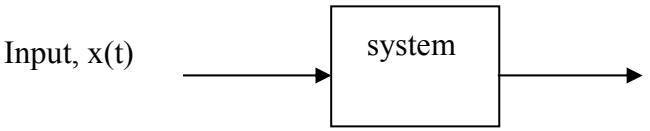
Dynamic Systems

The general problem we will consider can be represented diagrammatically as follows:



There can be more than one input and more than one output

Linear systems

Linearity:

Input	Output
$x_1(t)$	$y_1(t)$
$ax_1(t)$	$ay_1(t)$
$x_2(t)$	$y_2(t)$
$x_1(t)+x_2(t)$	$y_1(t)+y_2(t)$
$ax_1(t)+bx_2(t)$	$ay_1(t)+by_2(t)$

For LINEAR systems the applicability of the Principles of Superposition enable us to develop very general analytical methods for solving the prediction problem- e.g. Laplace transform, frequency response functions etc.

For NON-LINEAR systems no such general theoretical framework exits and we must solve the governing equations NUMERICALLY This can be done, for example, by using standard numerical techniques for solving ordinary differential equations. An alternativ approach is to use a SIMULATION SOFTWARE package.

System: an organised set of things (or ideas or theories) or a particular way of action

Dynamic: movement, change, interaction involved, compare static

Model: a description of a system to explain how it works or to calculate what might happen

Mathematical model: mathematical expression,
e.g. equation, to describe a system

Zero, first and second order dynamic models

Zero Order

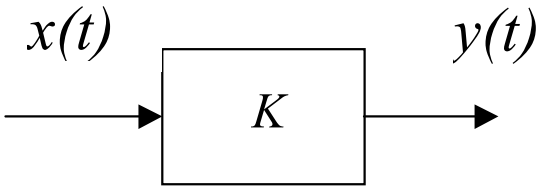
First Order

Second Order

Order - order of differential equations used to describe the system

Zero Order

$y(t) = K\,x(t)$

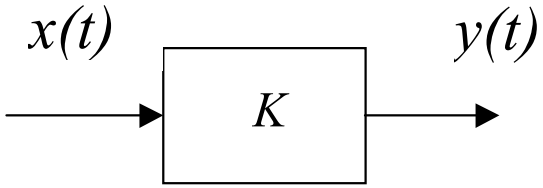


$x(t)$ - input
$y(t)$ - output

K - called gain, amplification coefficient, constant with appropriate units.

E.g. if $x(t) = \delta(t)$, $y(t)=K\,\delta(t)$
if $x(t) = u(t)$, $y(t)=K\,u(t)$
if $x(t) = A\,sin(\omega t)$, $y(t)=K\,A\,sin\,(\omega t)$

Zero Order



Real life examples:

Amplifier (ideal)	$v_{output}(t)=K\ v_{input}(t)$
Telephone (ideal)	$v_{output}(t)=K\ v_{input}(t)$
Gearboxes	$\theta_{output}(t)=K\ \theta_{input}(t)$
Steering wheel driving	$\theta_{output}(t)=K\ \theta_{input}(t)$

First Order Systems

$$a_1\ \dot{y}(t) + a_0\ y(t) = b_0\ x(t)$$

$$a_1 \frac{dy(t)}{dt} + a_0\ y(t) = b_0\ x(t)$$

Let $\tau = \frac{a_1}{a_0}$ and $K = \frac{b_0}{a_0}$,

$$\tau \dot{y}(t) + y(t) = K\ x(t)$$

Specify the input $x(t) = Xu(t)$

$$\tau \dot{y}(t) + y(t) = K\ X\ u(t)$$

Homogeneous equation

$$\tau \dot{y}(t) + y(t) = 0$$

The solution to the H. equation is

$$-c_1 \frac{1}{\tau} e^{-t/\tau} \tau + c_1\, e^{-t/\tau} = 0 \qquad y_1(t) = c_1 e^{-t/\tau}$$

The particular solution $y_2 = c_2 u(t)$

$$\tau\ 0 + c_2\ u(t) = KXu(t) \quad \Rightarrow c_2 = KX$$

Genaral solution

$$y(t) = y_1(t) + y_2(t) = c_1 e^{-t/\tau} + KXu(t)$$

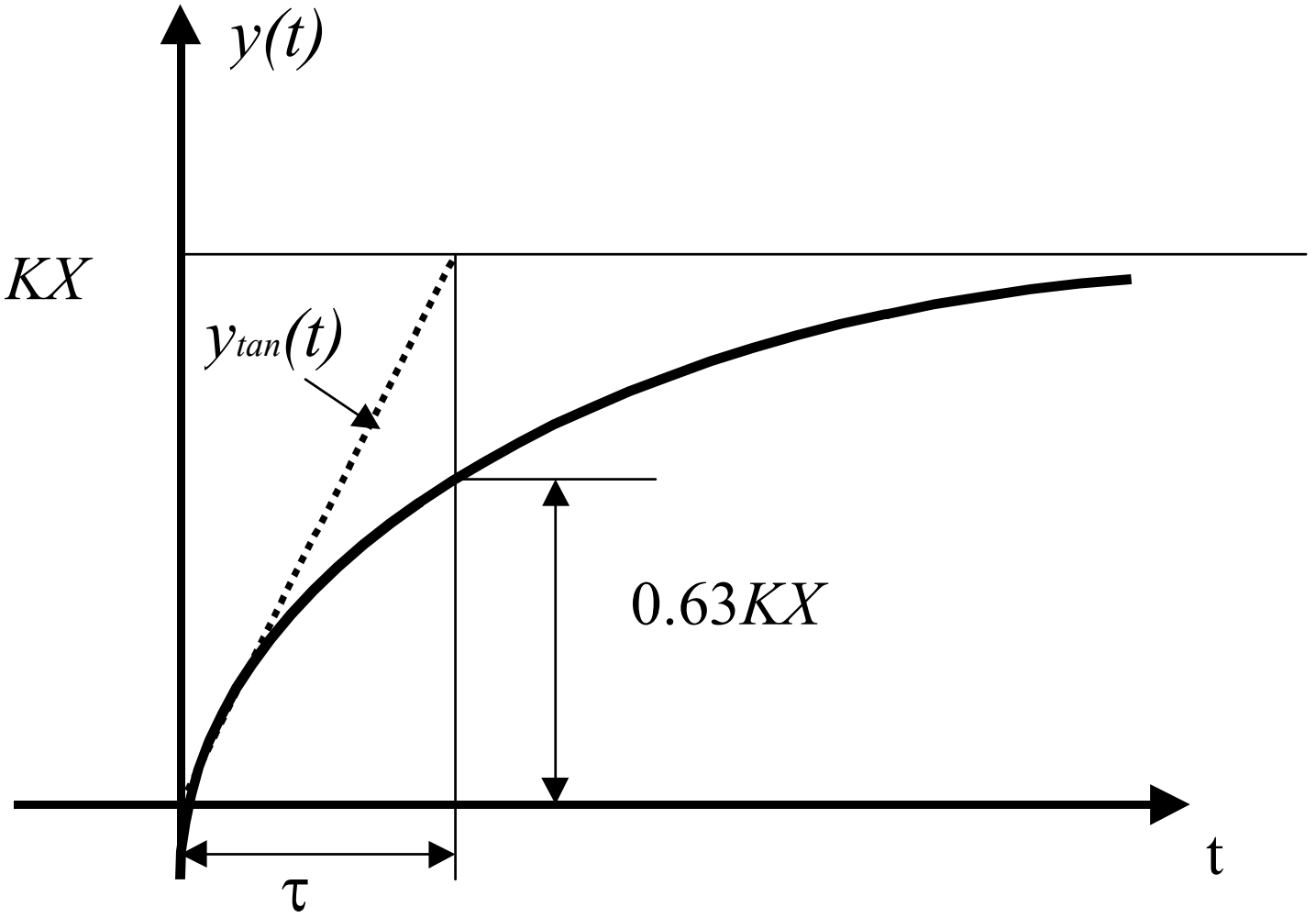
Consider initial condition $y(0)=0$

$$y(0) = c_1 + KX = 0 \quad \Rightarrow \quad c_1 = -KX$$

The solution

$$y(t) = -KXe^{-t/\tau} + KXu(t) = KX\left(1 - e^{-t/\tau}\right)$$

First Order Systems

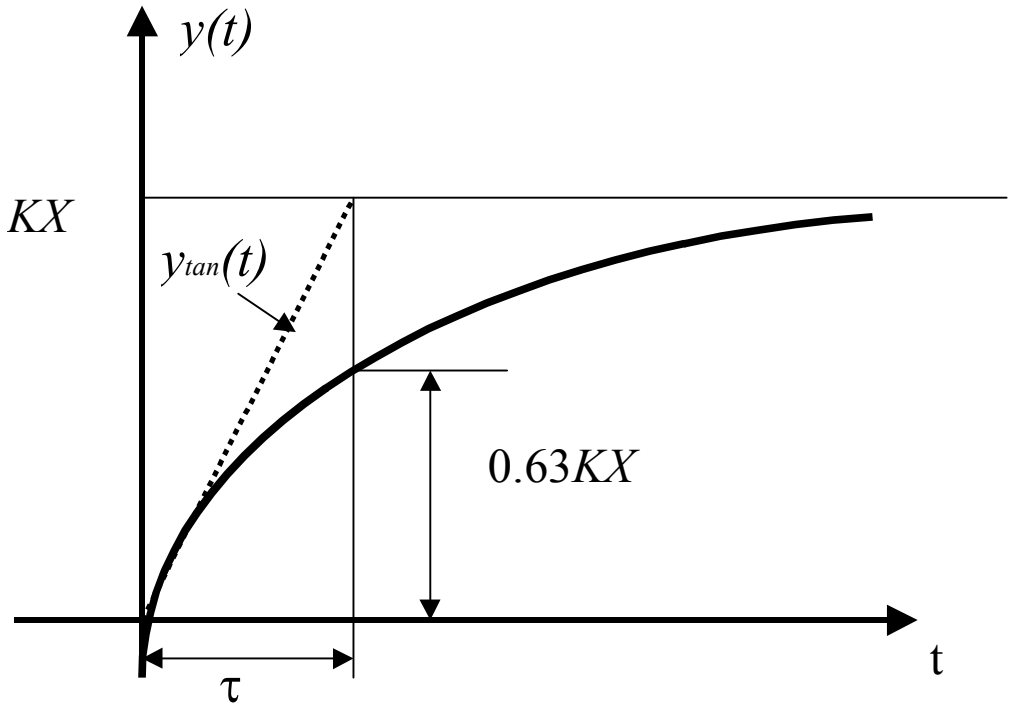


Gradiant : $\frac{dy(t)}{dt} = KX\frac{1}{\tau}e^{-t/\tau}$ $at\ \ t=0:$ $\frac{dy(0)}{dt} = KX\frac{1}{\tau}$

Tangent line through origin:

$$y_{\tan}(t) = KX\frac{t}{\tau}$$

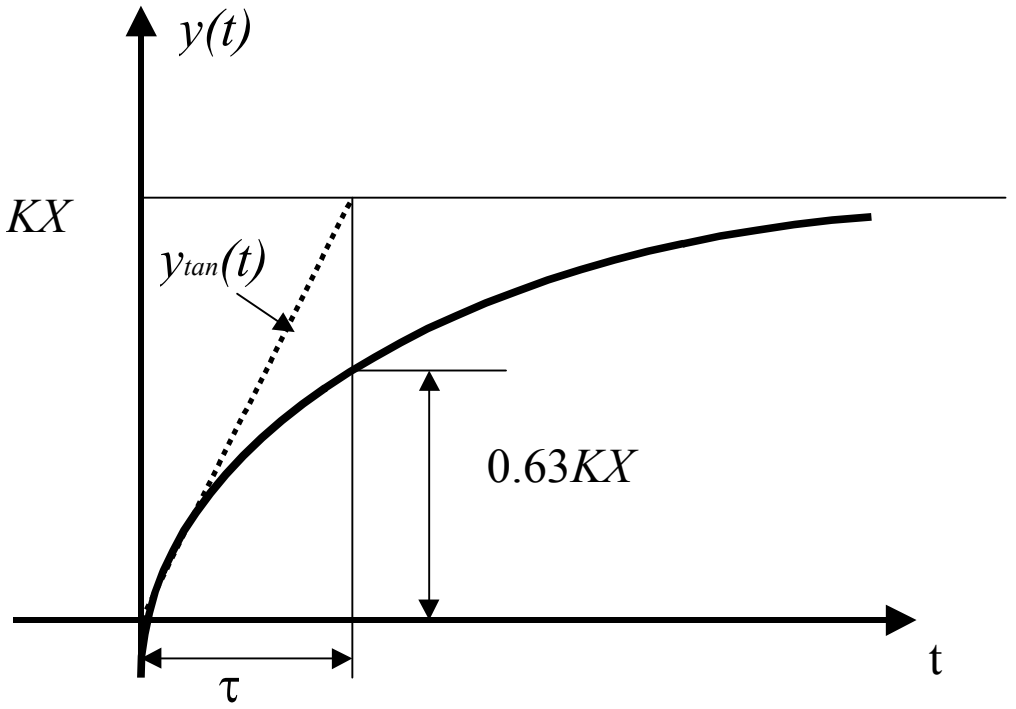
$$y_{\tan}(\tau) = KX\frac{\tau}{\tau} = KX$$



$$y(t) = KX\left(1 - e^{-t/\tau}\right)$$

At $t = \tau$

$$y(\tau) = KX\left(1 - e^{-\tau/\tau}\right) = KX(1 - e^{-1}) = KX(1 - 0.37) = 0.63KX$$



τ - time constant determine the rising rate (or decaying rate.

When t= τ, the response is to rise to 0.63 of the final value.

Decay case

$$\tau \dot{y}(t) + y(t) = K\ x(t)$$

Specify the input $x(t) = 0$

$$\tau \dot{y}(t) + y(t) = 0$$

It is already Homogeneous equation

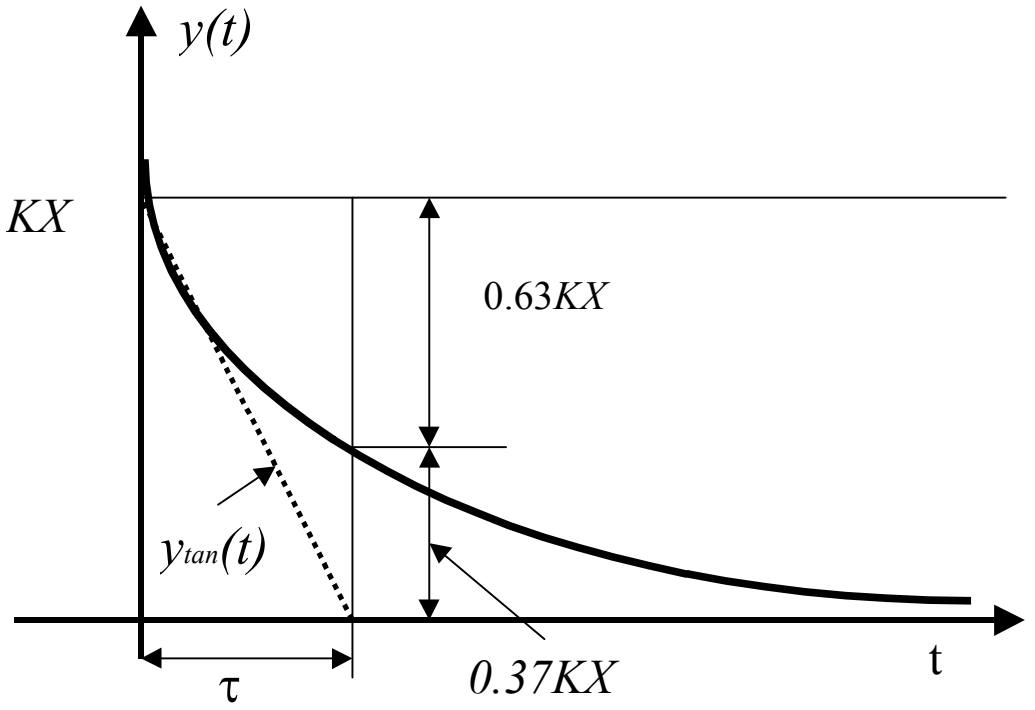
The solution to the H. equation is $y\ (t) = c\ e^{-t/\tau}$

Consider initial condition $y(0)=KX$

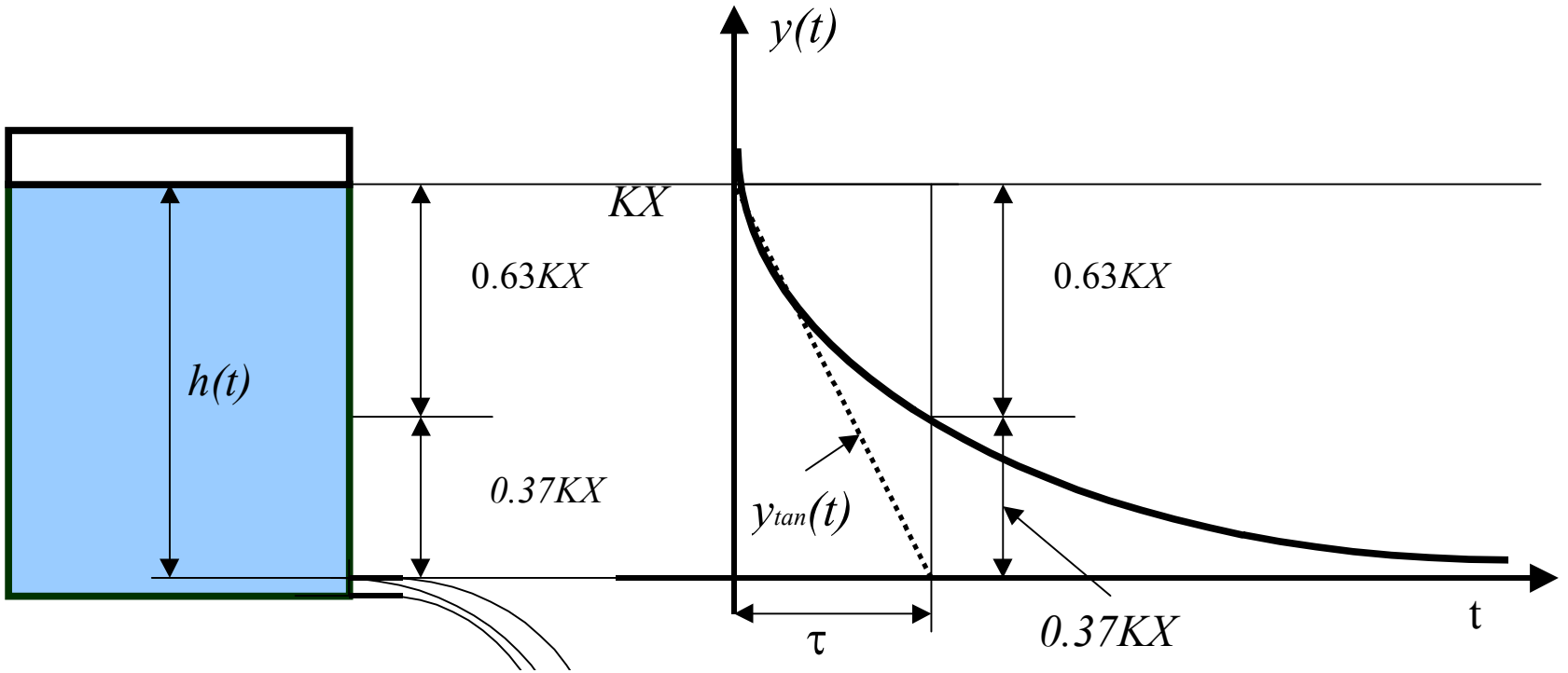
$$y(0) = \mathrm{c} = KX \qquad \Rightarrow \quad \mathrm{c} = KX$$

The solution

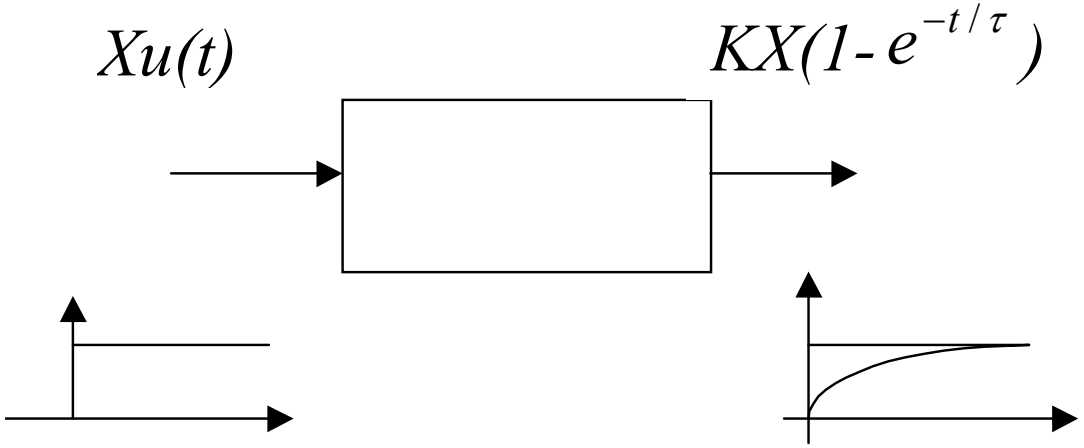
$$\mathrm{y(t)} = \mathrm{KX\,e^{-t/\tau}}$$



Water level of a storage vessel $h(t)$



$$h(t) = KX\, e^{-t/\tau}$$


Xu(t)
KX(1- e^{-t/τ})

Second Order System

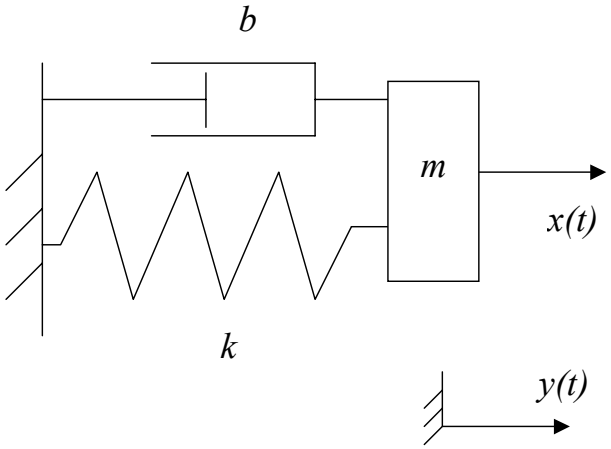
$$a_2 \frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

or

$$a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 x(t)$$

In mechanical system, using

Newton's law



$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = x(t)$$

$$\omega_0^{\,2} = \frac{k}{m},$$

$$2\zeta\omega_0 = \frac{b}{m} \qquad K = \frac{1}{m}$$

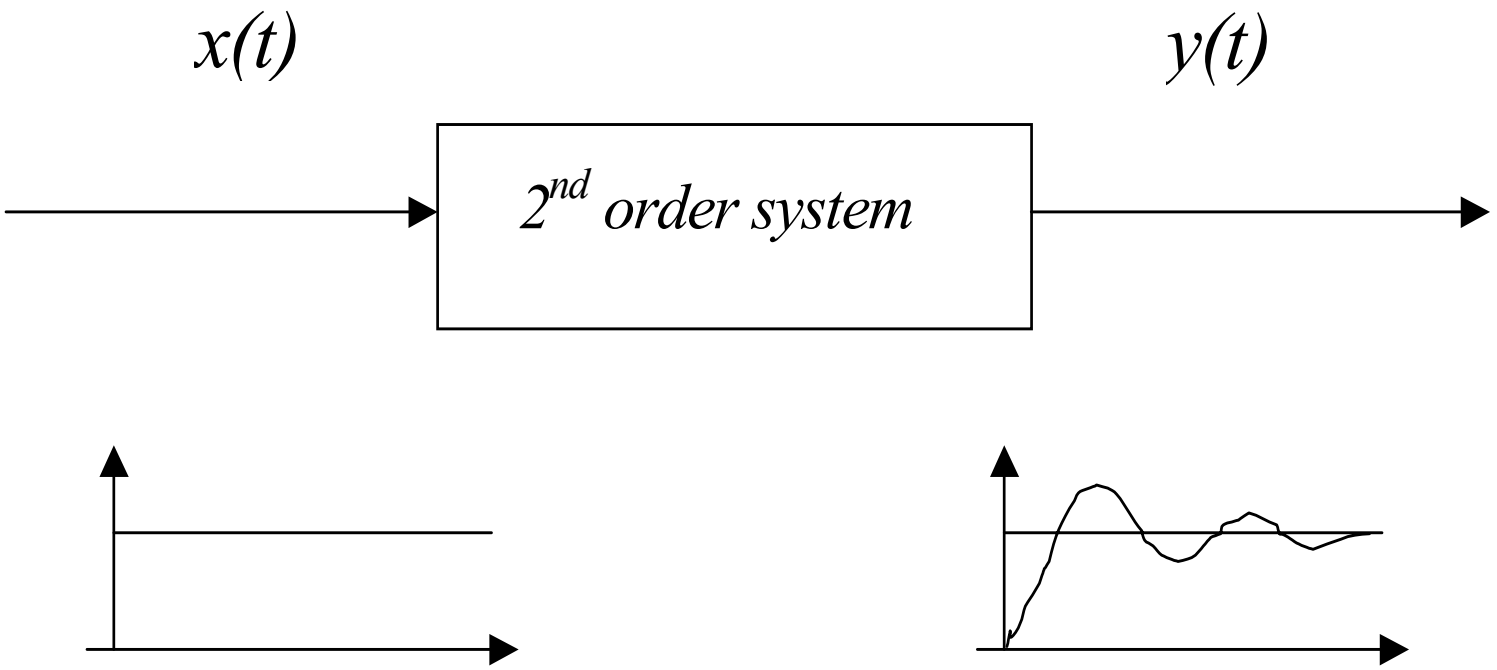
$$\zeta = \frac{b}{2m\omega_0} = \frac{b}{2\sqrt{km}}$$

$\omega_0 - natural\ frequency$

$\zeta = - damping\ ratio$

$K - constant_gain$

$$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = Kx$$


x(t)
y(t)
2nd order system

Eigen-equation (auxiliary equation)

$$r^2 + 2\zeta\omega_0 r + \omega_0^2 = 0$$

$$\mathrm{r}_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

$$General\ solution: \quad y_1(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$Particular\ solution: \quad y_2(t) = \ldots$$

$$y(t) = y_1(t) + y_2(t)$$

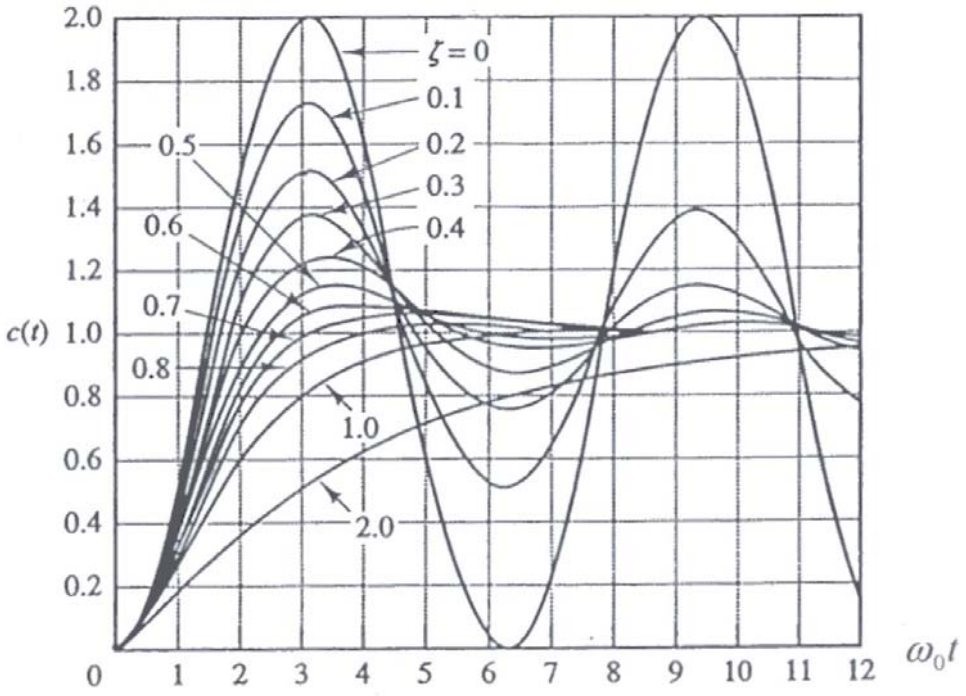
Step response

1) $\zeta = 0$ imaginary $\pm j\omega_0$ – *undamped oscillation*

2) $\zeta < 1$ complex $-\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$ – *damped oscillatio*

3) $\zeta = 1$ real $-\zeta\omega_0$ – *monotonic* rising

4) $\zeta > 1$ real $-\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$ – *monotonic* rising



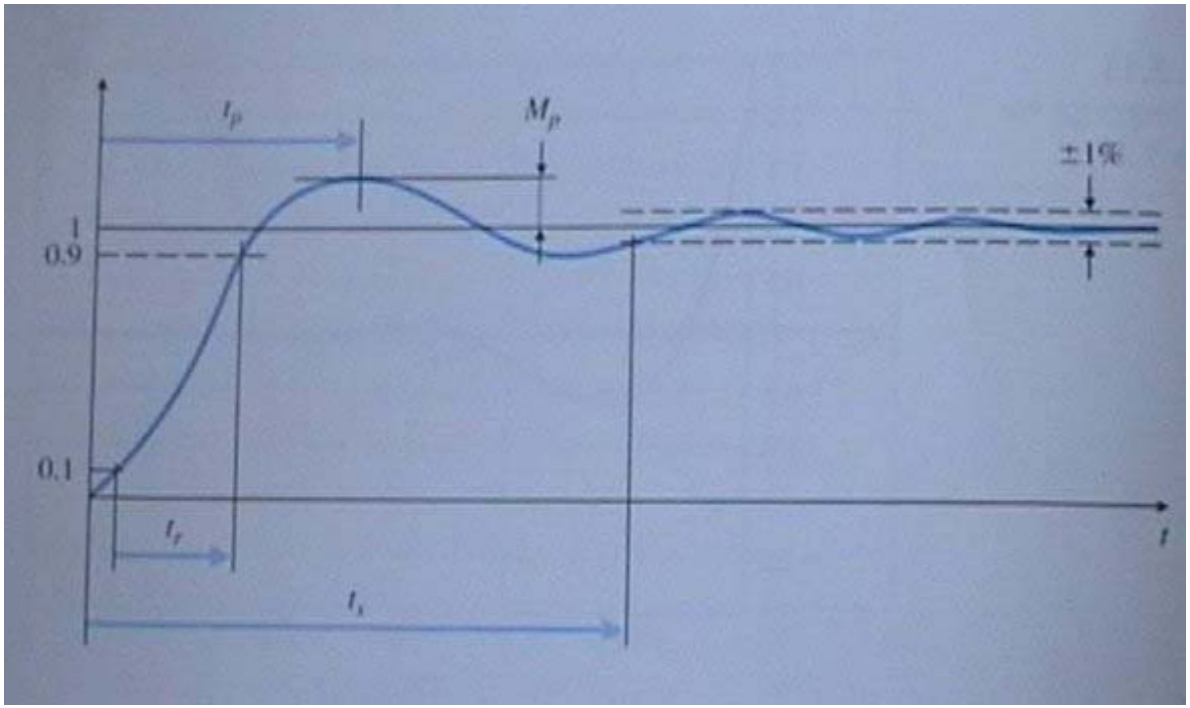
The response features:-

1) For $\zeta \geq 1$ no overshoot.

2) The fastest response with no overshoot is for $\zeta = 1$.

3) For $0 < \zeta < 1$ the response has an overshoot (M_p)

t_p - first peak time = the half cycle of damped oscillation



Overshoot

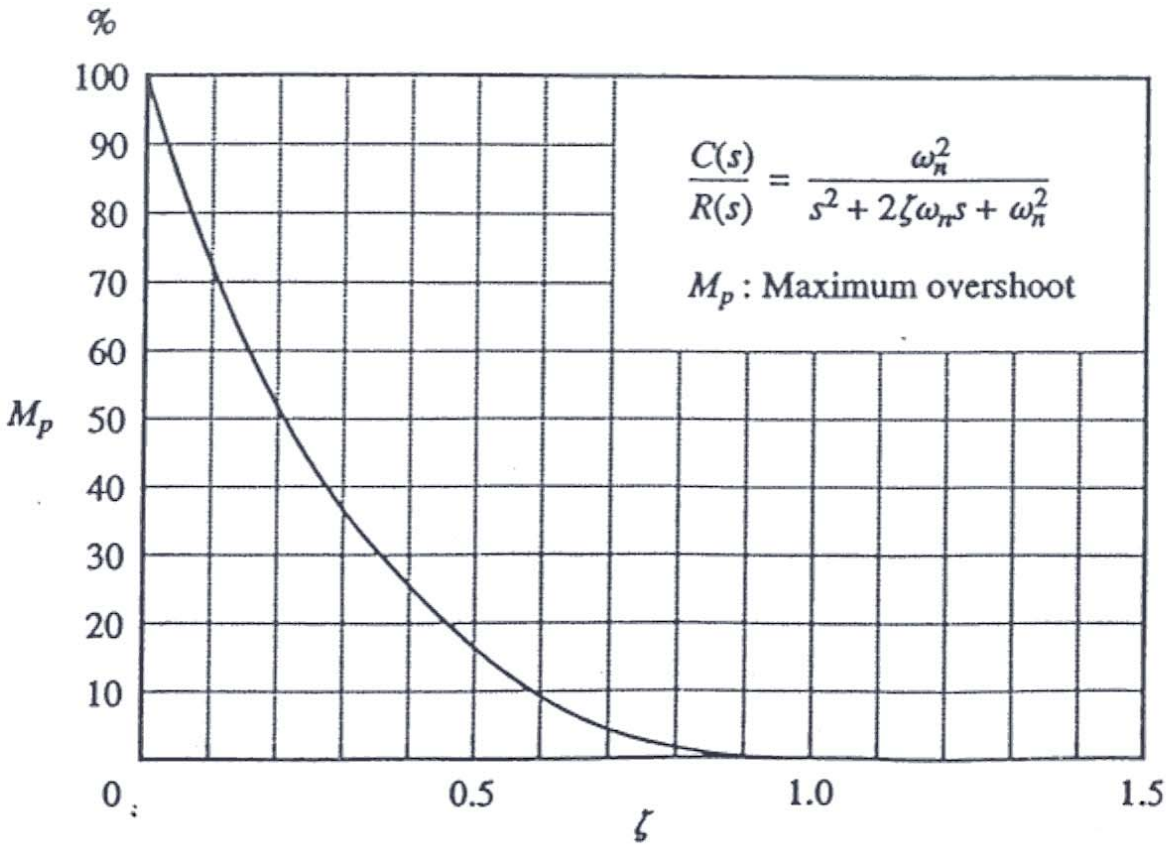
$$t_p = \frac{1}{2}\frac{2\pi}{\sqrt{1-\zeta^2}\omega_0} = \frac{\pi}{\sqrt{1-\zeta^2}\omega_0}$$

First peak at half of period

$$Overshoot\ \Delta = e^{-\zeta\omega_0 t_p} = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

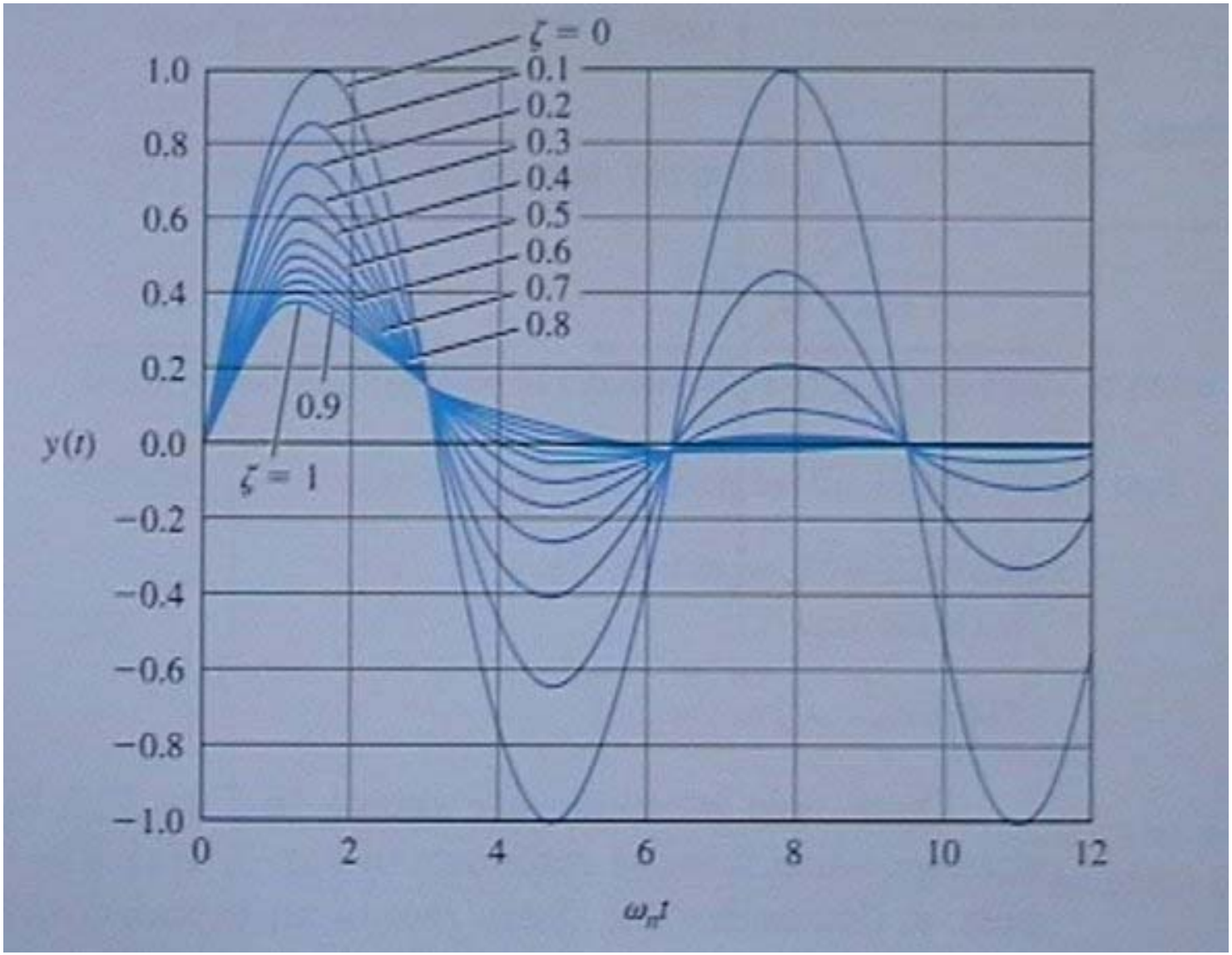
$$M_p(\%) = 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Overshoot Mp

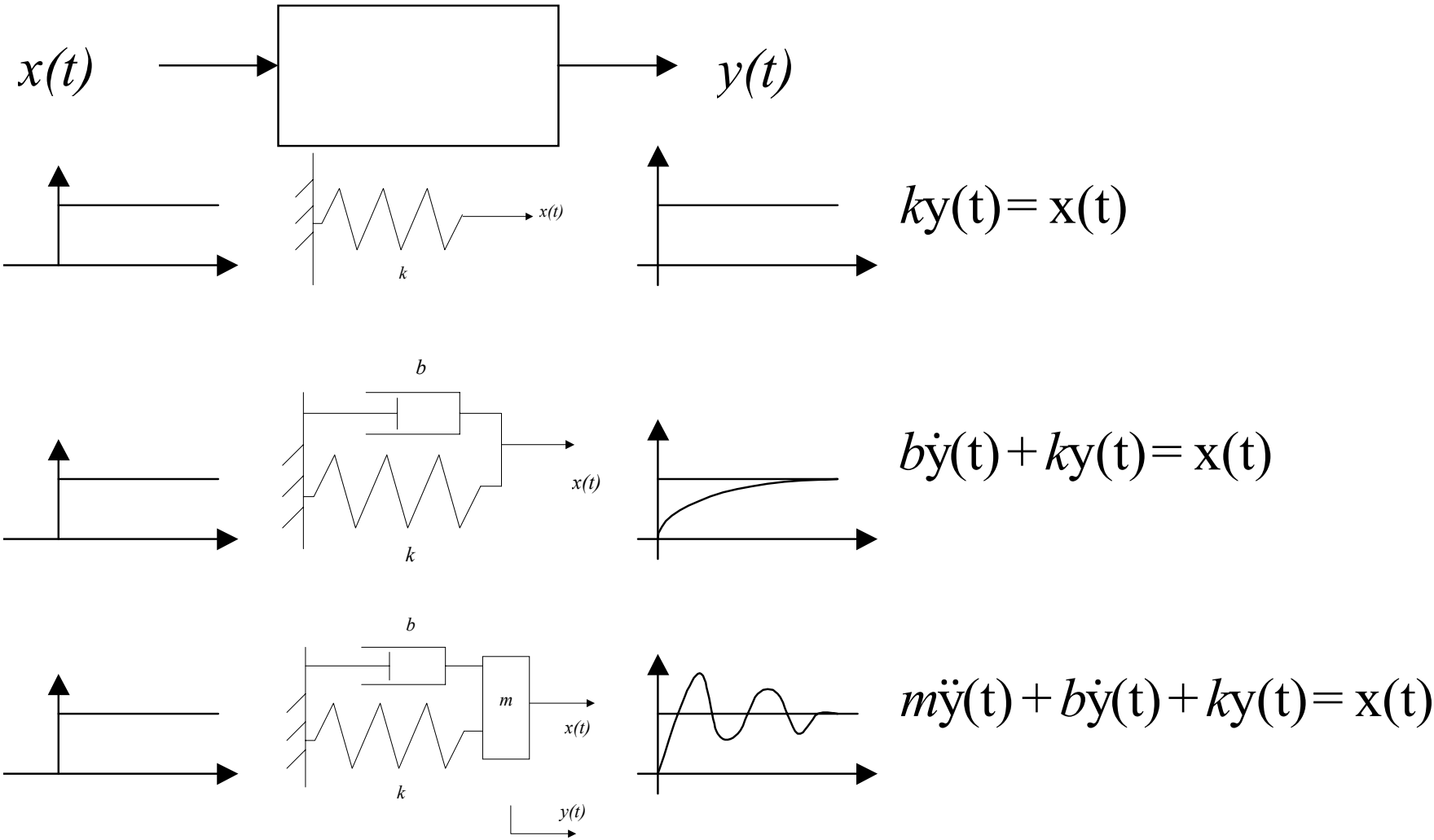


$$\omega_n = \omega_0(natural_frequency)$$

Impulse response

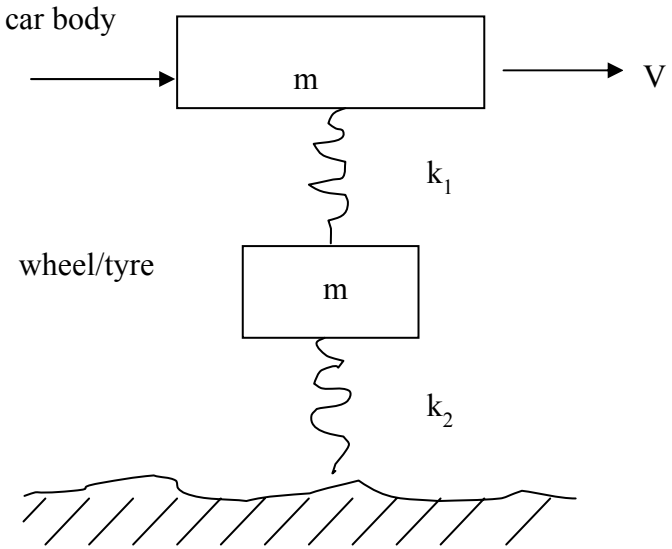


3 Types of Systems - Summery



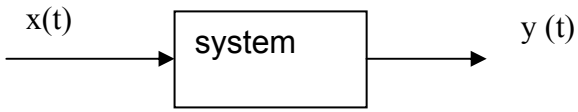
Lumped-mass model

In some cases the lumped-mass model is a natural approximation to a real system – e.g. consider a very simple model of a car suspension system:



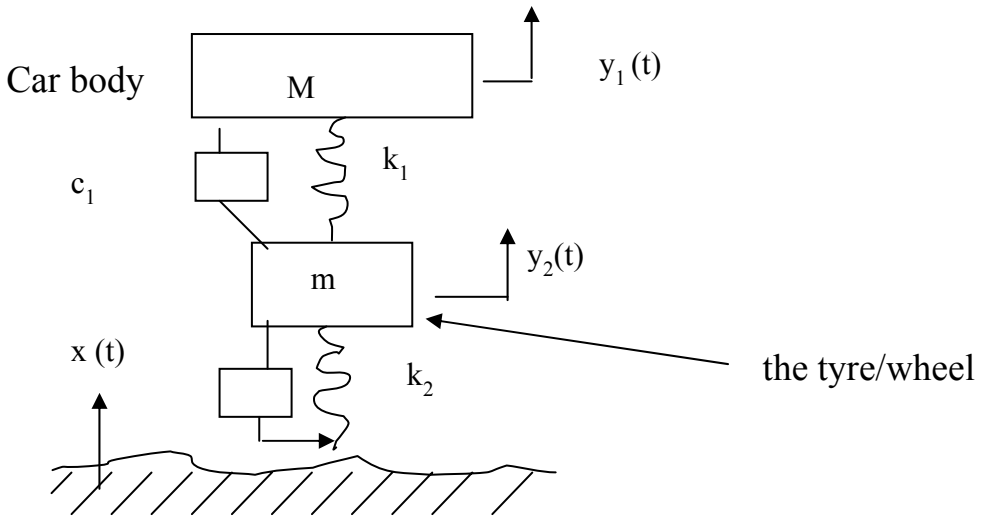
Identification of Systems

In many applications the system model may be difficult to determine by entirely theoretical means. In particular, even if the FORM of the model is known then some of the parameters may be difficult to determine theoretically.
In such situations experimental work can be carried out to IDENTIFY the system.



By substantially processing records of x(t) at the corresponding y(t) it is, in principle, possible to identify the system.
The identification can range from a completely black-box approach (i.e. we have no a-priori knowledge of the system model) to a parametric-identification-here the model structure is known and one or more parameters are identified.

As a reasonable first approximation we can regard the car body as a rigid body (i.e. no flexibility), the tyre/wheel as another and k1 , k2 are simple massless springs. We would also add idealised dampers



Here the "input is the real elevation and the output are the vertical displacements of the two masses.