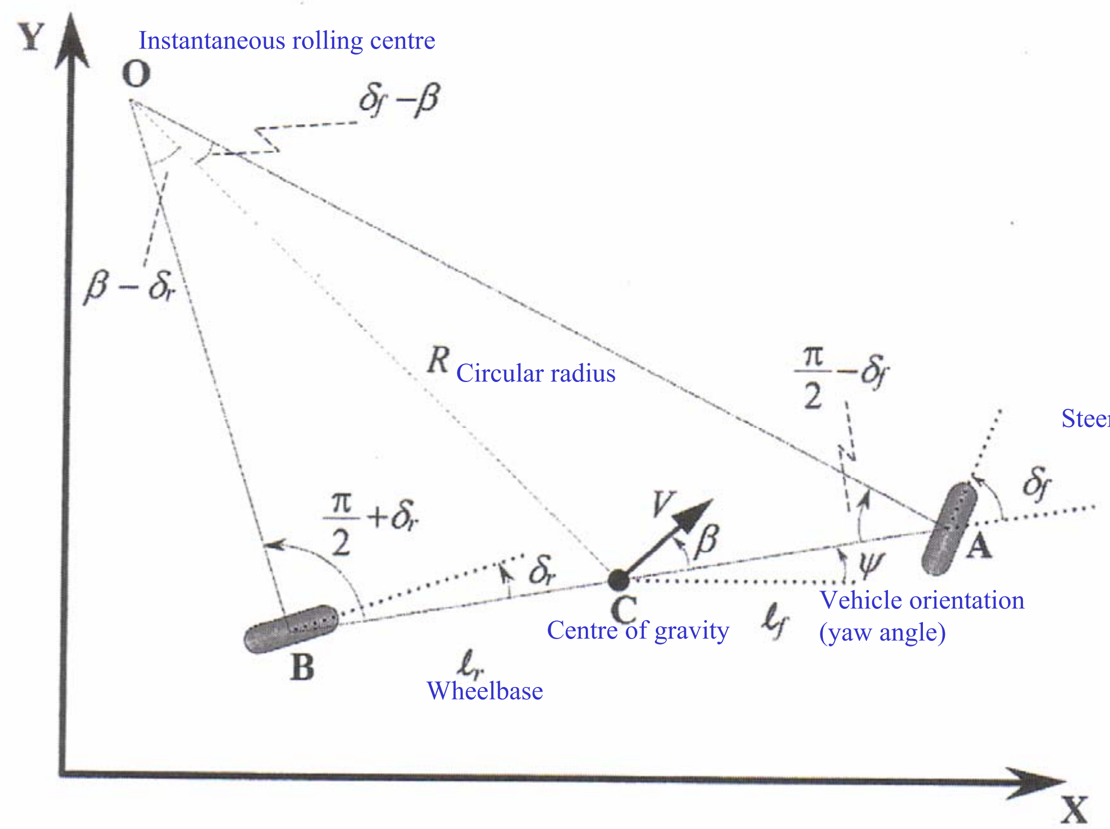
Lateral Vehicle Dynamics

Kinematic Model of Lateral Vehicle Motion: Bicycle Model



Kinematics of lateral vehicle motion

Major assumption : velocities at A and B are in the direction of wheels orientation, i.e. the wheel's slip angles are 0.

Bicycle Model

2 front wheels are represented by a single wheel at A
2 rear wheels are represented by a single wheel at B

Steering angles

δ_f

δ_r

Assumed both front and rear wheels can be steered.

For only front steering case, $\delta_r = 0$

C - centre of gravity

$L = l_f + l_r$: Wheelbase of the vehicle

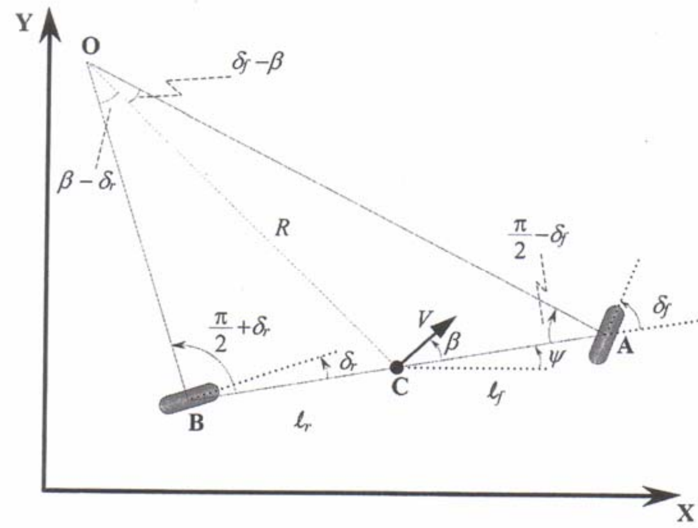
Ψ : Orientation of the vehicle - heading angle of vehicle

V : velocity of c.g.

β : Slip angle of the vehicle (angle between the motion direction and vehicle orientation)

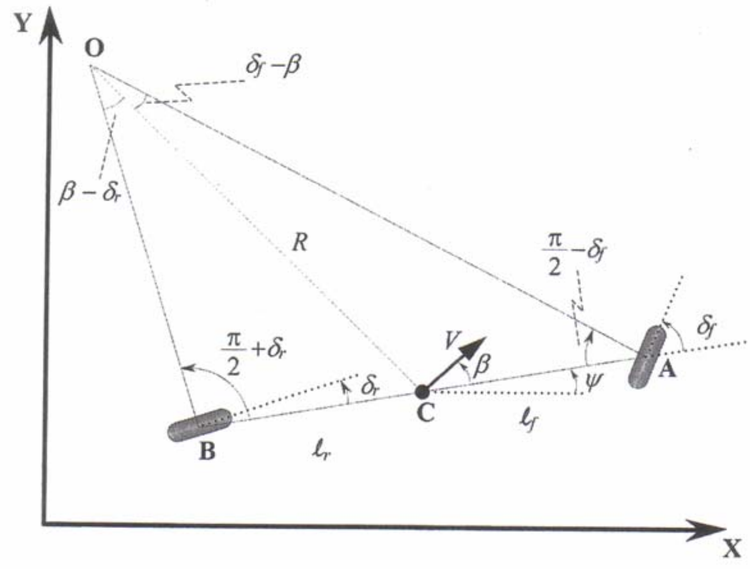
R : Circular radius

O : Instantaneous rolling centre



Kinematics of lateral vehicle motion

Bicycle Model



Kinematics of lateral vehicle motion

Course angle of the vehicle

$$\gamma = \psi + \beta$$

Triangle OCA

$$\frac{\sin(\delta_f - \beta)}{l_f} = \frac{\sin\left(\frac{\pi}{2} - \delta_f\right)}{R}$$

$$or \quad \frac{\sin\delta_f \cos\beta - \sin\beta\cos\delta_f}{l_f} = \frac{\cos\delta_f}{R} \qquad or \quad \tan\delta_f \cos\beta - \sin\beta = \frac{l_f}{R}$$

Triangle OCB

$$\frac{\sin(\beta - \delta_r)}{l_r} = \frac{\sin\left(\delta_r + \frac{\pi}{2}\right)}{R}$$

$$or \quad \frac{\sin\beta\cos\delta_r - \sin\delta_r \cos\beta}{l_r} = \frac{\cos\delta_r}{R} \qquad or \quad \sin\beta - \tan\delta_r \cos\beta = \frac{l_r}{R}$$

Add both

$$(\tan\delta_f - \tan\delta_r)\cos\beta = \frac{l_f + l_r}{R}$$

Bicycle Model

Slip angle β

From

$$\tan\delta_f \cos\beta - \sin\beta = \frac{l_f}{R} \qquad l_r \tan\delta_f \cos\beta - l_r \sin\beta = \frac{l_f l_r}{R}$$

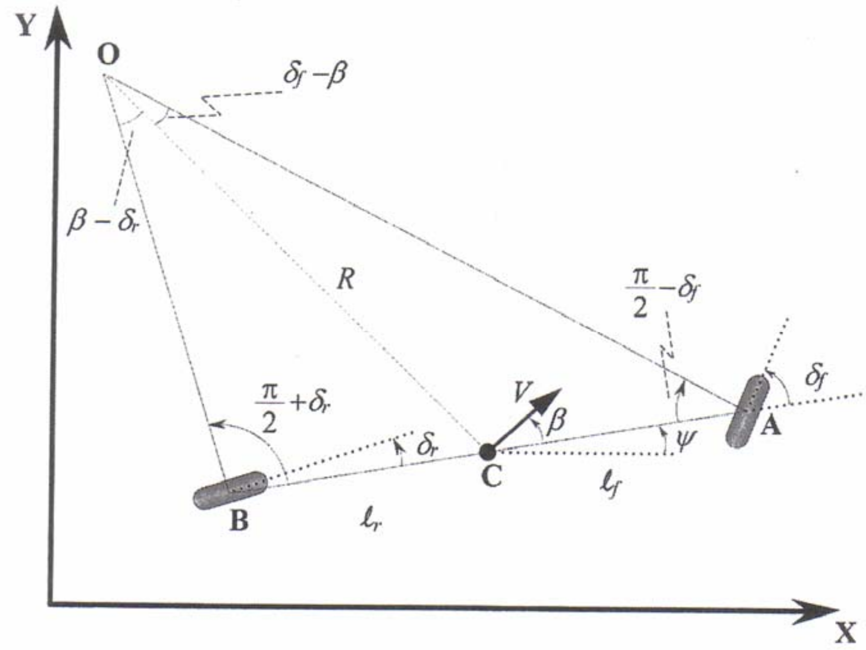
$$\sin\beta - \tan\delta_r \cos\beta = \frac{l_r}{R} \qquad l_f \sin\beta - l_f \tan\delta_r \cos\beta = \frac{l_r l_f}{R}$$

$$\left(l_f \tan\delta_r + l_r \tan\delta_f\right)\cos\beta - \left(l_f + l_f\right)\sin\beta = 0$$

$$\tan\beta = \frac{l_f \tan\delta_r + l_r \tan\delta_f}{l_f + l_f}$$

$$\beta = \tan^{-1}\left(\frac{l_f \tan\delta_r + l_r \tan\delta_f}{l_f + l_f}\right)$$

i.e. the slip angle is represented by steering angles and wheelbases.



Kinematics of lateral vehicle motion

Kinematic Model of Lateral Vehicle

$$\left(\tan\delta_f - \tan\delta_r\right)\cos\beta = \frac{l_f + l_r}{R}$$

Angular velocity

$$\dot{\psi} = \frac{V}{R}$$

Therefore

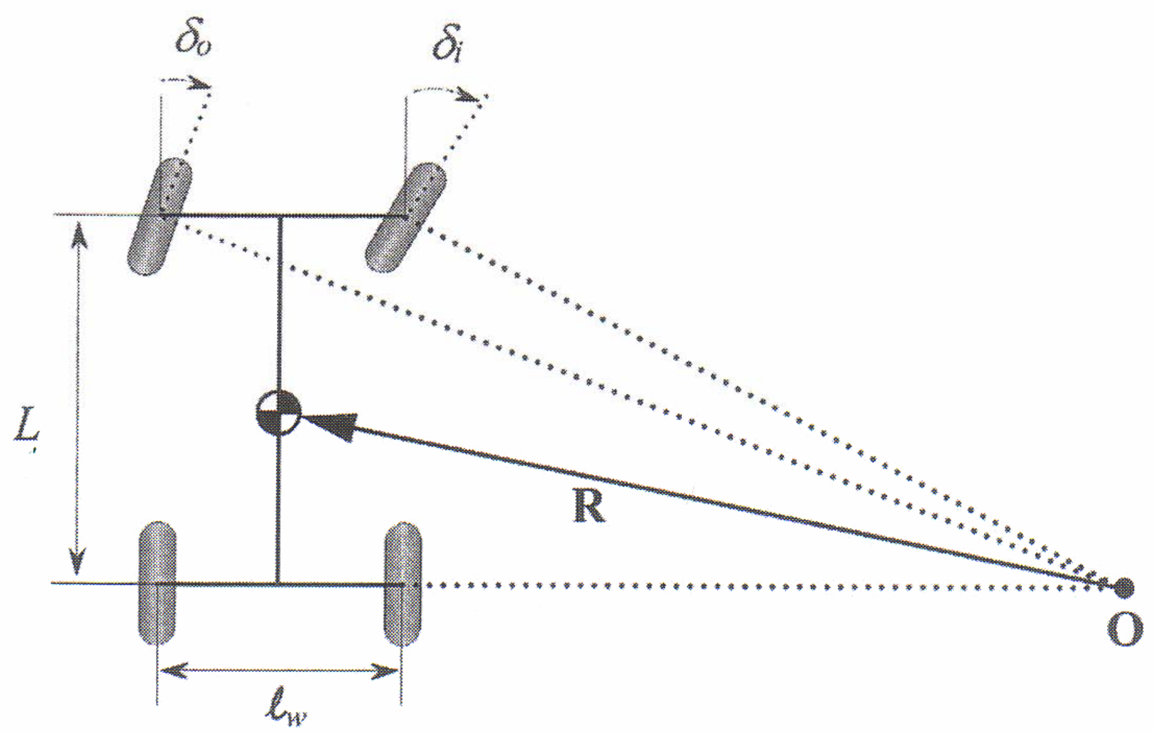
$$\dot{\psi} = \frac{V\left(\tan\delta_f - \tan\delta_r\right)\cos\beta}{l_f + l_r}$$

$$\dot{X} = V\cos(\psi + \beta)$$
$$\dot{Y} = V\sin(\psi + \beta)$$

Kinematic Model of Lateral Vehicle: consider vehicle width

Limitation of the bicycle model:

δ_o and δ_i in fact are different.



Ackerman turning geometry

Kinematic Model of Lateral Vehicle: consider vehicle width

In the bicycle model

$$\dot{\psi} = \frac{V(\tan\delta_f - \tan\delta_r)\cos\beta}{l_f + l_r}$$

If the slip angle β is small, $\delta_r = 0$

$$\dot{\psi} = \frac{V\delta_f}{l_f + l_r} = \frac{V\delta_f}{L}$$

on the other hand

$$\dot{\psi} = \frac{V}{R}$$

Therfore

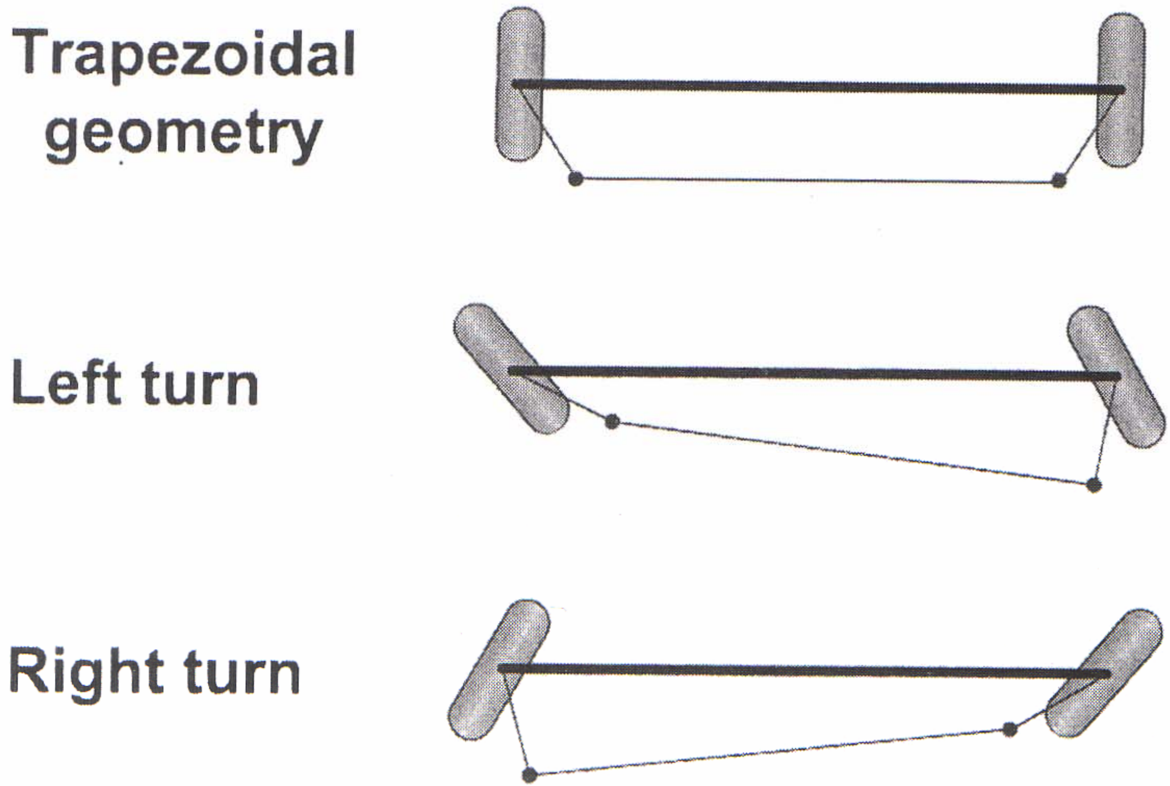
$$\delta_f = \frac{L}{R}$$

Let $\delta_f = \frac{\delta_o + \delta_i}{2} = \frac{L}{R}$

$$\delta_o = \frac{L}{R + \frac{l_w}{2}}, \qquad \delta_i = \frac{L}{R - \frac{l_w}{2}}$$

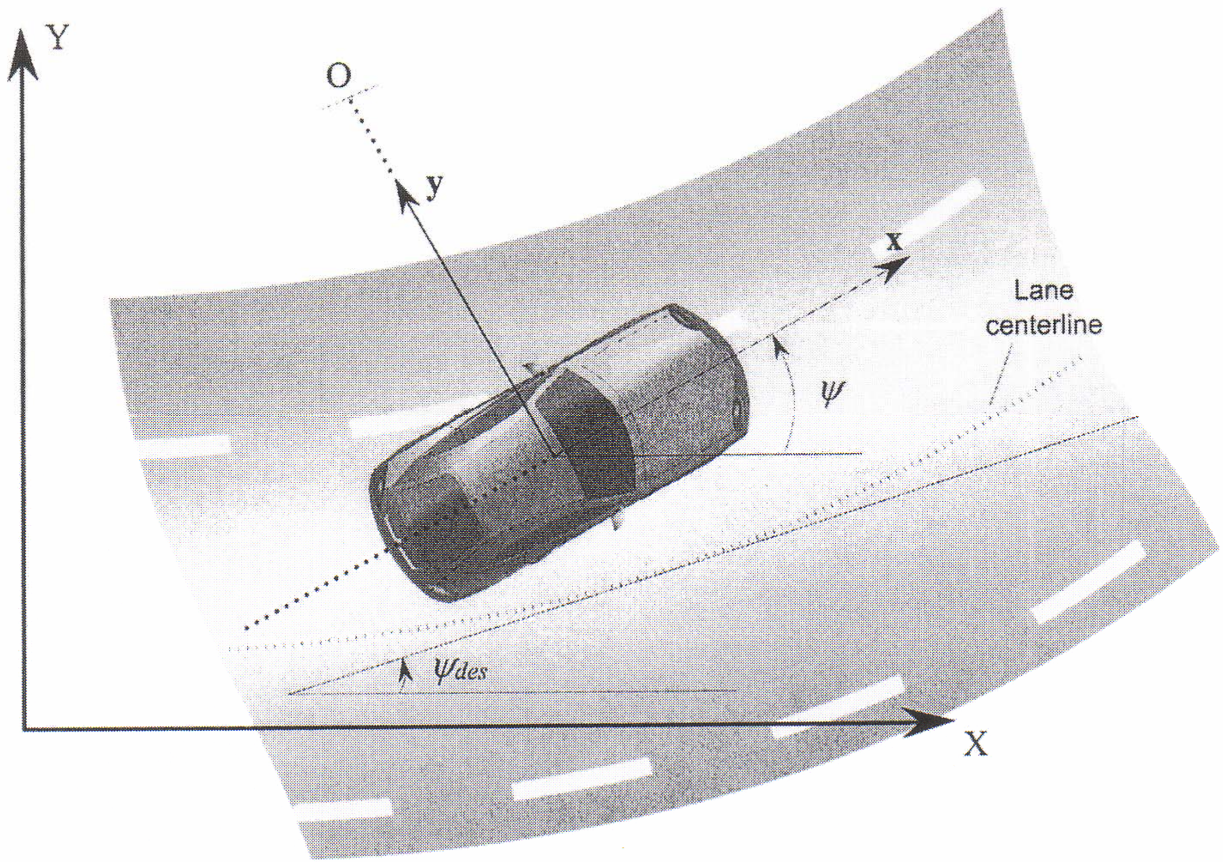
Kinematic Model of Lateral Vehicle: consider vehicle width

Trapezoidal tie rod arrangement to realize $\delta_i > \delta_o$



Differential steer from a trapezoidal tie-rod arrangement

Dynamics of Bicycle Model



Lateral vehicle dynamics

Dynamics of Bicycle Model

x – vehicle longitudinal axis

y - vehicle lateral position, measured along the vehicle lateral axis to the point O which is the centre of rotation

$X - Y$ global coordinates

ψ - yaw angle

$\mathrm{V_x}$ – vehicle longitudinal velocity

Using Newton's Law

$$\mathrm{ma_y} = F_{yf} + F_{yr}$$

$$a_y = \ddot{y} + \dot{\psi}^2 R = \ddot{y} + V_x \dot{\psi}$$

$$m(\ddot{y} + V_x \dot{\psi}) = F_{yf} + F_{yr}$$

$$I_z \ddot{\psi} = l_f F_{yf} - l_r F_{yr}$$

Experimental shows the lateral tyre force is proportional to the slip angle (when slip angle is small).

Dynamics of Bicycle Model

Experimental shows the lateral tyre force is proportional to the slip angle (when slip angle is small).

Slip angle of tyre :

$$\alpha_f = \delta - \theta_{vf}$$

where δ front tyre steering angle

Rear tyre $\alpha_r = -\theta_{vr}$

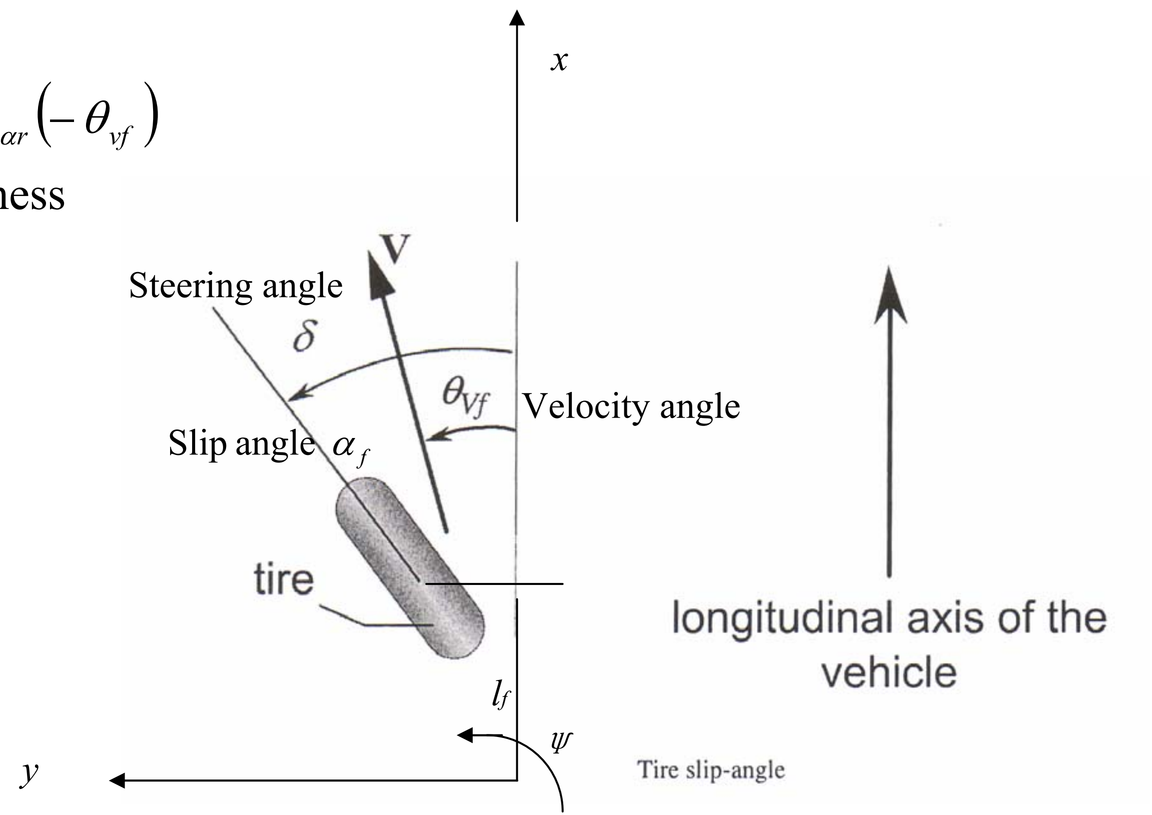
Forces

$$F_{yf} = 2C_{\alpha f}\left(\delta - \theta_{vf}\right) \qquad F_{yr} = 2C_{\alpha r}\left(-\theta_{vf}\right)$$

where $C_{\alpha f}, C_{\alpha r}$ are cornering stiffness

$$\tan\theta_{vf} = \frac{V_y + l_f\dot{\psi}}{V_x},$$

$$\tan\theta_{vr} = \frac{V_y - l_r\dot{\psi}}{V_x},$$



Tire slip-angle

Dynamics of Bicycle Model

When θ_{vf}, θ_{vr} are small

$$\theta_{vf} = \frac{\dot{y} + l_f \dot{\psi}}{V_x}$$

$$\theta_{vr} = \frac{\dot{y} - l_r \dot{\psi}}{V_x}$$

$$m\left(\ddot{y} + V_x \dot{\psi}\right) = F_{yf} + F_{yr}$$

$$I_z \ddot{\psi} = l_f F_{yf} - l_r F_{yr}$$

$$F_{yf} = 2C_{\alpha f}\left(\delta - \frac{\dot{y} + l_f \dot{\psi}}{V_x}\right)$$

$$F_{yr} = -2C_{\alpha r}\frac{\dot{y} - l_r \dot{\psi}}{V_x}$$

$$\ddot{y} + V_x\dot{\psi} = \frac{2C_{\alpha f}\delta}{m} - \frac{2C_{\alpha f}\left(\dot{y} + l_f\dot{\psi}\right)}{mV_x} - \frac{2C_{\alpha r}\left(\dot{y} - l_r\dot{\psi}\right)}{mV_x}$$

$$\ddot{\psi} = \frac{l_f}{I_z}\left(2C_{\alpha f}\delta - \frac{2C_{\alpha f}\left(\dot{y} + l_f\dot{\psi}\right)}{V_x}\right) + \frac{l_r}{I_z}\frac{2C_{\alpha r}\left(\dot{y} - l_r\dot{\psi}\right)}{V_x}$$

i.e.

$$\ddot{y} = \frac{2C_{\alpha f}\delta}{m} - \frac{2\left(C_{\alpha f} + C_{\alpha r}\right)}{mV_x}\dot{y} - \left(V_x + \frac{2\left(C_{\alpha f}l_f - C_{\alpha r}l_r\right)}{mV_x}\right)\dot{\psi}$$

$$\ddot{\psi} = \frac{l_f 2C_{\alpha f}\delta}{I_z} - \frac{2\left(C_{\alpha f}l_f - C_{\alpha r}l_r\right)}{I_zV_x}\dot{y} - \frac{2\left(C_{\alpha f}l_f^2 + C_{\alpha r}l_r^2\right)}{I_zV_x}\dot{\psi}$$

Introduce state space variable,

$$\begin{Bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{Bmatrix}$$

The lateral equations of motion become

$$\frac{d}{dt}\begin{Bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{2\left(C_{\alpha f}+C_{\alpha r}\right)}{mV_x} & 0 & -V_x-\dfrac{2\left(C_{\alpha f}l_f-C_{\alpha r}l_r\right)}{mV_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{2\left(C_{\alpha f}l_f-C_{\alpha r}l_r\right)}{I_zV_x} & 0 & -\dfrac{2\left(C_{\alpha f}l_f^2+C_{\alpha r}l_r^2\right)}{I_zV_x} \end{bmatrix}\begin{Bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{2C_{\alpha f}}{m} \\ 0 \\ \dfrac{2l_fC_{\alpha f}}{I_z} \end{bmatrix}\delta$$