima of the stability boundary

$$\frac{1}{\sqrt{1-p}} - \frac{1}{\sqrt{1+p}} - \frac{\zeta_1^2 - \zeta_2^2}{p}\left(\frac{1}{\sqrt{1+p}} + \frac{1}{\sqrt{1-p}}\right) < 2\pi/\tau. \tag{17}$$

The expression (17) can be further simplified for large time delay in a manner similar to (11), which gives $p = 2\sqrt{(\zeta_1^2 - \zeta_2^2)(1 - \zeta_1^2 + \zeta_2^2)} \approx 2\sqrt{\zeta_1^2 - \zeta_2^2}$.

It is noteworthy that the inequality (17) provides a good approximation for the stability boundary even when actual values of damping coefficients ζ_1 and ζ_2 are large, as long as the difference $(\zeta_1^2 - \zeta_2^2)$ is small by the absolute value. Figure 4 shows an excellent agreement between the asymptotic approximation (17) and the exact stability boundary, especially for sufficiently large time delay. In the Appendix it is shown how the exact

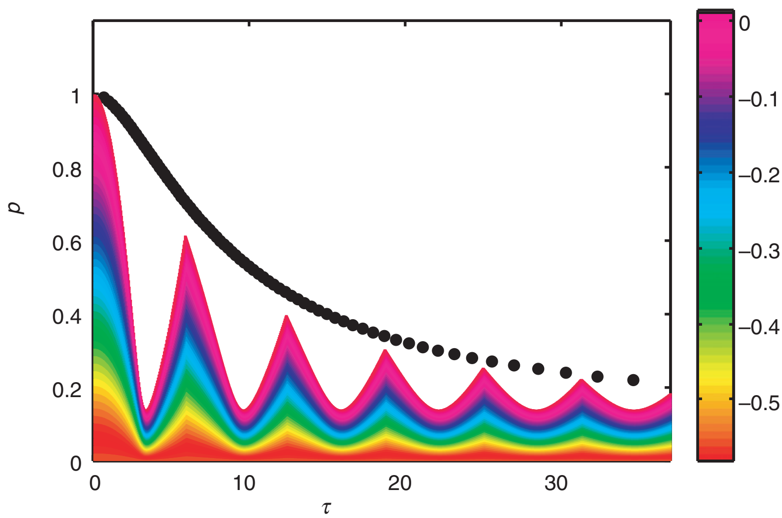

Figure 4. (Colour online) Comparison of the approximate upper bound of stability according to Equation (17) (dotted line) with an exact stability boundary for $\zeta_1 = 0.25$ and $\zeta_2 = 0.24$ in the (τ, p) plane. The grayscale (colour code) encodes the value of the largest real part of the complex eigenvalues λ.

stability boundary varies depending on parameters, and in particular on the relation between the two damping coefficients.

3. Conclusions

Time delays are an intrinsic feature of many physical, biological and engineering systems, and in recent years the analysis of such systems has led to many interesting and important findings. There are systems where the time delay is present intrinsically due to processing times, mechanical inertia etc., and there are those where the time delay is introduced externally in order to stabilize unstable periodic orbits and steady states. Therefore, a better understanding of delay differential equations will provide a clear picture of the system's stability and controllability. In this article we have concentrated on the analysis of two neutral delay differential equations. We have shown that depending on the time delay τ, the systems exhibit stability switches, where stability is lost/regained depending on the time delay. In the case of delayed velocity feedback, the interplay between the time delay and the two damping coefficients gives different stability regimes in the parameter plane, and for some parameter values the stability area collapses into separate islands. We have derived an asymptotic approximation of the stability peaks for large time delays, based upon universal scaling arguments, and have compared this approximation with the exact stability boundary. The results agree quite well even when the time delay is not too large, and give excellent agreement for large delays. The results presented in this article include numerical simulations of the characteristic spectrum and constitute the first attempt to approximate stability peaks for neutral DDEs. As has already been mentioned in the introduction, neutral DDEs arise naturally in a wide range of physical problems, which makes the approach developed in this article a useful tool for the stability analysis of such systems.

Acknowledgements

Y.K. was supported by an EPSRC Postdoctoral Fellowship (Grant EP/E045073/1). The authors would like to thank Dimitri Breda for providing assistance with numerical computation of the roots of characteristic equations. This work was partially supported by Deutsche Forschungsgemeinschaft in the framework of Sfb 555.

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Appendix: Exact stability boundary

To find an exact analytical expression for the stability boundary, one has to substitute $\lambda = i\omega$ into the characteristic Equation (3). After separating real and imaginary parts, this gives [54]

$$\begin{aligned} 1 - \omega^2 - p\omega^2 \cos(\omega\tau) = 0, \\ 2\zeta\omega + p\omega^2 \sin(\omega\tau) = 0. \end{aligned} \tag{A1}$$

Squaring and adding these equations gives

$$(1 - p^2)\omega^4 + (4\zeta^2 - 2)\omega^2 + 1 = 0. \tag{A2}$$

This equation can be solved as

$$\omega_{1,2}^2 = \frac{1}{1 - p^2}\left[1 - 2\zeta^2 \pm \sqrt{(1 - 2\zeta^2)^2 - 1 + p^2}\right]. \tag{A3}$$

In fact, Equation (A2) provides an expression for stability boundary value of p as parametrized by the Hopf frequency ω:

$$p(\omega) = \frac{1}{\omega^2}\sqrt{\omega^4 + 2\omega^2(2\zeta^2 - 1) + 1}. \tag{A4}$$

The corresponding value of the time delay at the stability boundary is derived from Equation (A1)

$$\tau(\omega) = \frac{1}{\omega}\left[\mathrm{Arctan}\frac{2\zeta\omega}{\omega^2 - 1} \pm \pi k\right], \tag{A5}$$

where $k = 0, 1, 2, \ldots$ and Arctan denotes the principal value of arctan. Figure A1 illustrates the dependence of critical mass ratio p on the time delay τ which ensures the stability of the steady state. It is noteworthy that if $|p| > 1$, the steady state is unstable for any positive time delay τ; on the other hand, if $|p| < 1$ and $\zeta > 1/\sqrt{2}$, then the steady state is asymptotically stable for any positive time

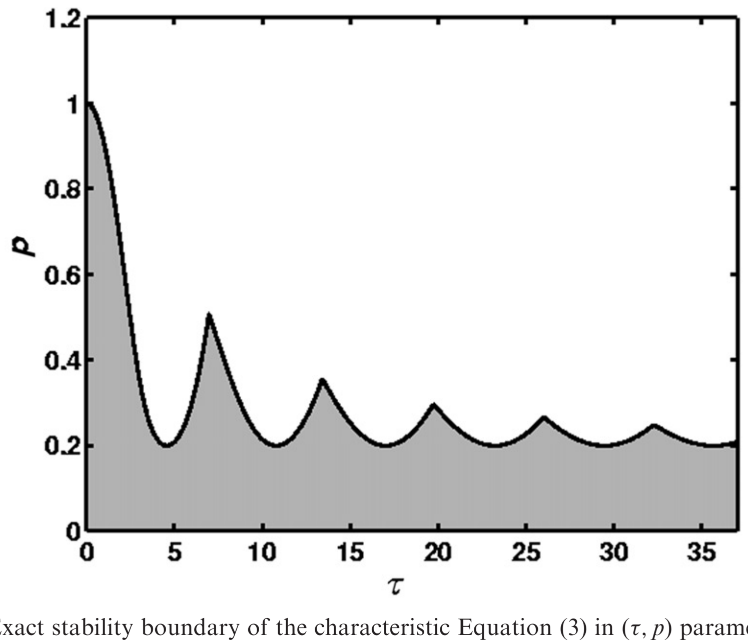

Figure A1. Exact stability boundary of the characteristic Equation (3) in (τ, p) parameter plane for $\zeta = 0.1$. The steady state is stable in the shaded area.

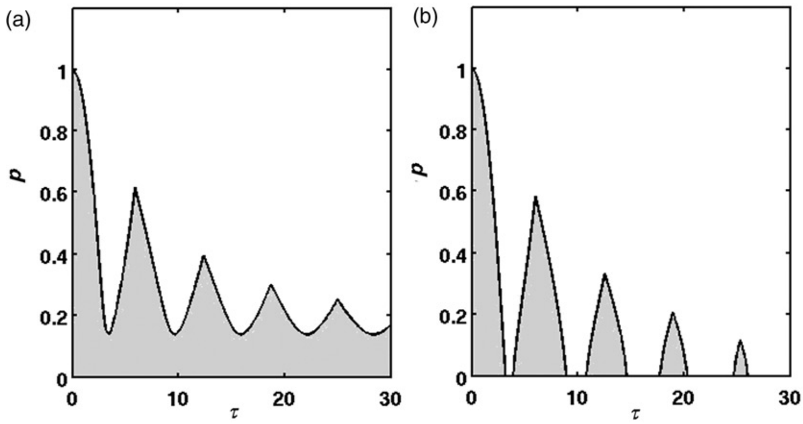

Figure A2. Exact stability boundary of the characteristic Equation (13) in (τ, p) parameter plane. The steady state is stable in the shaded area. Parameter values are: (a) $\zeta_1 = 0.25$ and $\zeta_2 = 0.24$, (b) $\zeta_1 = 0.23$ and $\zeta_2 = 0.25$.

delay τ. For $\zeta < 1/\sqrt{2}$, there is a lower bound on the value of $p_{\min} = 2\zeta\sqrt{1-\zeta^2}$, so that an asymptotic stability is guaranteed for all $\tau > 0$ provided $p < p_{\min}$ [42].

In the case of time-delayed viscous damping, the characteristic Equation (13) at the points of stability changes can be written as

$$\begin{aligned} 1 - \omega^2 - p\omega^2\cos\omega\tau + 2\zeta_2\omega\sin\omega\tau = 0, \\ 2\zeta_1 + p\omega\sin\omega\tau + 2\zeta_2\cos\omega\tau = 0. \end{aligned} \tag{A6}$$

Squaring and adding these two equations gives the following parametrization of p by the Hopf frequency:

$$\omega_{1,2}^2 = \frac{1}{1-p^2}\left[1 - 2\zeta_1^2 + 2\zeta_2^2 \pm \sqrt{\left(1 - 2\zeta_1^2 + 2\zeta_2^2\right)^2 - 1 + p^2}\right]. \tag{A7}$$

Similar to the previous case, one can derive parametric expressions for $p(\omega)$ and $\tau(\omega)$ from Equation (A6)

$$p(\omega) = \frac{1}{\omega^2}\sqrt{(\omega^2-1)^2 + 4\left(\zeta_1^2 - \zeta_2^2\right)\omega^2}. \tag{A8}$$

The corresponding value of the time delay at the stability boundary can be found as

$$\tau(\omega) = \frac{1}{\omega}\left[2\pi n - \arccos\frac{p(1-\omega^2) - 4\zeta_1\zeta_2}{p^2\omega^2 + 4\zeta_2^2}\right], \qquad n = 1, 2, 3, \ldots . \tag{A9}$$

Figure A2 demonstrates how stability boundary is affected by the relation between ζ_1 and ζ_2. In particular, we note that when $\zeta_1 = \zeta_2$, the stability boundary touches the τ-axis ($p = 0$), and for $\zeta_2 > \zeta_1$, the stability area consists of non-overlapping death islands, inside which the oscillations are damped, and the steady state is stable.