Maths for Computing

Lecture 3

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$\mathbb{C}$: The complex numbers

- Every complex number can be written as

$$z = x + iy \qquad \begin{matrix} x \in \mathbb{R} \\ y \in \mathbb{R} \end{matrix}$$

Real part **Imaginary part**

- We can operate on complex numbers like usual (BODMAS), with the additional rule
$$i^2 = -1$$

BB Dividing complex numbers

$$\frac{a+ib}{c+id}=?$$

Trick:

$$(c+id)\cdot(c-id)=c^2-i^2d^2-cid+cid=c^2+d^2$$

Therefore,

$$\frac{a+ib}{c+id}=\frac{a+ib}{c+id}\cdot\frac{c-id}{c-id}=\frac{1}{c^2+d^2}(a+ib)(c-id)$$

$$=\frac{1}{c^2+d^2}(ac+bd-iad+icb)$$

Summation notation ($\sum$ notation)

upper limit

Definition: $$\sum_{j=1}^{3} x_j := x_1 + x_2 + x_3$$

summation index

lower limit

Note: Increment always by 1!

It is like a “for” loop:

```
a= 0;
for ( j=1; j <= 3; j= j + 1) {
    a= a+x_j
}
```

Some alternative notations **BB**

BB Some alternative notations

$$\sum_{i \in \{1,\ldots,N\}} x_i = \sum_{i=1}^{N} x_i$$

$$\sum_{i \in \{1,\ldots,N\}, i \neq 2} x_i = \left(\sum_{i=1}^{N} x_i \right) - x_2$$

$$\sum_{\substack{i \in \{1,\ldots,N\} \\ j \in \{1,\ldots,M\}}} x_i x_j = \sum_{i=1}^{N} \sum_{j=1}^{M} x_i x_j$$

Summation notation

Summation notation is extremely convenient and is used everywhere.

Note: Empty sums are zero:

$$\sum_{k=2}^{1} k^2 = 0$$

Examples: **BB**

BB "Normal" examples

$$\sum_{i=1}^{4} 3 \cdot i = 3 + 6 + 9 + 12$$

$$\sum_{j=3}^{5} \sum_{k=1}^{2} j \cdot k = \sum_{j=3}^{5} (j \cdot 1 + j \cdot 2)$$

$$= 3 \cdot 1 + 3 \cdot 2 + 4 \cdot 1 + 4 \cdot 2 + 5 \cdot 1 + 5 \cdot 2$$

$$= 3 + 6 + 4 + 8 + 5 + 10$$

BB More examples

$$\sum_{j=3}^{5}\sum_{k=1}^{2} j\cdot k = \sum_{k=1}^{2} 3\cdot k + \sum_{k=1}^{2} 4\cdot k + \sum_{k=1}^{2} 5\cdot k$$

$$= 3\cdot 1 + 3\cdot 2 + 4\cdot 1 + 4\cdot 2 + 5\cdot 1 + 5\cdot 2$$

$$= 3 + 6 + 4 + 8 + 5 + 10$$

Note how the order of expansion does not matter!
Reason:

$a + (b + c) = (a + b) + c$ (Associativity)

$a + b = b + a$ (Commutativity)

Product notation

Definition: $$\prod_{j=1}^{5} a_j := a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$$

Think about a for loop, but multiplication inside, instead of summation.

Example: $$\prod_{j=1}^{5} j = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

Product notation

Product notation is somewhat less common but also still used frequently (and useful)

Note: Empty products are 1:

$$\prod_{i=5}^{1} x_i = 1$$

This makes a lot of sense:

Empty sum

```
a= 0;
for ( j=1; j <= 3; j= j + 1) {
   a= a+x_j
}
```

If the loop is never executed -> sum is 0.

Empty product

```
a= 1;
for ( j=1; j <= 3; j= j + 1) {
   a= a*x_j
}
```

If the loop is never executed -> product is 1.

Why one implies the other: **BB**

BB … and why one implies the other

For $a, b \in \mathbb{R}$

$$a \cdot b = \exp(\log(a \cdot b)) = \exp(\log(a) + \log(b))$$

$$\prod_{i=1}^{n} x_i = \exp\left(\sum_{i=1}^{n} \log(x_i)\right)$$

If the product is empty, then the sum is empty and

$$\prod_{i=1}^{n} x_i = \exp(0) = 1$$

Summary

Sum: $$\sum_{j=1}^{3} x_j := x_1 + x_2 + x_3$$, empty sum is 0.

Product: $$\prod_{j=1}^{5} a_j := a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$$, empty product is 1

Multiple sums or products:

$$\sum_{j=3}^{5} \sum_{k=1}^{2} j \cdot k$$

… order of evaluation does not matter

“The domino effect”

INDUCTION

Mathematical induction

Let $P(n)$ be the predicate (or assumption).

$P(n)$ is true for particular values of *n,* e.g:

$P(n)$: “$n^3 - n$ is divisible by 3 “

$$n = 1 \ : \ 1^3 - 1 = 0$$

$$n = 2 \ : \ 2^3 - 2 = 8 - 2 = 6$$

$$n = 3 \ : \ 3^3 - 3 = 27 - 3 = 24$$

Proof by induction

$P(n)$: '$n^3 - n$ is divisible by 3'

Seems alright for the examples.

But we would like to show:

$P(n)$ is true for all $n \in \mathbb{N}$

In "maths speak": $\forall n \in \mathbb{N} \; : \; P(n)$

"for all"

Proof by induction

implies

Intuitively …

1. Show that P(n) $\Rightarrow$ P(n+1) for any n
2. Show that P(1) is true
3. Then from 1. and 2., P(2) is true
4. … and from 1. and 3., P(3) is true
5. … and from 1. and 4., P(4) is true
6. …

Mathematical induction

In “maths speak”:

“and”

$$P(1) \wedge \Big(\forall n \in \mathbb{N} : \big(P(n) \Rightarrow P(n+1)\big)\Big)$$
$$\Rightarrow \forall n \in \mathbb{N} : P(n)$$

implies

- If P(1) is true
  … and for all n, P(n) $\Rightarrow$ P(n+1),
  then P(n) is true for all values of n.

- This is the principle of **mathematical induction.**

Example 1

$P(n)$: '$n^3 - n$ is divisible by 3'

- Case n=1: $1^3 - 1 = 0$ … is divisible by 3
- Now **assume** that P(n) is true
  - Need to **show** P(n+1) is true
  - I.e. we need to show that

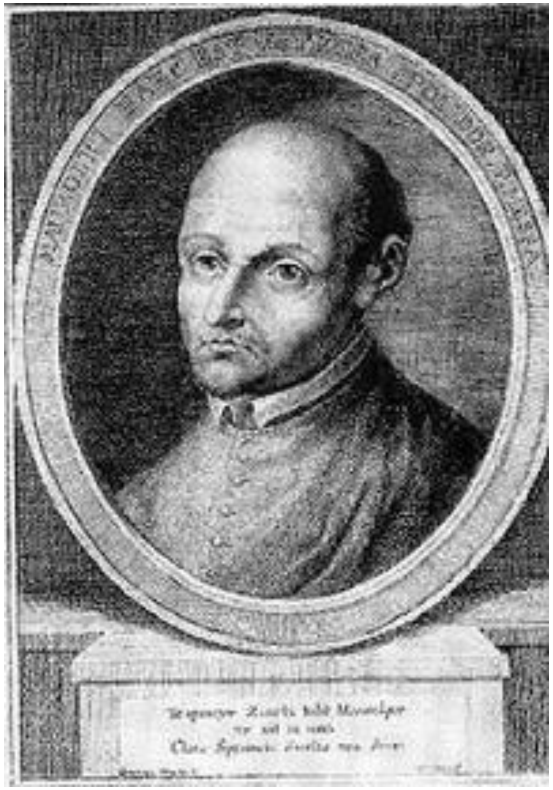
    $(n+1)^3 - (n+1)$ is divisible by 3.

    But we can use that $n^3 - n$ is divisible by 3.

BB

Example 2

- Francesco Maurolico (1575)
- First known proof by induction.



$$P(n)\ :\ \sum_{i=1}^{n}(2i-1) = 1+3+5+\ldots+2n-1 = n^2$$

Example 2 continued

- $P(1) \ : \ 1 = 1^2$ … is true.
- Assume P(n) is true, need to show P(n+1) is then also true.
- Write P(n+1) in terms of P(n):

$$P(n+1) \ : \ \sum_{i=1}^{n+1} 2i - 1 = \underbrace{\sum_{i=1}^{n} 2i - 1}_{=n^2 \text{ by Ind.}} + 2(n+1) - 1$$

$$= n^2 + 2n + 1 = (n+1)^2$$

Example 2 concluded

- So, if P(n) is true, P(n+1) is true
- And P(1) is true.

- Therefore, **by mathematical induction**,

$$\Rightarrow \forall n \in \mathbb{N} : P(n)$$

BB Example 3

“Geometric series”

Claim:

$$P(n) \ : \ \sum_{i=0}^{n} q^i = 1 + q + q^2 + q^3 + \ldots + q^n$$

$$= \frac{1 - q^{n+1}}{1 - q}$$

BB Example 3: Start

$$P(1) \ : \ \frac{1-q^2}{1-q} = \frac{(1-q)\cdot(1+q)}{1-q}$$

$$= 1 + q = \sum_{i=0}^{1} q^i$$

… so P(1) is true.

Assume now P(n) is true.

BB Example 3: Induction step

$$P(n+1)\ :\ \sum_{i=0}^{n+1} q^i = \underbrace{\sum_{i=0}^{n} q^i}_{=\frac{1-q^{n+1}}{1-q}} + q^{n+1}$$

by assumption

$$= \frac{1-q^{n+1}}{1-q} + q^{n+1}\frac{1-q}{1-q}$$

$$= \frac{1-q^{n+1}+q^{n+1}-q^{n+2}}{1-q}$$

$$= \frac{1-q^{n+2}}{1-q} = \frac{1-q^{(n+1)+1}}{1-q}$$

BB Example 3: Conclusion

By induction it follows that

$$\sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \quad \forall \quad n \in \mathbb{N}$$

Proof complete.
Qed
(quod erat
demonstrandum)

Things about induction

- Mathematical induction can <u>only</u> be used to prove arguments for positive, whole numbers – i.e., the **natural numbers** $\mathbb{N}$.

- No need to start with P(1). Sometimes it is easier to start with P(3) or P(4) etc.

- But then the inductive proof will only be valid for P(>=3) or P(>=4) etc.

Induction and recursion

- Mathematical induction is important in computer science because it allows us to prove things about **recursive** programs.
- A recursive function is a **function that calls itself**.
- Recursion is the basis of several programming languages, e.g., PROLOG.

metro
Woman spotted yesterday reading today's paper

Recursive programming

- A technique for simplifying a problem by dividing it into subproblems of the same type.
- For example, consider this function of the factorial:

```
function n= Factorial(nm)
  if (nm <= 1)
    n= 1;
  else
    n= nm*Factorial(nm-1);
  end
end
```

Proof of correctness

Proof by induction:

1. n=1:

```
if (nm <= 1)
    n= 1;
else ...
```

$1! = 1$ Ok!

2. Now assume it is correct for n;

3. Induction: ...

is correct by assumption $= (n-1)!$

```
      else
         n= nm*Factorial(nm-1);
      end
```

Proof of correctness complete

Furthermore,

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n \cdot (n-1)!$$

…

```
  else
     n= nm*Factorial(nm-1);
  end
```

$$= (n-1)!$$

Therefore, n= nm! … the function does always return the correct value. Done.

Practical tips about Induction

- If you don’t use the assumption, you are not doing proof by induction! BB1
- Sometimes it is useful to “work from two sides” BB2
- If you forget any of the steps it is not a proof! BB3
  (especially the start is often forgotten)

BB1 Using the assumption

Example 2 again:

We are trying to do the Induction step to show

$$\sum_{i=1}^{n+1} 2i - 1 = (n+1)^2$$

Induction step:

This is "n" -> we can use the assumption here!

$$\sum_{i=1}^{n+1} 2i - 1 = \sum_{i=1}^{n} 2i - 1 + \text{something}$$

BB2 “Coming from two sides”

We need to show

$$\sum_{i=1}^{n+1} 2i - 1 = (n+1)^2$$

“From the left”:

$$\sum_{i=1}^{n+1} 2i - 1 = \left(\sum_{i=1}^{n} 2i - 1\right) + 2(n+1) - 1$$

using the assumption simplifying

$$= n^2 + 2(n+1) - 1 = n^2 + 2n + 1$$

BB “Coming from two sides …”

“From the right”:

$$(n+1)^2 = n^2 + 2n + 1$$

simplifying

“From the left”:

$$n^2 + 2n + 1$$

“From the right”:

$$n^2 + 2n + 1$$

It’s equal – success!

Wrong Wrong

BB Forgotten start

“Proof” that for all numbers $n = n - 1$:

1. (forgotten start)

2. Assumption: True for n, i.e. $n = n - 1$

3. Induction:

$$n + 1 = \underset{\uparrow}{(n - 1)} + 1 = n \qquad \text{q.e.d}$$

Using assumption:

$n = n - 1$

Wrong Wrong

Summary

Proof by induction always has 3 parts:

1. Start: e.g. show P(n) for n=0, or n=1

2. Assumption: P(n) is true

3. Induction step: show that P(n+1) is true using P(n)