Mathematical Concepts (G6012)

Lecture 24

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REVISIONS III

PROBABILITY THEORY

Probability space

Set of events — probability measure

(Ω, P) is a probability space, if the following conditions hold:

$$P : \mathcal{P}(\Omega) \to [0,1]$$
$$\omega \mapsto P(\omega)$$

$$P(\Omega) = 1$$

$$P(\emptyset) = 0$$

$$P(A \cup B) = P(A) + P(B) \quad \text{if } A \cap B = \emptyset$$

(Additivity)

Properties of probabilities

- **Probability of the complement:**

Let $A \subset \Omega$ be an event, then $A^C = \Omega \backslash A$
and $P(A^C) = 1 - P(A)$

- **Independence:**

Definition: Two events A and B are independent if
$P(A \cap B) = P(A) \cdot P(B)$

- **Conditional probabilities:**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Read: probability of A given B

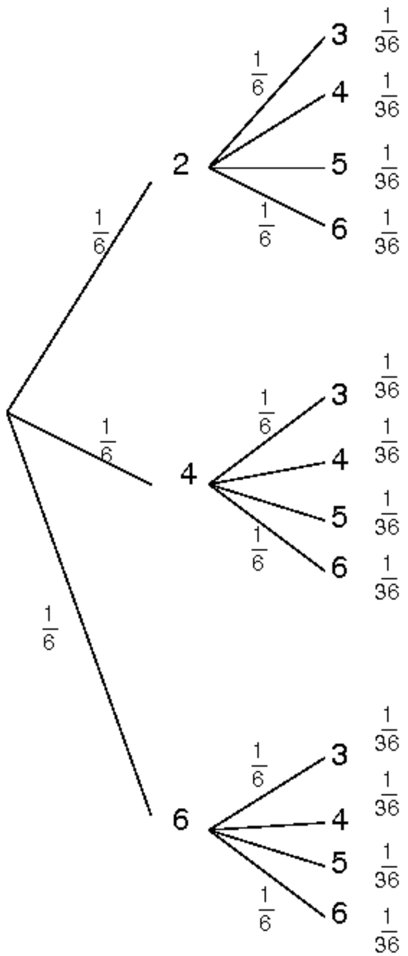
Properties of probabilities

- **Bayes rule:**

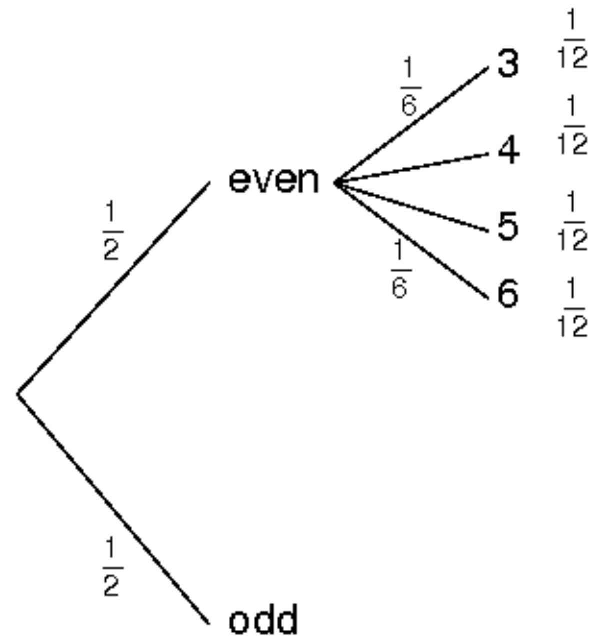
$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Useful tool: Tree graphs, e.g. 2 dice

$$P(\{(i,j) : i \text{ even}, j \geq 3\})$$



$$12 \cdot \frac{1}{36} = \frac{1}{3}$$



$$4 \cdot \frac{1}{12}$$

In both cases the result is $\frac{1}{3}$ (as it should)

Random variable

A random variable is a function

$$X : \Omega \to \mathbb{R}$$
$$\omega \mapsto X(\omega)$$

Examples:

$$\Omega = \{(i,j) : i = 1, \ldots 6, j = 1 \ldots 6\}$$ (two dice)

$$X(i,j) = i + j \qquad Z(i,j) = i^2 + j^2$$

$$Y(i,j) = i \cdot j$$ … and so on

Expectation value

- The **expectation value** is defined as

$$\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega) P(\omega) \quad \text{or} \quad \mathbb{E}X = \sum_{k \in \mathcal{X}} k P(X = k)$$

- It is additive:

$$\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y$$

$$\mathbb{E}(k \cdot X) = k \cdot \mathbb{E}X$$

- For the expectation value of **independent random variables**

$$\mathbb{E}(X \cdot Y) = \mathbb{E}X \cdot \mathbb{E}Y$$

Variance, Standard Deviation

- Variance is how much a RV **varies** around its expectation value

$$Var(X) = \mathbb{E}((X - \mathbb{E}X)^2)$$
$$= \mathbb{E}X^2 - (\mathbb{E}X)^2$$

(please prove in worksheet 5)

- Standard deviation is the square root of variance $std(X) = \sigma_X = \sqrt{Var(X)}$

Covariance, Correlation

- Covariance for two RV X and Y

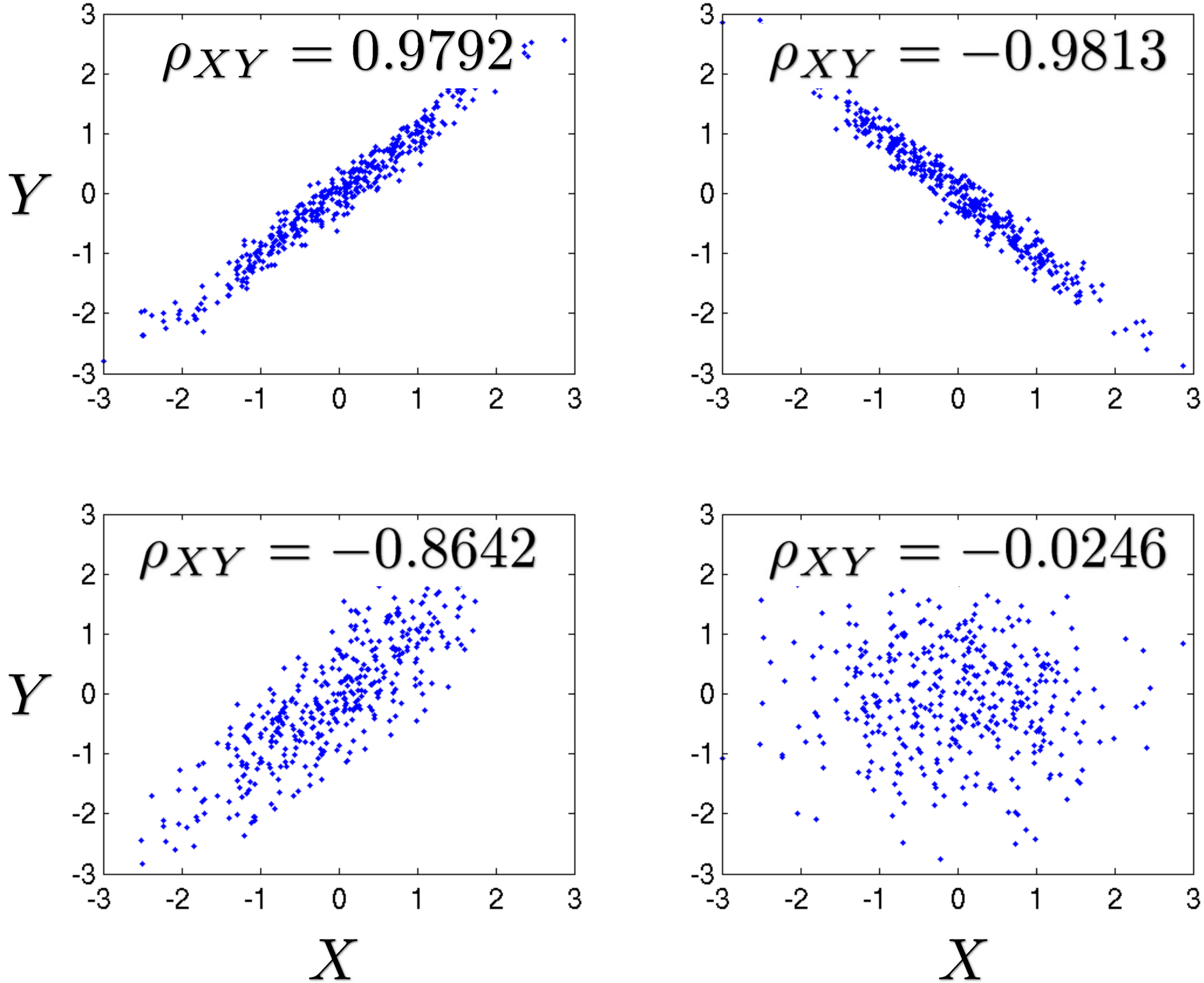
$$\mathrm{Cov}(X,Y) = \mathbb{E}\big((X - \mathbb{E}X)(Y - \mathbb{E}Y)\big)$$
$$= \mathbb{E}(XY) - \mathbb{E}X\mathbb{E}Y$$

- Correlation is the normalized covariance:

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \, \sigma_Y}$$

where σ_X and σ_Y are the standard deviations.

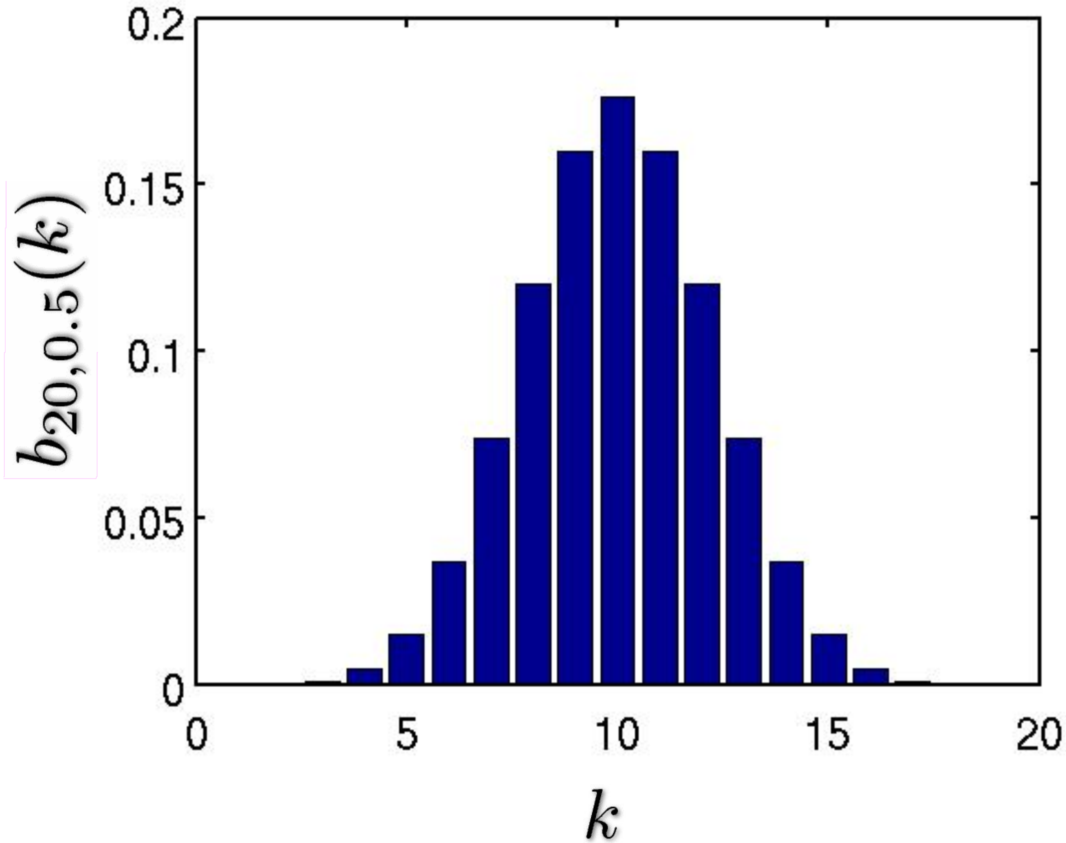
Examples for correlation



Binomial distribution

Probability of k successes (of probability p) in n trials:

$$P(\{X = k\}) = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} = b_{n,p}(k)$$



Expectation value

$$\mathbb{E} \sum_{i=1}^{N} x_i = N \cdot p$$

Standard deviation

$$\sigma_{\sum_{i=1}^{N} x_i} = \sqrt{Np(1-p)}$$

Law of large numbers

For any $\epsilon > 0$

$$\lim_{n\to\infty} P(|x_n - \mathbb{E}x_n| \geq \epsilon) = 0$$

The probability for x_n to be more than ϵ away from its expectation value $\mathbb{E}x_n = p$ converges to 0 for $n \to \infty$.

STATISTICS

Statistical quantities

- The **mean** of a set of observations $\{x_i\},\ i = 1, \ldots, N$ is $\overline{x} = \langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i$

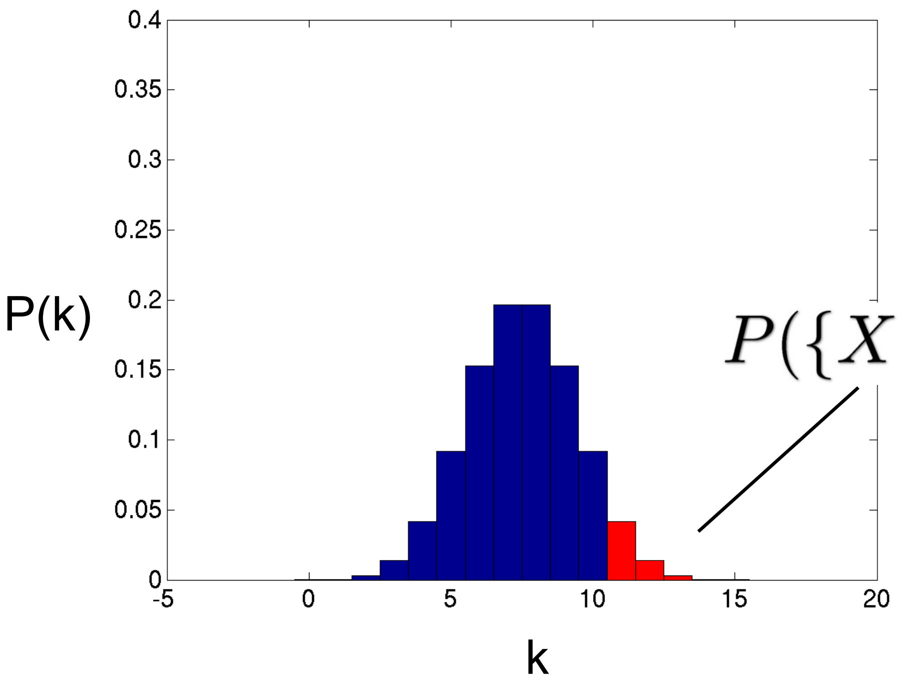
- The **standard deviation** is

$$std(x) = \sigma_x = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \overline{x})^2}$$

- The **median** of a set of observations is the value "in the middle"

Statistical Tests (in a nutshell)

- Null-Hypothesis (e.g. p= 0.5)
- Test statistic X (e.g. number of wins)
- Probability distribution of X
- Find the location of the tail with α probability – rejection boundary



$$P(\{X \geq 11\}) \approx \alpha$$

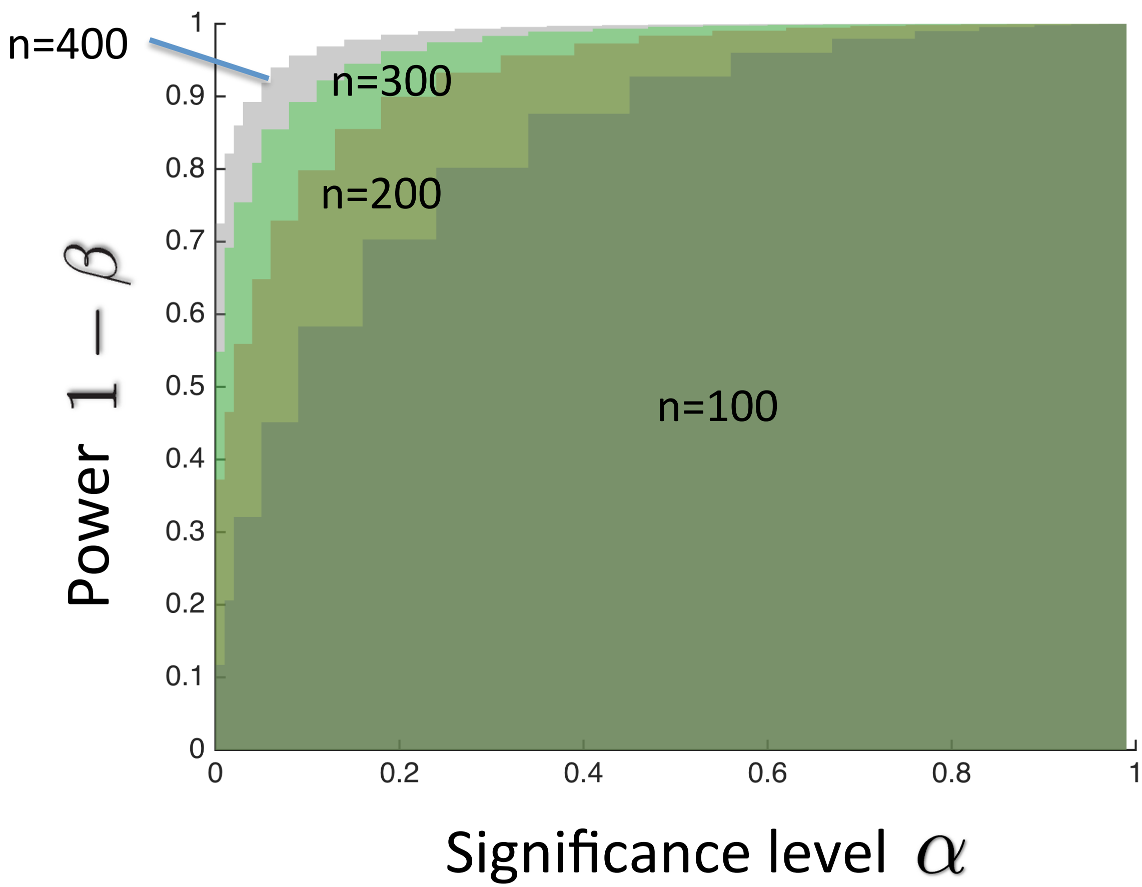
Reject H_0 if observation is greater than 10

Possible errors

1. Errors of the first kind: The null-Hypothesis was true but we reject it accidentally. The probability for this type of error is limited by the significance level α
2. Errors of the second kind: The null-Hypothesis is false but we cannot reject it. The probability of this type of error is not controlled in basic test design and may be large.
   This is one of the reasons why not rejecting the null-Hypothesis should not be interpreted as accepting it.
   This error is sometimes denoted

Area under the curve (AUC)

- The area under the α - 1- β curve can be used as an indicator of the quality of a test:



Central limit Theorem

- For independent, identically distributed (i.i.d) random variables X_i with expectation μ and variance σ^2 the distribution of

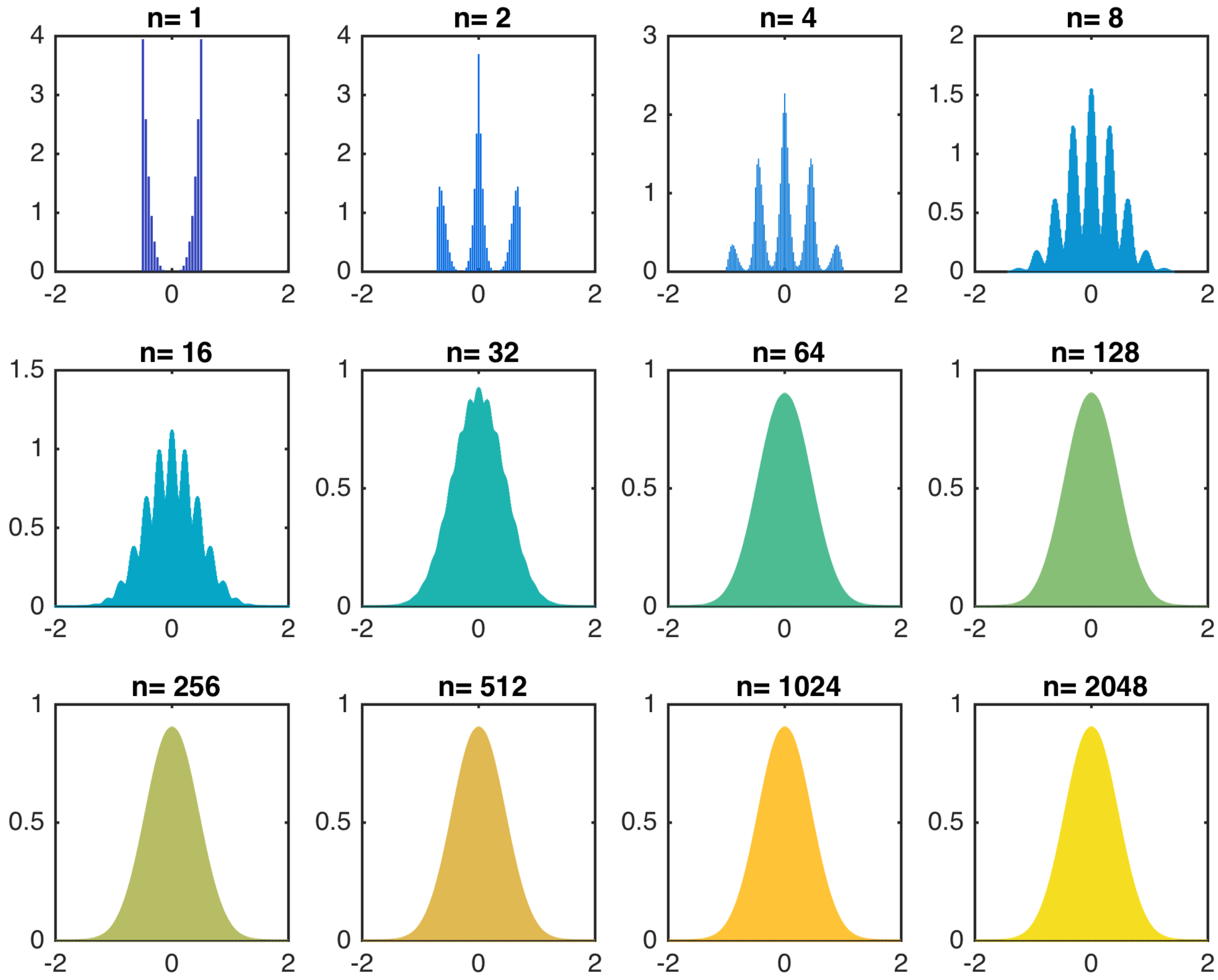
$$s_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

converges to

$$\mathcal{N}(\mu, \sigma^2/n) = \frac{1}{\sqrt{2\pi}\sigma/\sqrt{n}} \exp\Big(- \frac{(x-\mu)^2}{2\sigma^2/n} \Big)$$

(independent of the probability distribution of X_i)

Example 2



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THE END