Principle of Virtual Work

Degrees of Freedom

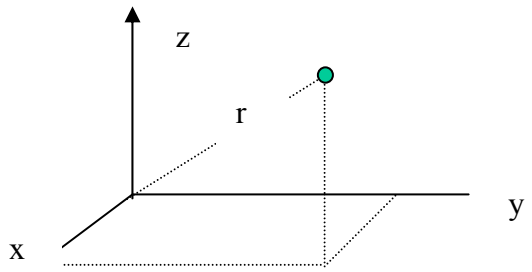
Associated with the concept of the lumped-mass approximation is the idea of the NUMBER OF DEGREES OF FREEDOM.

This can be defined as
“the number of independent co-ordinates required to specify the configuration of the system”.

The word “independent” here implies that there is no fixed relationship between the co-ordinates, arising from geometric constraints.

Degrees of Freedom of Special Systems

A particle in free motion in space has 3 degrees of freedom



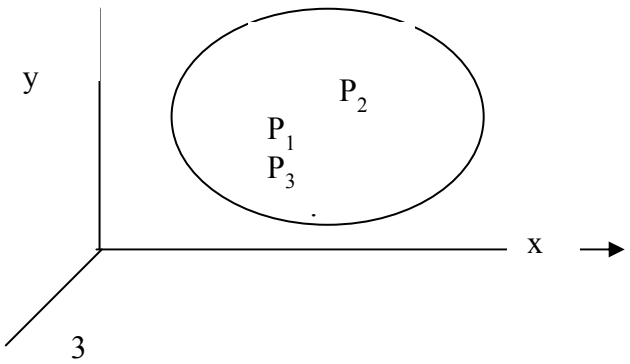
particle in free motion in space
has 3 degrees of freedom

3

If we introduce one constraint – e.g. r is fixed then the number of degrees of freedom reduces to 2.
note generally:

no. of degrees of freedom = no. of co-ordinates –no. of equations of constraint

Rigid Body



This has 6 degrees of freedom
3 translation
3 rotation

e.g. for partials P1, P2 and P3 we have 3 x 3 = 9 co-ordinates but the distances between these particles are fixed – for a rigid body – thus there are 3 equations of constraint.

The no. of degrees of freedom = no. of co-ordinates (9) - no. of equations of constraint (3)
= 6.

Formulation of the Equations of Motion

Two basic approaches:

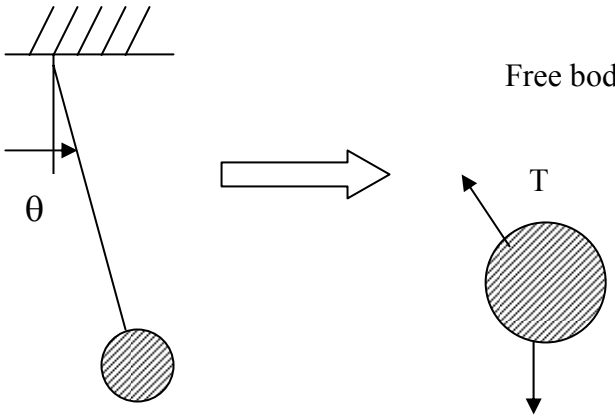
1. application of Newton's laws of motion to free-body diagrams

Disadvantages of Newton's law approach are that we need to deal with vector quantities – force and displacement.

thus we need to resolve in two or three dimensions – choice of method of resolution needs to be made. Also need to introduce all internal forces on free-body diagrams – these usually disappear when the final equation of motion is found.

2. use of work

with work based approach we deal with scalar quantities – e.g. work – we can develop a routine method – no need to take arbitrary decisions.

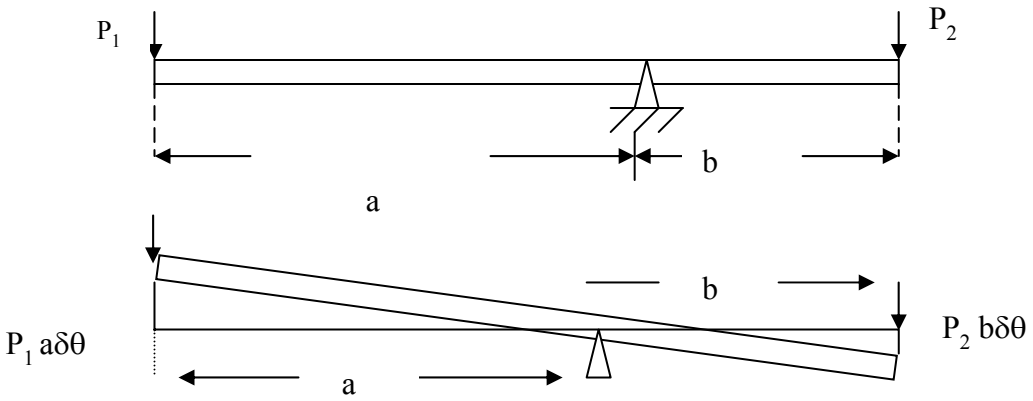


Free body diagram

Principle of Virtual Work

The work done by all the forces acting on a system, during a small virtual displacement is ZERO.

Definition A virtual displacement is a small displacement of the system which is compatible with the geometric constraints.



e.g.

This is a one-degree of freedom system, only possible movement is a rotation.

work done by P_1 = $P_1(- a\delta\theta)$
work done by P_2 = $P_2(b\delta\theta)$

Total work done = $P_1(- a\delta\theta) + P_2(b\delta\theta) = \delta W$

By principle of Virtual Work

$$\delta W = 0$$

therefore:

$P_1 \ (- a\delta\theta) + P_2(b\delta\theta) = 0$

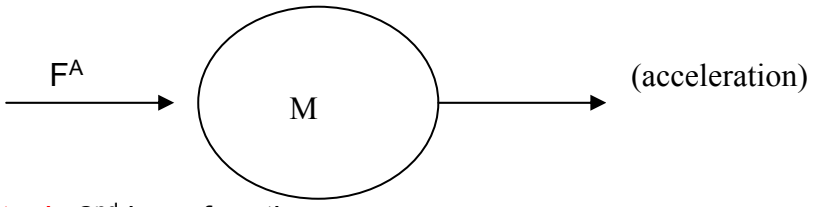
$$- a\,P1 + b\,P2 = 0$$

$$P_1 a = P_2 b$$

D'Alembert's Principle

Consider a rigid mass, M, with force F^A applied

$$A = \ddot{x}$$



From Newton's 2nd law of motion

$$F^A = Ma = M\ddot{x}$$

or

$$F^A - M\ddot{x} = 0$$

Now, the term ($-M\ddot{x}$) can be regarded as a force – we call it an inertial force, and denote it F^I – thus

we can then write:

$$F^I = -M\ddot{x}$$

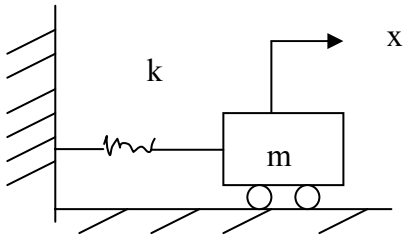
$F^A + F^I = 0$

In words – the sum of all forces acting on a body (including the inertial force) is zero – this is a statics principle.
In fact all statics principles apply if we include inertial forces, including the Principle of Virtual Work.

Virtual Work and Displacements

Using the concept of virtual displacements, and virtual work, we can derive the equations of motion of lumped parameter systems.

Example 1
Mass/Spring System



Here number of degrees of freedom =1
Co-ordinate to describe the motion is x
Now consider free-body diagram, at some time t

$$Total_force: -m\ddot{x} - kx$$

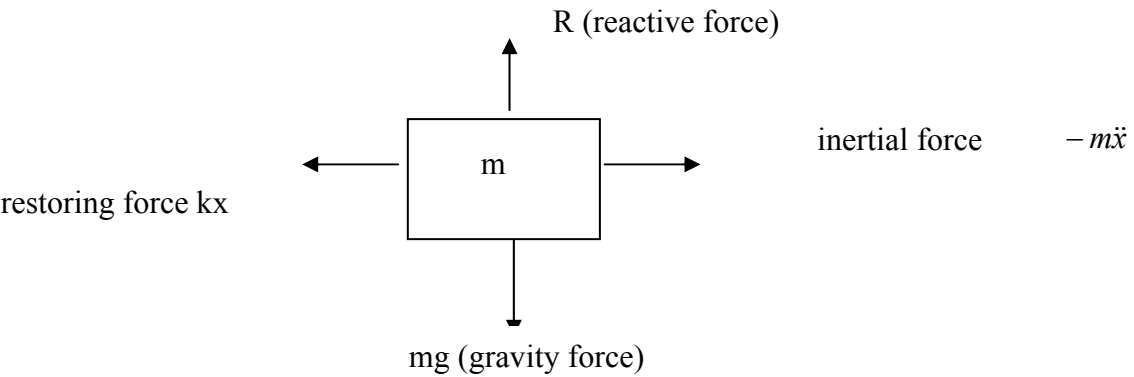
$$\delta W = \left(-m\ddot{x} - kx\right)\delta x = 0$$

Hence

$$-m\ddot{x} - kx = 0$$

or

$$m\ddot{x} + kx = 0$$



General one degree of freedom system

If q_1 is the co-ordinate used to describe the movement then the general form of δW is as follows:

$$\delta \mathbf{W} = \mathbf{Q}_1 \delta_{\mathbf{q}1}$$

we call δ_{q1} – generalised displacement

Q_1 – generalised force.

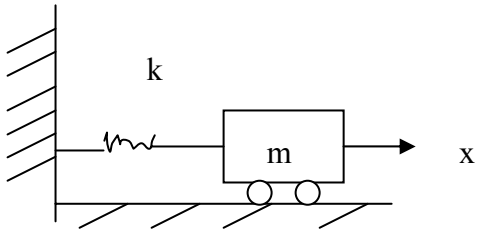
From principle of virtual work $\delta \mathbf{W} = \mathbf{Q}_1 \delta_{\mathbf{q}1} = 0$

$$\therefore \mathbf{Q}_1 = 0$$

Example

Referring to the mass/spring system again

$x = q_1$



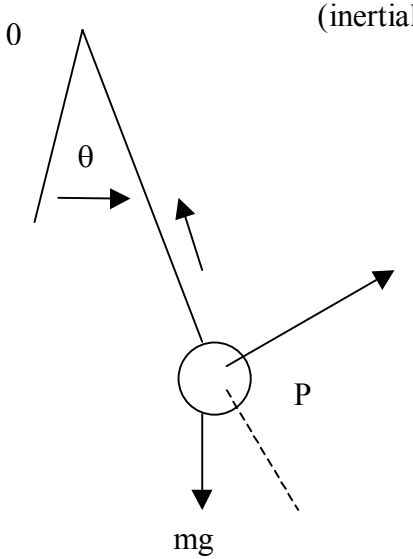
$$\delta W = \left(-m\ddot{q}_1 - kq_1\right)\delta q_1 = 0$$

$$\therefore Q_1 = -m\ddot{q}_1 - kq_1 = 0$$

$$m\ddot{q}_1 + kq_1 = 0$$

Example 2

Simple pendulum



$-ml\dot{\theta}^2$

(inertial force –radial)

$-ml\ddot{\theta}$ (inertial force – tangential)

This is another one degree of freedom system.
During a virtual displacement, $\delta\theta$, the virtual work done is

$$\delta W = (-\mathbf{ml}\ddot{\theta} - \mathbf{mg}\sin\theta)\,\mathbf{l}\delta\theta = 0$$

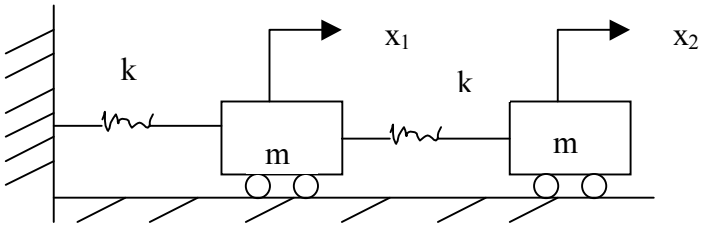
δW = 0 (PVW)

$$\therefore \mathbf{ml}\ddot{\theta} + \mathbf{mg}\sin\theta = 0$$

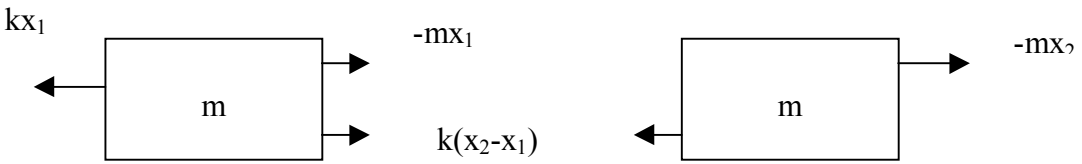
or

$$\ddot{\theta} + \frac{\mathbf{g}}{\mathbf{l}}\sin\theta = 0$$

Example 3 – two degrees of freedom system



free-body diagrams.



For LH mass:

$$[-kx_1 - m\ddot{x}_1 + k(x_2 - x_1)]\delta x_1 = \delta W_1$$
$$[-k(x_2 - x_1) - m\ddot{x}_2]\delta x_2 = \delta W_2$$
$$\delta W = \delta W_1 + \delta W_2 = (Q_1)\delta x_1 + (Q_2)\delta x_2$$

For $\delta W = 0$ for all $\delta x_1, \delta x_2$ the Q_i quantities must be zero.

Hence:

$$m\ddot{x}_1 + kx_1 - k(x_2 - x_1) = 0$$

$$m\ddot{x}_2 + k(x_2 - x_1) = 0$$

or

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

n degree of freedom systems

Having discussed single and two degree of freedom systems, and introduced the concept of generalised forces we can now consider the general case of an n degree of freedom system.

A virtual displacement must be consistent with the constraints on the system. The motion can be described by n independent, generalised co-ordinates, $q_1, q_2, \ldots, q_n$.
Hence a virtual displacement can be represented by small changes in these co-ordinates:-

$$\delta_{q1}, \delta_{q2}, \ldots, \delta_{qn}$$

Suppose only one co-ordinate, $q_i \left(1 \le i \le n\right)$ is given a small, imaginary displacement, δ_{qi}. As a result every particle in the system will be, in general, displaced a certain amount. The virtual work done will be of the form $\delta W = Q_i \delta_{qi}$ where Q_i is an expression relating directly to the forces acting on the system. Q_i is the generalised force associated with q_i.

From the principle of virtual work

$$\delta W = Q_i \delta_{qi} = 0$$

Since, δ_{qi} is finite, we get

$$Q_i = 0$$

This must be true for $i = 1,2,...,n.$

I.e.

$$\left.\begin{aligned} Q_1 &= 0 \\ Q_2 &= 0 \\ &\vdots \\ Q_n &= 0 \end{aligned}\right\}$$

these are the equations of motion of the system

The generalised forces have component parts

1) inertial forces (mass x acceleration)
2) elastic or restraining forces
3) damping forces (energy dissipation)
4) external forces
5) constraint forces

$$\delta W = Q_i \delta_{qi} = \delta W^I + \delta W^E + \delta W^D + \delta W^A$$

inertial elastic damping external

$$= \left(Q_i^I + Q_i^E + Q_i^D + Q_i^A \right) \delta_{qi}$$

(Noting, as before, that the constraint forces do no virtual work)

Then the equations of motion are:

These are the n equations of motion.

We will examine each of these components now, in more detail. The aim is to relate these component forces to the generalised co-ordinates $q_1, q_2, \ldots, q_n$.

Inertial Forces (See also Handout)

The position of the ith particle of mass, in the system, is, in general, related to the n generalised co-ordinates, and time (if the constraints are independent of time) then the position of the ith particle depends only on the n generalised co-ordinates. Thus

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_n, t) \qquad (1)$$

Now we suppose that the system is in motion and that we represent the inertial force on the ith particle (using D'Alembert's Principle) as

$$-m_i \ddot{\vec{r}}_i \qquad (2)$$

We now give the system an arbitrary virtual displacement – this can be represented in terms of generalised co-ordinates by $\delta q_1, \delta q_2, \ldots, \delta q_n$. The virtual displacement of the I[th]particle can be represented by

$$\delta \vec{r} \qquad (3)$$

and the virtual work done by the inertia force on the Ith particle is simply

$$(-m_i \ddot{\vec{r}}_i).\ \delta \vec{r}_i \qquad (4)$$

(note that this is a scalar product). From this result we get the total virtual work as

$$\delta W^I = \sum_i (-m_i \ddot{\vec{r}}_i).\ \delta \vec{r}_i \qquad (5)$$

Using equation (1) we have

$$\delta\vec{r}_i = \sum_{i=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j \qquad (6)$$

Hence

$$\delta W^I = \sum_i (-m_i \ddot{\vec{r}}_i) \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j \qquad (7)$$

and re-arranging

$$\delta W^I = \sum_{j=1}^{n} \delta q_j \left\{ \sum_i \left(- m_i \ddot{\vec{r}}_i \right) \cdot \frac{\partial r_i}{\partial r_j} \right\} \qquad (8)$$

However, the generalised inertial forces, Q_j, are effectively defined by

$$\delta W^I = \sum_i (-m_i \ddot{\vec{r}}_i) \cdot \sum_{j=1}^{n} Q_j^I \delta q_j \qquad (9)$$

Comparing (8) and (9) we have

$$Q_j^I = \sum_i (-m_i \ddot{\vec{r}}_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j} \qquad (10)$$

It is shown in the handout notes that

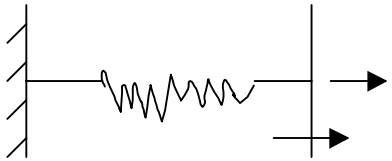
$$Q_j^I = -\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial T}{\partial q_j} \qquad (11)$$

where T is the total KE

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \qquad (12)$$

Elastic Forces

Consider a simple spring:

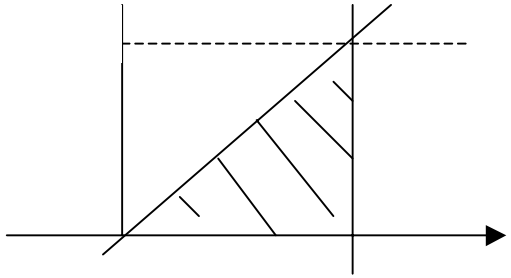


For static equilibrium

$$F_E = F_S$$

(external force) = (internal spring force)

Suppose we define the POTENTIAL ENERGY, V, as the work done by the external force to extend the spring a distance χ



external work done = $\frac{1}{2}kx^2 = V$

$\therefore\ V = f(x) = V(x) -$ a function of x

Here $V = \frac{1}{2}kx^2$

The work done by the internal spring force, W, is equal and opposite to V

$$W = -V = -\frac{1}{2}kx^2$$

Now consider a small, virtual displacement, δx . Corresponding changes to W and V are as follows:

$$\delta V = \frac{\delta V}{\delta x} \cdot \quad \delta x = -\delta W$$

Comparing with standard form

$$\delta W = Q_E^1 \delta q_1 \qquad (q_1 = x, here)$$

$$then \qquad Q_1^E = -\frac{\partial V}{\partial q_1} = -\frac{\partial V}{\partial x \cdot}$$

Generally,

$$V = V(q_1, q_2, ..., q_n)$$

$$\delta V = -\delta W = \sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j$$

Compare with

$$\delta W = \sum Q_j^E \delta q_j$$

$$\therefore \qquad Q_j^E = -\frac{\partial V}{\partial q_j}$$

Lagrange's equation

Suppose that no damping forces are present, and there are no externally applied forces. Then

$$Q_i^D = Q_i^A = 0 \qquad (i = 1,2,\ldots,n)$$

We have found that

$$\left.\begin{aligned} Q_i^I &= -\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i} \\ Q_i^E &= -\frac{\partial V}{\partial q_i} \end{aligned}\right\} \quad (i = 1,2,\ldots,n)$$

If we collect these results we get

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \qquad (i = 1,2,\ldots,n)$$

This is LAGRANGE's EQUATION

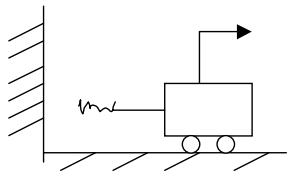
If we define

L=T-V

And assume that V does not depend on the $\dot{q}_i$'s, then Lagrange's equation can be written as:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0$$

Example 1- mass/spring system



This is a single degree of freedom system. Here

$$q_1 = x$$

and

$$T = \frac{1}{2} m \dot{q}_1^2$$

$$V = \frac{1}{2} k q_1^2$$

The Lagrange equation is (n=1 so only one equation)

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_1}\right)-\frac{\partial T}{\partial q_1}+\frac{\partial V}{\partial q_1}=0$$

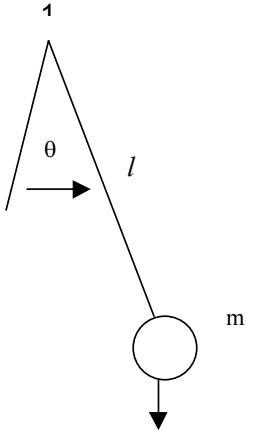
Here $$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_1}\right)=\frac{\partial}{\partial t}\left(M\dot{q}_1\right)=M\ddot{q}_1$$

$$\frac{\partial T}{\partial q_1}=0$$

$$\frac{\partial V}{\partial q_1}=kq_1$$

Hence

Example 2-simple pendulum



Here $q_1 = \theta$

In terms of q_1 we have

T= $\frac{1}{2} m \left(l \dot{q}_1\right)^2$

V= $mgl\left(1 - \cos q_1\right)$

Hence

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{\partial}{\partial t}\left(ml^2 \dot{q}_1\right) = ml^2 \ddot{q}_1$$

$$\frac{\partial T}{\partial q_1} = 0$$

$$\frac{\partial V}{\partial q_1} = mgl \sin q_1$$

$$ml^2 \ddot{q}_1 + mgl \sin q_1 = 0$$

or $$\ddot{q}_1 + \frac{g}{l} \sin q_1 = 0$$