University of Sussex

# Design, Modelling and Manufacture 3

# H7082

2

# Core of Finite Element Method

A structural can be considered as being composed of many interconnected 'spring' elements.

Assembly of the Global System of Equations

$$[K]=\sum_{e=1}^{E}[K_e]$$

$$\{F\}=\sum_{e=1}^{E}\{f_e\}$$

For equilibrium and compatibility

$$\{F\}=[K]\{u\}$$

where

$\{F\}$- external nodal forces

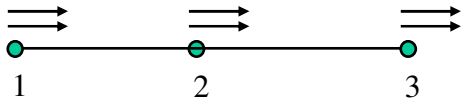
$[K]$- structural stiffness matrix

square, symmetrical, inverse–able if constrained

$\{u\}$- nodal displacements of structures

Example: Two rod element structure

Element 1,2; Nodes 1,2,3




Elemental stiffness matrices in local co-ordinates

$$\begin{matrix} & u_1 & u_2 \\ \end{matrix}$$

$$\left[K_1^{\,e}\right] = \frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} u_1 \\ u_2 \end{matrix}$$

$$\begin{matrix} & u_2 & u_3 \\ \end{matrix}$$

$$\left[K_2^{\,e}\right] = \frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} u_2 \\ u_3 \end{matrix}$$

Elemental stiffness matrices in global co-ordinates

$$\begin{matrix} & u_1 & u_2 & u_3 \\ \end{matrix}$$

$$\left[K_1^{\,G}\right] = \frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

$$\begin{matrix} & u_1 & u_2 & u_3 \\ \end{matrix}$$

$$\left[K_2^{\,G}\right] = \frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

Assemble elemental stiffness matrices to structural stiffness matrix

$$\begin{matrix} & u_1 & u_2 & u_3 \\ \end{matrix}$$

$$[K] = \left[K_1^{\,G}\right] + \left[K_2^{\,G}\right] = \begin{bmatrix} \frac{A_1E_1}{l_1} & -\frac{A_1E_1}{l_1} & 0 \\ -\frac{A_1E_1}{l_1} & \frac{A_1E_1}{l_1} + \frac{A_2E_2}{l_2} & -\frac{A_2E_2}{l_2} \\ 0 & -\frac{A_2E_2}{l_2} & \frac{A_2E_2}{l_2} \end{bmatrix}\begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} \frac{A_1E_1}{l_1} & -\frac{A_1E_1}{l_1} & 0 \\ -\frac{A_1E_1}{l_1} & \frac{A_1E_1}{l_1} + \frac{A_2E_2}{l_2} & -\frac{A_2E_2}{l_2} \\ 0 & -\frac{A_2E_2}{l_2} & \frac{A_2E_2}{l_2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

# Two Dimensional Rod Elements

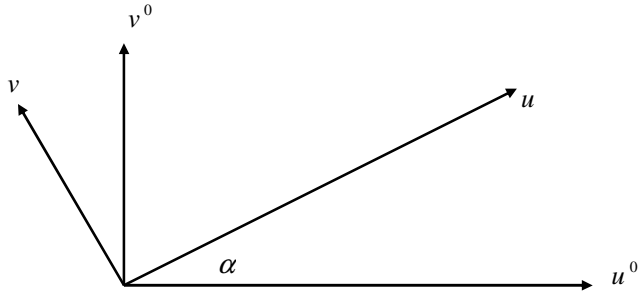
Relationship between local displacement, $u, v$ and global displacement $u^0, v^0$

$$u = u^0 \cos\alpha + v^0 \sin\alpha$$
$$v = v^0 \cos\alpha - u^0 \sin\alpha$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u^0 \\ v^0 \end{Bmatrix}$$

or

$$\{u\} = [\zeta]\{u^0\}$$




$$\begin{Bmatrix} u^0 \\ v^0 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

or

$$\{u^0\} = [\zeta]^T \{u\}$$

$$[\zeta]^{-1} = [\zeta]^T$$

Elements of a structure may be inclined at various angles to the horizontal.
Consider 2 D element Node 1:

$$u_1^0, v_1^0, X_1^0, Y_1^0$$

Node 2: $u_2^0, v_2^0, X_2^0, Y_2^0$

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u_1{}^0 \\ v_1{}^0 \end{Bmatrix} \qquad \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u_2{}^0 \\ v_2{}^0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u_1^0 \\ v_1^0 \\ u_2^0 \\ v_2^0 \end{Bmatrix}$$

$$= \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix} \begin{Bmatrix} u_1^0 \\ v_1^0 \\ u_2^0 \\ v_2^0 \end{Bmatrix} = [\eta] \begin{Bmatrix} u_1^0 \\ v_1^0 \\ u_2^0 \\ v_2^0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = [\eta] \begin{Bmatrix} u_1^0 \\ v_1^0 \\ u_2^0 \\ v_2^0 \end{Bmatrix} \qquad \textit{where} \qquad [\eta] = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix}_{4\times 4}, \qquad [\eta]^{-1} = [\eta]^T$$

Same co-ordinate relationship for forces

$$\begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{Bmatrix} = [\eta] \begin{Bmatrix} X_1^0 \\ Y_1^0 \\ X_2^0 \\ Y_2^0 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \left[K^e\right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1^0 \\ Y_1^0 \\ X_2^0 \\ Y_2^0 \end{Bmatrix} = [\eta]^T \begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{Bmatrix} = [\eta]^T \left[K^e\right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = [\eta]^T \left[K^e\right] [\eta] \begin{Bmatrix} u_1^0 \\ v_1^0 \\ u_2^0 \\ v_2^0 \end{Bmatrix}$$

Elemental stiffness in global co-ordinates:

$$\left[K_G^e\right]=\left[\eta\right]^T\left[K^e\right]\left[\eta\right]$$

$$=\frac{AE}{l}\begin{bmatrix}\cos\alpha & -\sin\alpha & 0 & 0\\ \sin\alpha & \cos\alpha & 0 & 0\\ 0 & 0 & \cos\alpha & -\sin\alpha\\ 0 & 0 & \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha & 0 & 0\\ -\sin\alpha & \cos\alpha & 0 & 0\\ 0 & 0 & \cos\alpha & \sin\alpha\\ 0 & 0 & -\sin\alpha & \cos\alpha\end{bmatrix}$$

i.e.

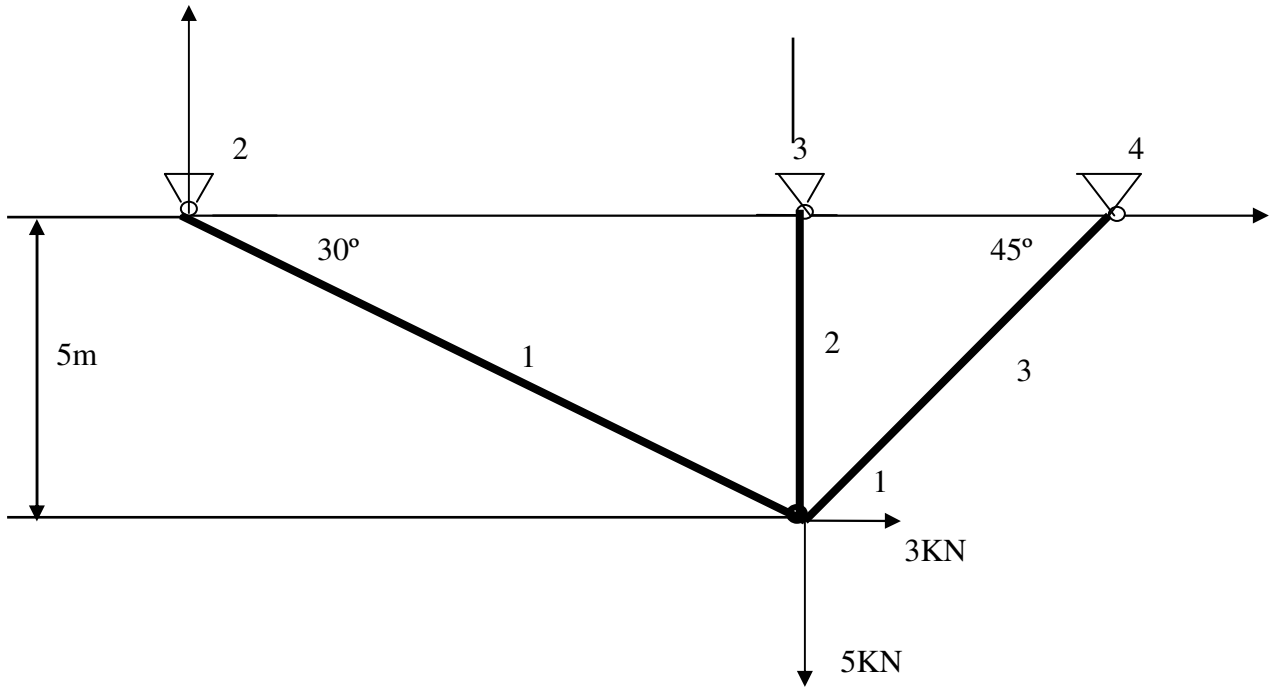
$$\left[K_G^e\right]=\frac{AE}{l}\begin{bmatrix}c^2 & cs & -c^2 & -cs\\ cs & s^2 & -cs & -s^2\\ -c^2 & -cs & c^2 & cs\\ -cs & -s^2 & cs & s^2\end{bmatrix}$$

If two nodes of a rod element are marked by $i, j$; global co-ordinates are $\left(x_i^0, y_i^0\right), \left(x_j^0, y_j^0\right)$

$$l=\sqrt{\left(x_j^0-x_i^0\right)^2+\left(y_j^0-y_i^0\right)^2} \qquad \cos\alpha=\frac{x_j^0-x_i^0}{l}, \qquad \sin\alpha=\frac{y_j^0-y_i^0}{l}$$

Example: A plane pin-jointed truss is composed of three uniform members (1) to (3), cross-sectional areas: $A_0, A_0, \sqrt{2}A_0$. $A_0 = 15\times10^{-6} m^2$ all $E = 2\times10^{11} PA.$

All members are pin-jointed, only sustain axial forces.



Element (1): $x_1^0 = 8.66m, \quad x_2^0 = 0, \quad y_1^0 = -5m, \quad y_2^0 = 0, \quad l = 10m.$

$$c = \frac{x_2^0 - x_1^0}{l} = \frac{0 - 0.866}{10} = -\frac{\sqrt{3}}{2} \qquad s = \frac{y_2^0 - y_1^0}{l} = \frac{0 - (-5)}{10} = \frac{1}{2}$$

$$\left[K_1^e\right] = \frac{A_0 E}{10}\frac{1}{4}\begin{bmatrix} 3 & -\sqrt{3} & -3 & \sqrt{3} \\ -\sqrt{3} & 1 & \sqrt{3} & -1 \\ -3 & \sqrt{3} & 3 & -\sqrt{3} \\ \sqrt{3} & -1 & -\sqrt{3} & 1 \end{bmatrix}$$

Element (2): $x_1^0 = 8.66m, \quad x_3^0 = 8.66, \quad y_1^0 = -5m, \quad y_3^0 = 0, \quad l_2 = 5m.$

$$c = \frac{x_3^0 - x_1^0}{l} = 0 \qquad s = \frac{y_3^0 - y_1^0}{5} = \frac{0 - (-5)}{5} = 1$$

$$\left[K_2^e\right] = \frac{A_0 E}{10}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix}$$

Element (3):

$$c = \frac{x_4^0 - x_1^0}{l} = \frac{(8.66 + 5) - 8.66}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad s = \frac{y_4^0 - y_1^0}{5\sqrt{2}} = \frac{0 - (-5)}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$x_1^0 = 8.66m, \; x_4^0 = 8.66 + 5, \qquad y_1^0 = -5m, \; y_4^0 = 0, \qquad l_3 = 5\sqrt{2}m.$

$$\left[K_3^e\right] = \frac{\sqrt{2}A_0E}{5\sqrt{2}}\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

Assembled Structure stiffness matrix

$$\left[K^0\right] = \frac{A_0E}{10}\begin{bmatrix} 1.75 & 0.567 & -0.75 & 0.433 & 0 & 0 & -1 & -1 \\ 0.567 & 3.25 & 0.433 & -0.25 & 0 & -2 & -1 & -1 \\ -0.75 & 0.433 & 0.750 & -0.433 & 0 & 0 & 0 & 0 \\ 0.433 & -0.25 & -0.433 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$u_2^0, v_2^0, u_3^0, v_3^0, u_4^0, v_4^0 = 0$$

$$\begin{Bmatrix} X_1^0 \\ Y_1^0 \\ X_2^0 \\ Y_2^0 \\ X_3^0 \\ Y_3^0 \\ X_4^0 \\ Y_4^0 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1^0 \\ v_1^0 \\ u_2^0 \\ v_2^0 \\ u_3^0 \\ v_3^0 \\ u_4^0 \\ v_4^0 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1^0 \\ v_1^0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1^0 \\ Y_1^0 \end{Bmatrix} = [K_{11}] \begin{Bmatrix} u_1^0 \\ v_1^0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1^0 \\ v_1^0 \end{Bmatrix} = [K_{11}]^{-1} \begin{Bmatrix} X_1^0 \\ Y_1^0 \end{Bmatrix} = \frac{10}{A_0 E} \begin{bmatrix} 0.606 & -0.106 \\ -0.106 & 0.326 \end{bmatrix} \begin{Bmatrix} 3000 \\ -5000 \end{Bmatrix} = \frac{10}{A_0 E} \begin{Bmatrix} 2348 \\ -1948 \end{Bmatrix} = \begin{Bmatrix} 0.0078 \\ -0.0065 \end{Bmatrix} m$$

# Force in each element:

Element (1) $$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = [\zeta] \begin{Bmatrix} u_1^0 \\ v_1^0 \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} u_1^0 \\ v_1^0 \end{Bmatrix} = \frac{10}{A_0 E} \begin{bmatrix} -0.866 & 0.5 \\ -0.5 & -0.866 \end{bmatrix} \begin{Bmatrix} 2.348 \\ -1.948 \end{Bmatrix} = \frac{10}{A_0 E} \begin{Bmatrix} -3 \\ 0.513 \end{Bmatrix}$$

$$F_{(1)} = \frac{A_0 E}{l_1}(u_2 - u_1) = \frac{A_0 E}{l_1} \frac{l_1}{A_0 E} = 3KN \qquad \text{(tensile)}$$

Element (2) $$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} u_1^0 \\ v_1^0 \end{Bmatrix}$$ $v_1$ is not of interest. The force only depends on $u_1$

$$u_1 = [c \quad s] \begin{Bmatrix} u_1^0 \\ v_1^0 \end{Bmatrix} = [0 \quad 1] \frac{10}{A_0 E} \begin{Bmatrix} 2.348 \\ -1.948 \end{Bmatrix} = -1.948 \frac{10}{A_0 E}$$

$$F_{(2)} = \frac{A_0 E}{l_2}(u_2 - u_1) = \frac{A_0 E}{5}\left(0 + 1.948 \frac{10}{A_0 E}\right) = 3.9KN \qquad \text{(tensile)}$$

Element (3) $$u_1 = [c \quad s] \begin{Bmatrix} u_1^0 \\ v_1^0 \end{Bmatrix} = [0.707 \quad 0.707] \frac{10}{A_0 E} \begin{Bmatrix} 2.348 \\ -1.948 \end{Bmatrix} = 2.83 \frac{10}{A_0 E}$$

$$F_{(3)} = \frac{\sqrt{2} A_0 E}{l_3}(u_2 - u_1) = \frac{\sqrt{2} A_0 E}{5\sqrt{2}}\left(0 - 2.83 \frac{10}{A_0 E}\right) = -0.563KN \qquad \text{(compressive)}$$

```
%Rod Example 1 MATLAB code
clear all; close all;
%Given
E0=2e11; F1x=3000; F1y=-5000;
A0=15e-6;
A(1)=A0; A(2)=A0; A(3)=A0*sqrt(2);
E(1)=E0; E(2)=E0; E(3)=E0;
x=[8.667    0      8.667  8.667+5];
y=[-5       0      0        0   ];

%Get truss lengths
L(1)=sqrt((x(2)-x(1))^2+(y(2)-y(1))^2);
L(2)=sqrt((x(3)-x(1))^2+(y(3)-y(1))^2);
L(3)=sqrt((x(4)-x(1))^2+(y(4)-y(1))^2);

%Get truss l and m (cosine and sine)
l(1)=(x(2)-x(1))/L(1); m(1)=(y(2)-y(1))/L(1);
l(2)=(x(3)-x(1))/L(2); m(2)=(y(3)-y(1))/L(2);
l(3)=(x(4)-x(1))/L(3); m(3)=(y(4)-y(1))/L(3);

%Element stiffness matrices 1-4
Ke=zeros(4,4,4);  %define an array to store element stiffness
matrices
for i=1:3
   c=l(i);
   s=m(i);
   Ke(:,:,i) =E(i)*A(i)/L(i)* [ c^2    c*s      -c^2    -c*s;
                                c*s    s^2      -c*s    -s^2;
                               -c^2   -c*s       c^2     c*s;
                               -c*s   -s^2       c*s     s^2]
end;
% Element Dof information

EDof(1,:)=[1 2 3 4]; EDof(2,:)=[1 2 5 6]; EDof(3,:)=[1 2 7 8];
```

```
%Assenble Element stiffness matrices to Global Stiffness matrix
K=zeros(2*4); % Define Global Stiffness matrices, 5 nodes, each node has 2 DOF

for k=1:3             % all element
   for i=1:4
      for j=1:4
         i1=EDof(k,i);
         j1=EDof(k,j);
         K(i1,j1)=K(i1,j1)+Ke(i,j,k);
      end;
   end;
end;
K %display K

% {F}=[K]{X}
% Displacement constraints at DOF
% b=[3 4 5 6 7 8], ignore column of 3 4 5 6 7 8,
% The reduced stiffness matrix 2x2
K1=[K(1,1) K(1,2);
   K(2,1) K(2,2)]
%Forces vector
F=[ F1x;
   F1y]
%Displacement at node 1
X=inv(K1)*F

%Forces in elements
u2=0;u3=0;u4=0;
%element 1
U=[l(1) m(1); -m(1) l(1)]*X;
u=U(1);v=U(2);
Fs=A(1)*E(1)/L(1)*(u2-u)
%element 2
U=[l(2) m(2); -m(2) l(2)]*X;
u=U(1);v=U(2);
Fs=A(2)*E(2)/L(2)*(u3-u)
%element 3
U=[l(3) m(3); -m(3) l(3)]*X;
u=U(1);v=U(2);
Fs=A(3)*E(3)/L(3)*(u4-u)
```

# Beam Elements (without axial forces)

$$\begin{Bmatrix} Y_1 \\ M_1 \\ Y_2 \\ M_2 \end{Bmatrix} = \left[K^e\right] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{32} & k_{33} & k_{34} \\ k_{14} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\left[K^e\right] = EI \begin{bmatrix} \frac{12}{l^3} & -\frac{6}{l^2} & -\frac{12}{l^3} & -\frac{6}{l^2} \\ -\frac{6}{l^2} & \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} \\ -\frac{12}{l^3} & \frac{6}{l^2} & \frac{12}{l^3} & \frac{6}{l^2} \\ -\frac{6}{l^2} & \frac{2}{l} & \frac{6}{l^2} & \frac{4}{l} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

# Rod-Beam Element

Superposition of road element and beam element. In local co-ordinates:

$$\begin{Bmatrix} X_1 \\ Y_1 \\ M_1 \\ X_2 \\ Y_2 \\ M_2 \end{Bmatrix}' = EI \begin{bmatrix} \frac{A}{l\ I} & 0 & 0 & -\frac{A}{l\ I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{A}{l\ I} & 0 & 0 & \frac{A}{l\ I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ Y_1 \\ M_1 \\ X_2 \\ Y_2 \\ M_2 \end{Bmatrix}'' = EI \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12}{l^3} & -\frac{6}{l^2} & 0 & -\frac{12}{l^3} & -\frac{6}{l^2} \\ 0 & -\frac{6}{l^2} & \frac{4}{l} & 0 & \frac{6}{l^2} & \frac{2}{l} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12}{l^3} & \frac{6}{l^2} & 0 & \frac{12}{l^3} & \frac{6}{l^2} \\ 0 & -\frac{6}{l^2} & -\frac{6}{l^2} & 0 & \frac{6}{l^2} & \frac{4}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ Y_1 \\ M_1 \\ X_2 \\ Y_2 \\ M_2 \end{Bmatrix} = EI \begin{bmatrix} \frac{A}{l\ I} & 0 & 0 & -\frac{A}{l\ I} & 0 & 0 \\ 0 & \frac{12}{l^3} & -\frac{6}{l^2} & 0 & -\frac{12}{l^3} & -\frac{6}{l^2} \\ 0 & -\frac{6}{l^2} & \frac{4}{l} & 0 & \frac{6}{l^2} & \frac{2}{l} \\ -\frac{A}{l\ I} & 0 & 0 & \frac{A}{l\ I} & 0 & 0 \\ 0 & -\frac{12}{l^3} & \frac{6}{l^2} & 0 & \frac{12}{l^3} & \frac{6}{l^2} \\ 0 & -\frac{6}{l^2} & -\frac{6}{l^2} & 0 & \frac{6}{l^2} & \frac{4}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

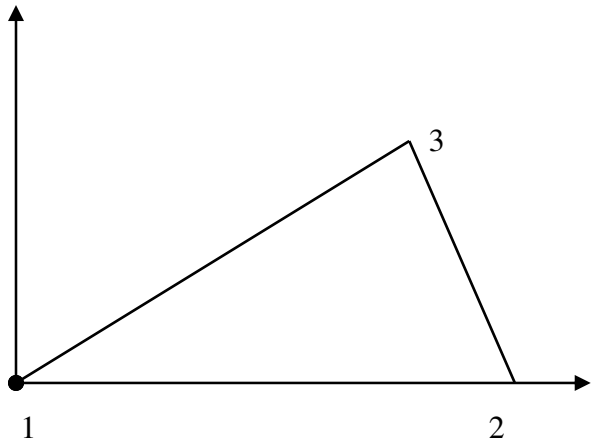
# In-plane Triangular Element

In-plane triangular elements are suitable for plane stress $\sigma_z = 0$ and plane strain $\varepsilon_z = 0$ problems.

Element: 3 nodes, constant stress, nodal displacements (shape function):

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$




Boundary conditions at 3 nodes

$$u_1 = \alpha_1 + \alpha_2 0 + \alpha_3 0 \quad u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 0 \quad u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$
$$v_1 = \alpha_4 + \alpha_5 0 + \alpha_6 0 \quad v_2 = \alpha_4 + \alpha_5 x_2 + \alpha_6 0 \quad v_3 = \alpha_4 + \alpha_5 x_3 + \alpha_6 y_3$$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

assumed displacement

$$\{u\} = [C]\{\alpha\}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_x \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_x \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

Geometrical relationship

$$\{\varepsilon\} = [B]\{\alpha\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_x \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_x \\ \gamma_{xy} \end{Bmatrix}$$

material properties

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$\{\sigma\} = [D]\{\varepsilon\} = [D][B]\{\alpha\} = [D][B][C]^{-1}\{u\}$$

Using virtual work principle

$$\{F\} = [C]^{-T}\left(\int_V [B]^T [D][B]dV\right)[C]^{-1}\{u\}$$

$$[K] = [C]^{-T}\left(\int_V [B]^T [D][B]dV\right)[C]^{-1}$$

Plane Strain Elemental Stiffness Matrix

$$[K^e] = \frac{E_1 t}{2x_2 y_3} \cdot \begin{bmatrix} y_3^2 + \gamma x_{32}^2 & & & & & \\ -(\gamma+\mu) y_3 x_{32} & \gamma y_3^2 + x_{32}^2 & & Sym & & \\ -y_3^2 - \gamma x_3 x_{32} & y_3(\gamma x_3 + \mu x_{32}) & y_3^2 + \gamma x_3^2 & & & \\ y_3(\gamma x_{32} + \mu x_3) & -\gamma y_3^2 - x_3 x_{32} & -x_3 y_3(\mu+\gamma) & \gamma y_3^2 + x_3^2 & & \\ \gamma x_2 x_{32} & -\gamma x_2 y_3 & -\gamma x_2 x_3 & \gamma x_2 y_3 & \gamma x_2^2 & \\ -\mu x_2 y_3 & x_2 x_{32} & \mu x_2 y_3 & -x_2 x_3 & 0 & x_2^2 \end{bmatrix}$$

$$E_1 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{\nu}{1-\nu}, \quad \gamma = \frac{1-2\nu}{2(1-\nu)}$$

Stiffness matrix in global co-ordinates

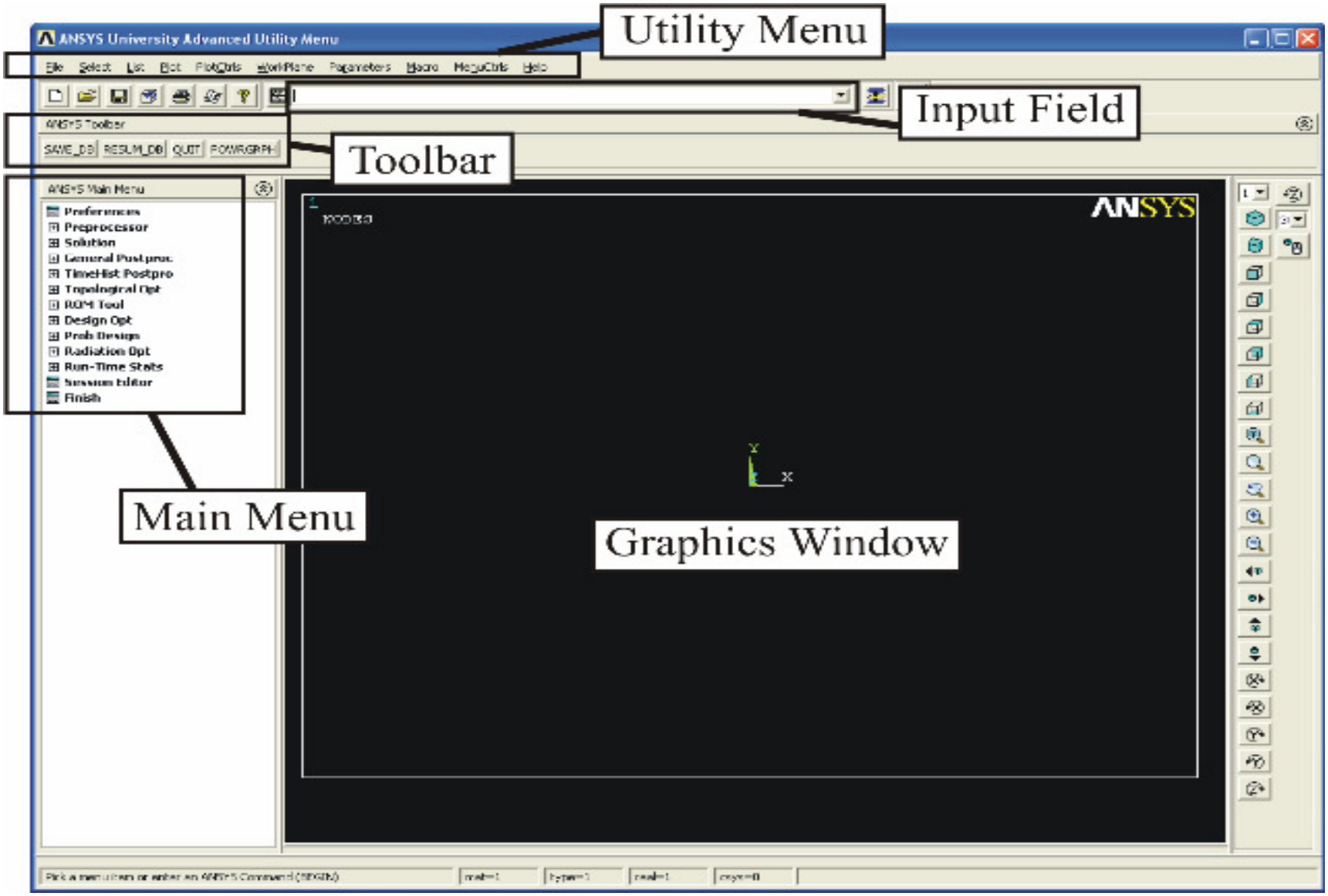
$$[K^e]_G = [\eta]^T [K][\eta]$$

# FEA Software

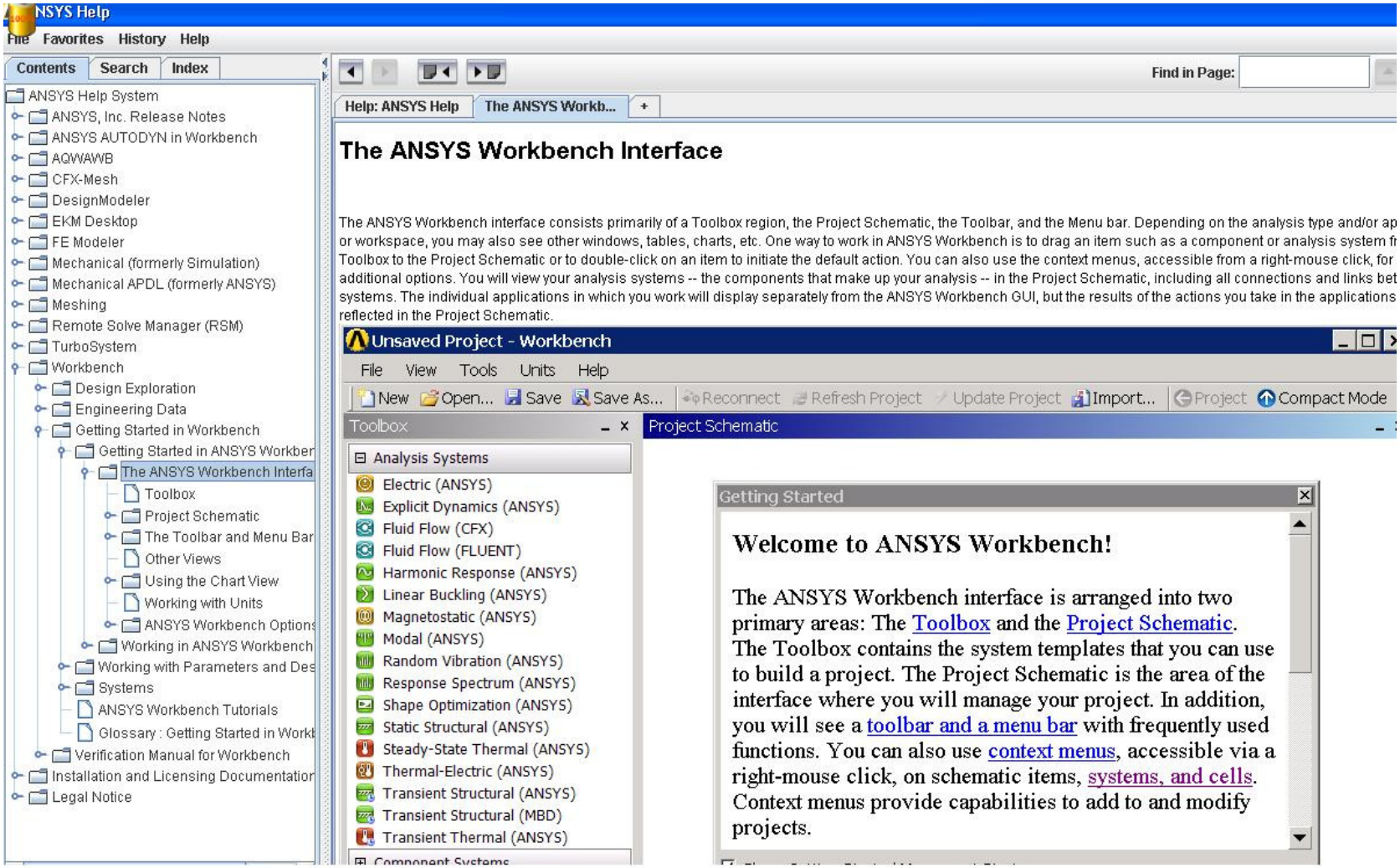
1. Self-programming with computer languages
MATLAB (e.g. Calfem MATLAB toolbox)
C
Fortran

2. Commercial software package
Ansys
Abacus
Nastran

# ANSYS Mechanical

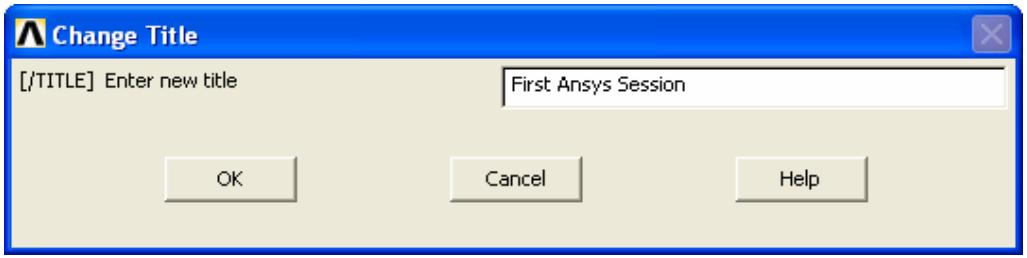


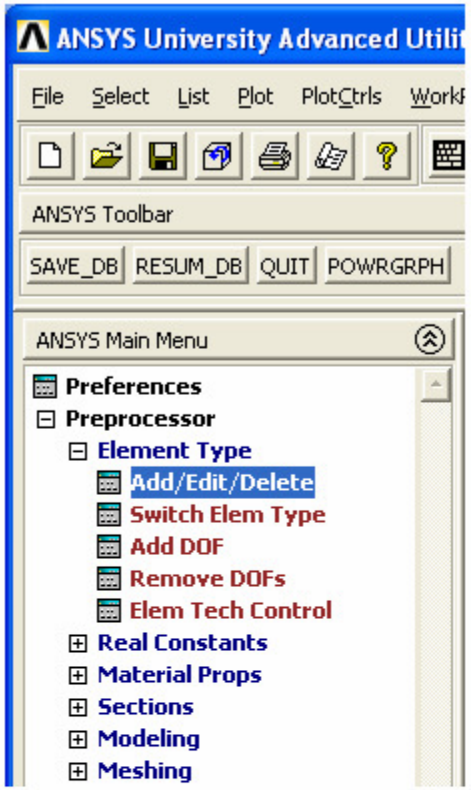
# ANSYS Workbench

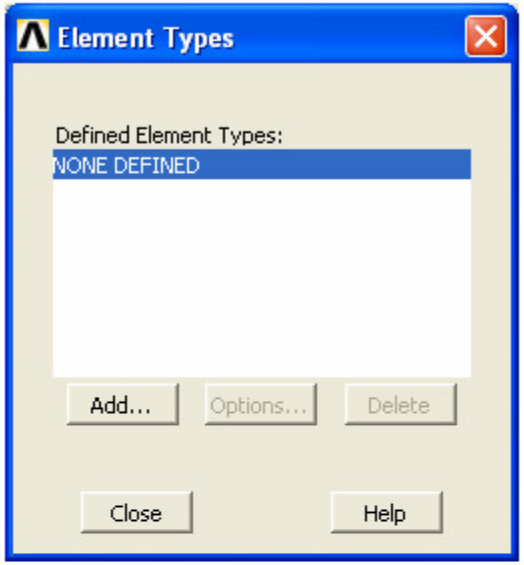


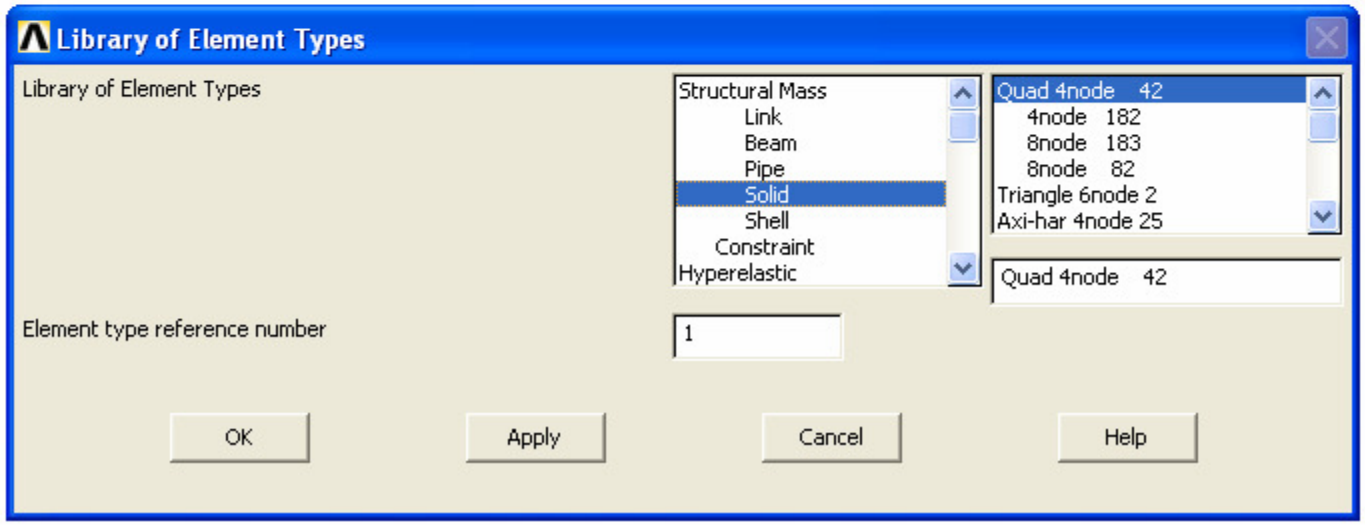
# Building the Model

In ‘File’
Specify the job name and type of analysis (units)
Specify the element types

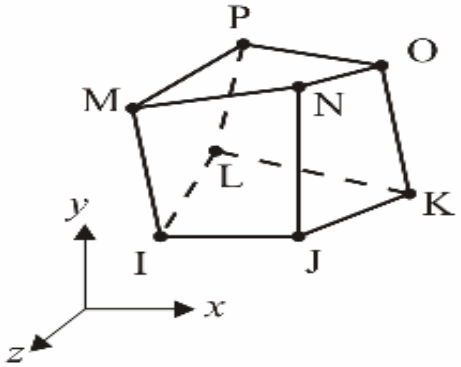



ANSYS University Advanced Utilit
File Select List Plot PlotCtrls Workl
ANSYS Toolbar
SAVE_DB RESUM_DB QUIT POWRGRPH
ANSYS Main Menu
Preferences
Preprocessor
Element Type
Add/Edit/Delete
Switch Elem Type
Add DOF
Remove DOFs
Elem Tech Control
Real Constants
Material Props
Sections
Modeling
Meshing

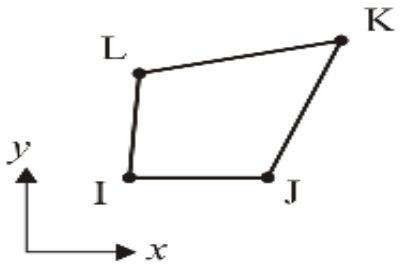

Element Types
Defined Element Types:
NONE DEFINED
Add...
Options...
Delete
Close
Help


Library of Element Types
Library of Element Types
Structural Mass
Link
Beam
Pipe
Solid
Shell
Constraint
Hyperelastic
Quad 4node 42
4node 182
8node 183
8node 82
Triangle 6node 2
Axi-har 4node 25
Quad 4node 42
Element type reference number
1
OK
Apply
Cancel
Help

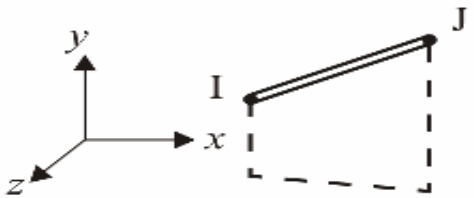
SOLID45 - 3-D BRICK
(DOF: UX, UY, UZ)



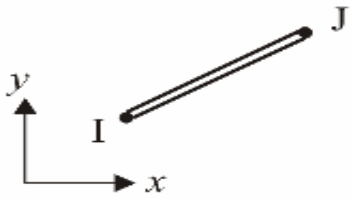

PLANE42 - 2-D PLANE
(DOF: UX, UY)



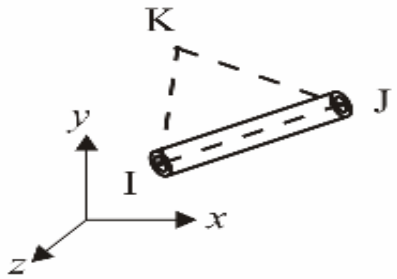

LINK8 - 3-D SPAR
(DOF: UX, UY, UZ)



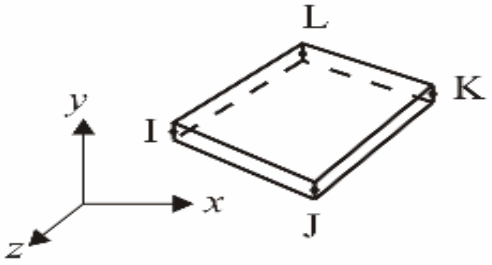

BEAM3 - 2-D BEAM
(DOF: UX, UY, ROTZ)

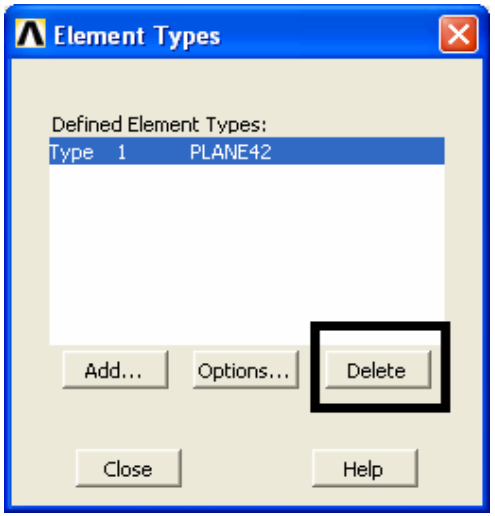



PIPE16 - 3-D STRAIGHT PIPE
(DOF: UX, UY, UZ,
ROTX, ROTY, ROTZ)

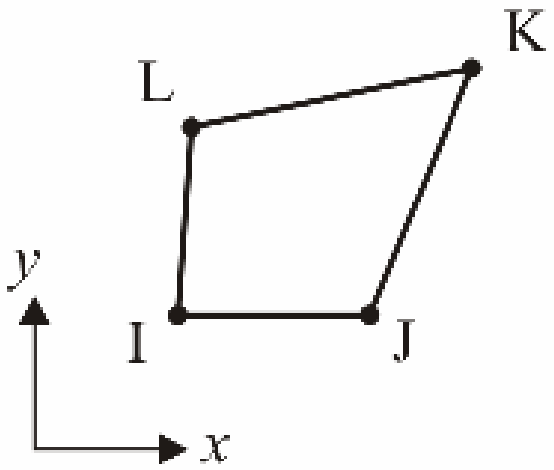



SHELL63 - 3-D SHELL
(DOF: UX, UY, UZ,
ROTX, ROTY, ROTZ)

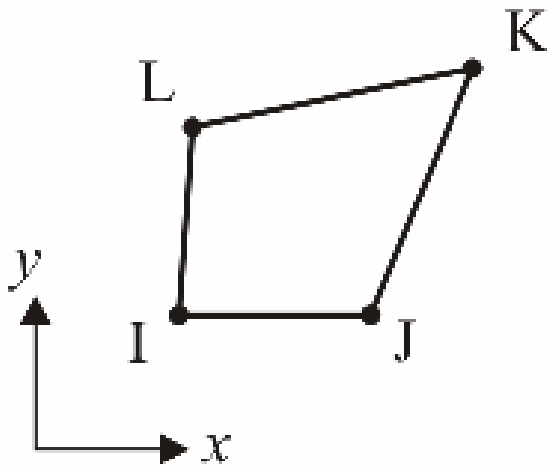


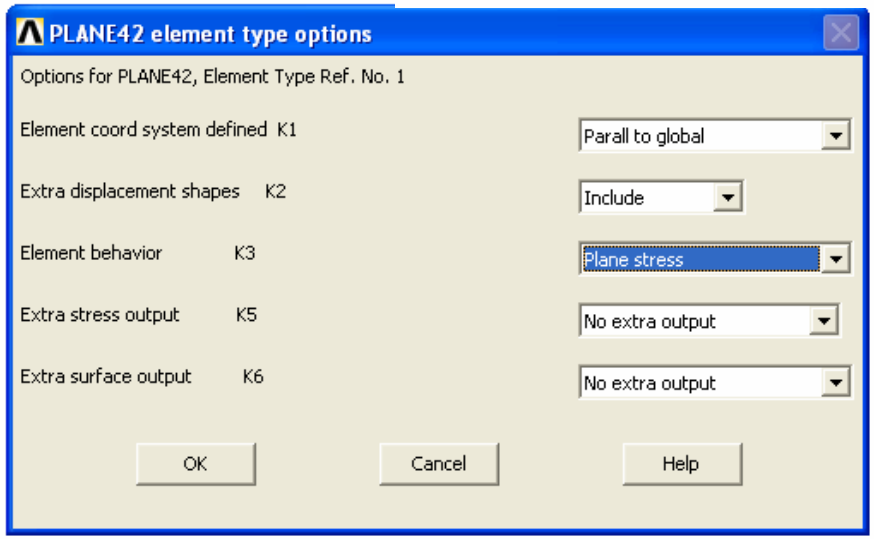


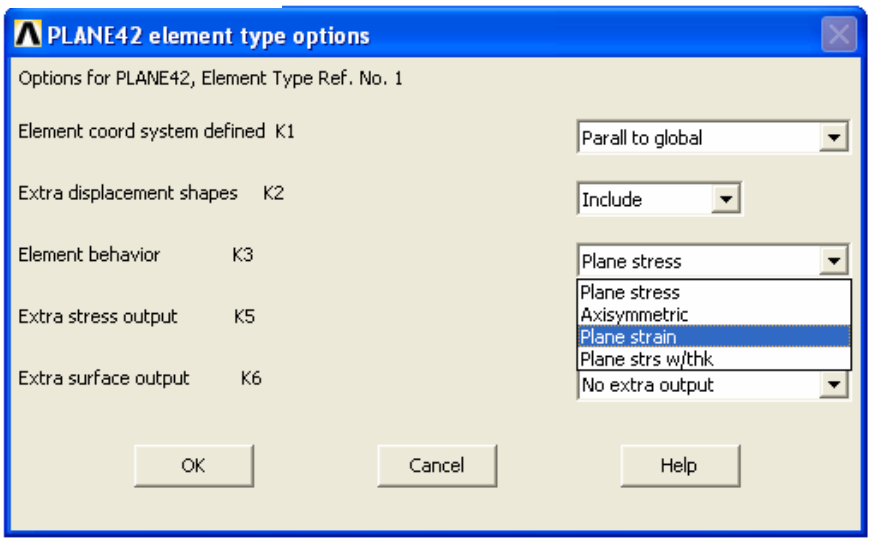

PLANE42 - 2-D PLANE
(DOF: UX, UY)



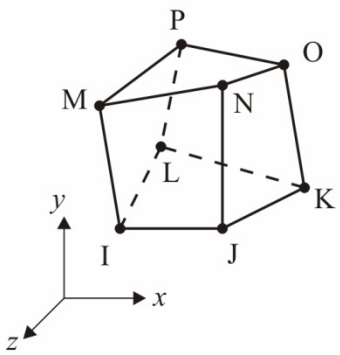
# PLANE42 - 2-D PLANE
# (DOF: UX, UY)



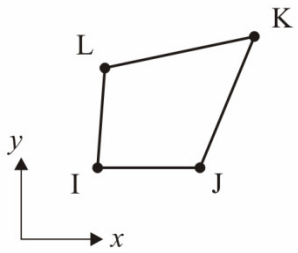



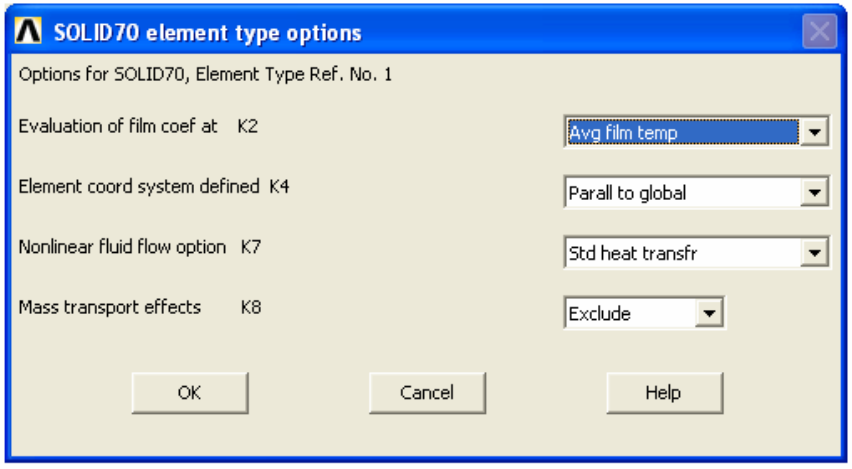



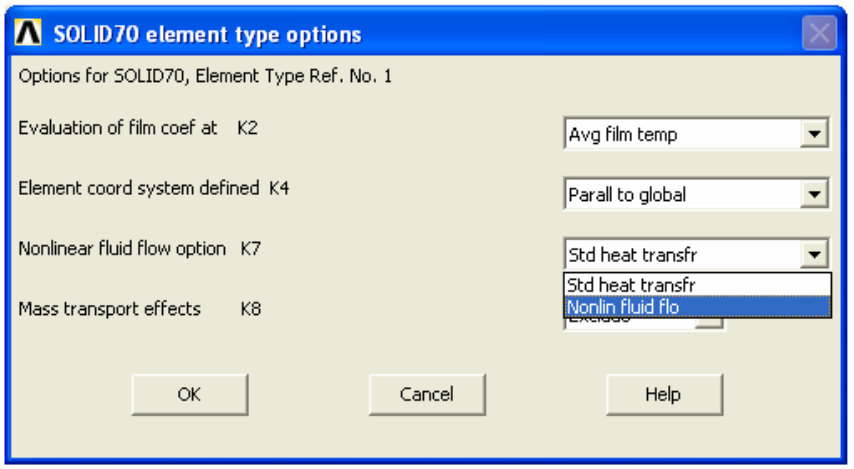
SOLID70 - 3-D THERMAL
(DOF: TEMPERATURE)



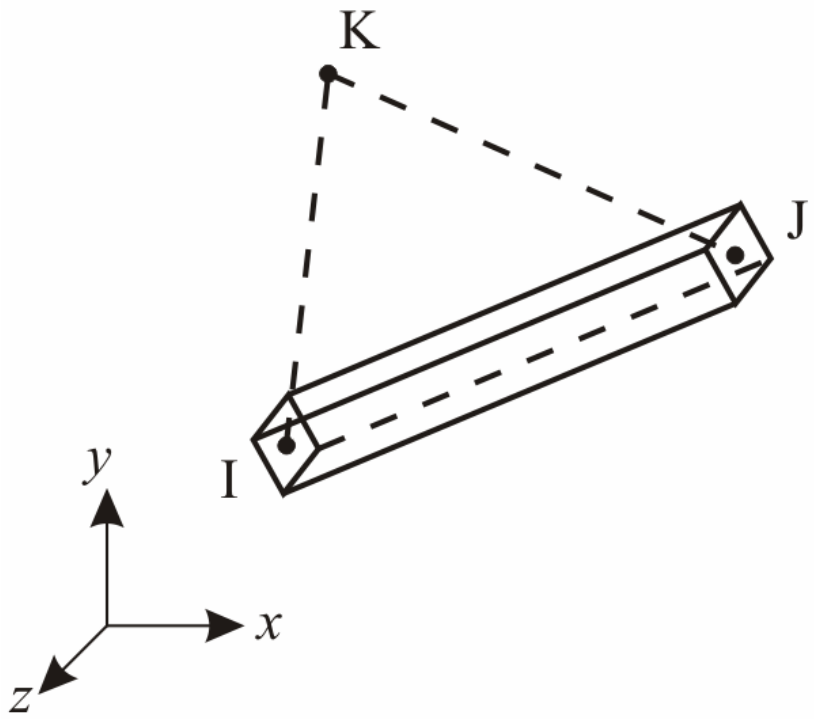

PLANE55 - 2-D THERMAL
(DOF: TEMPERATURE))



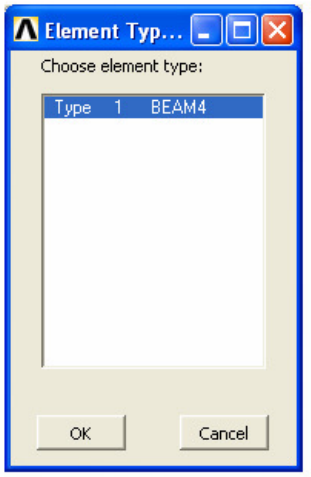



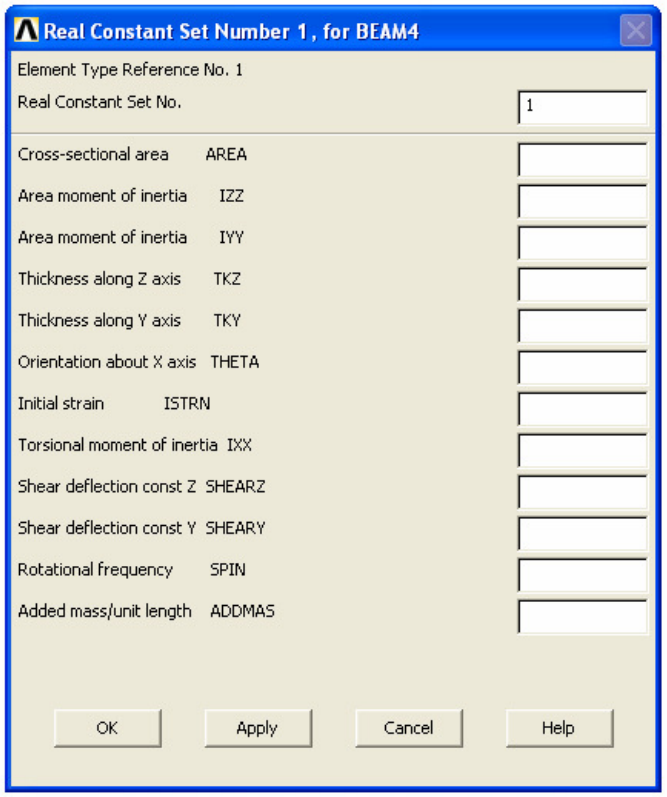
# BEAM4 - 3-D BEAM (DOF: UX, UY, UZ, ROTX, ROTY, ROTZ)

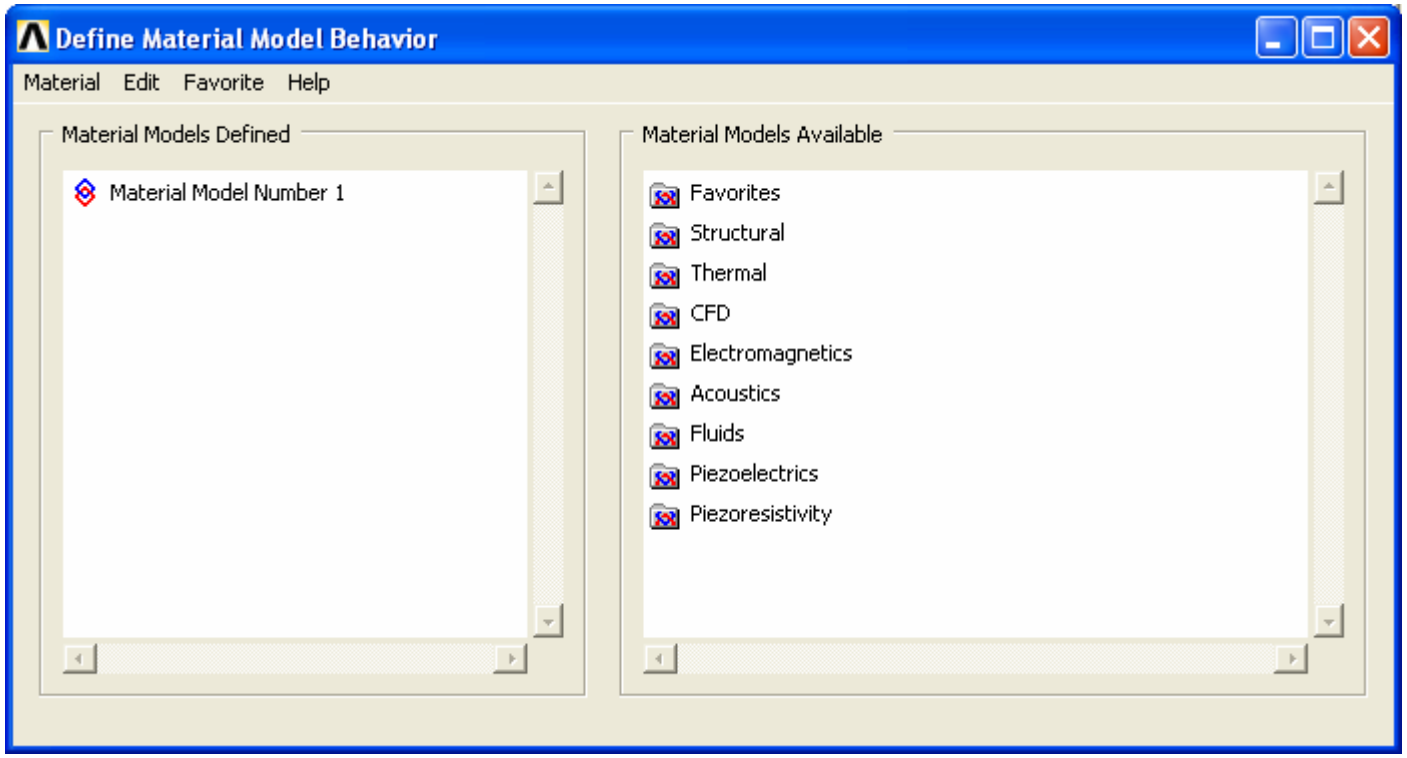


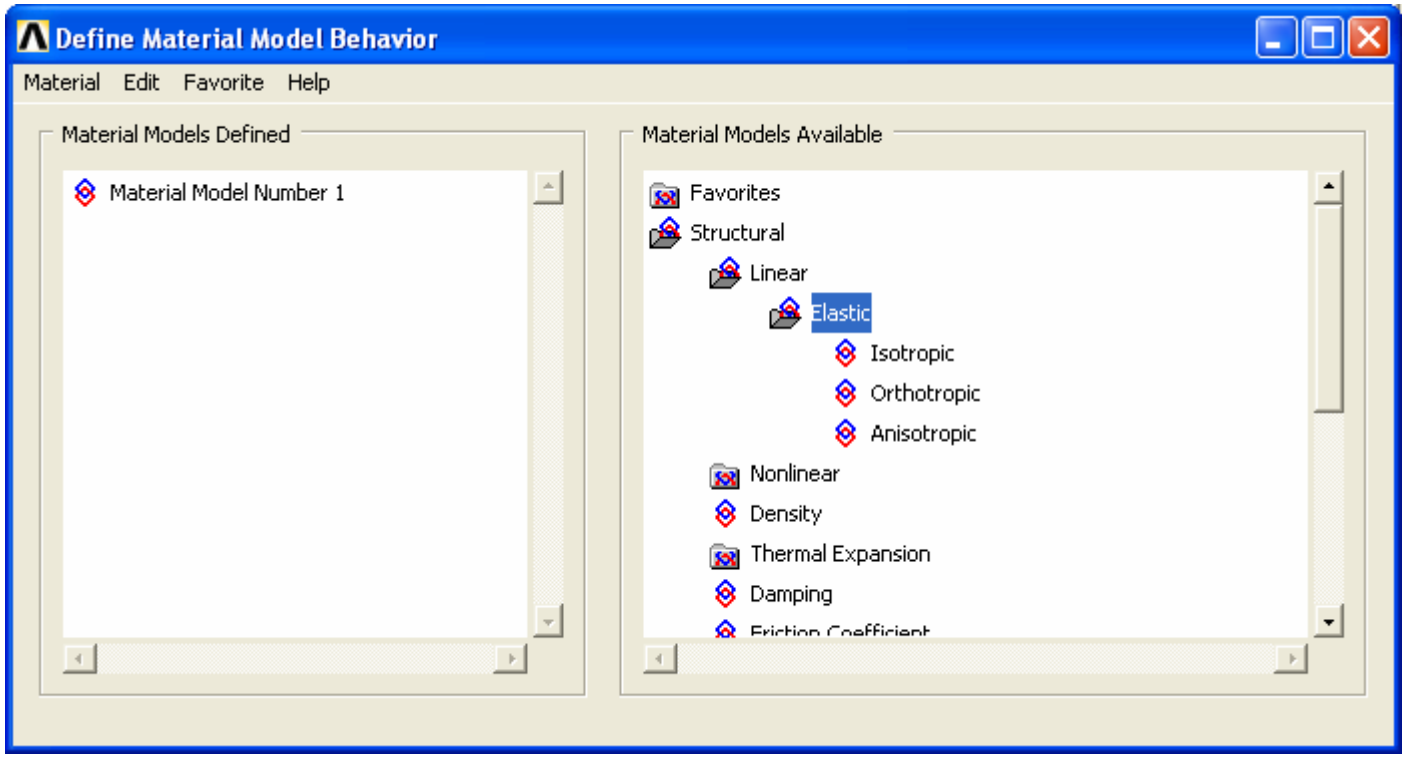


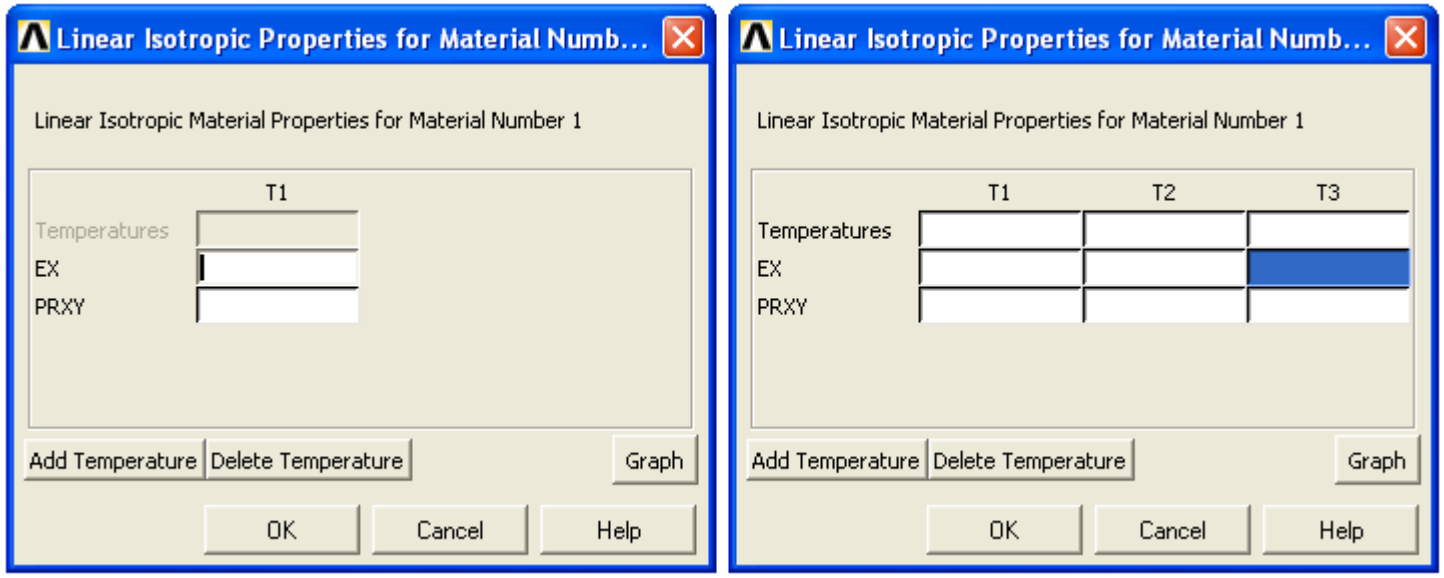




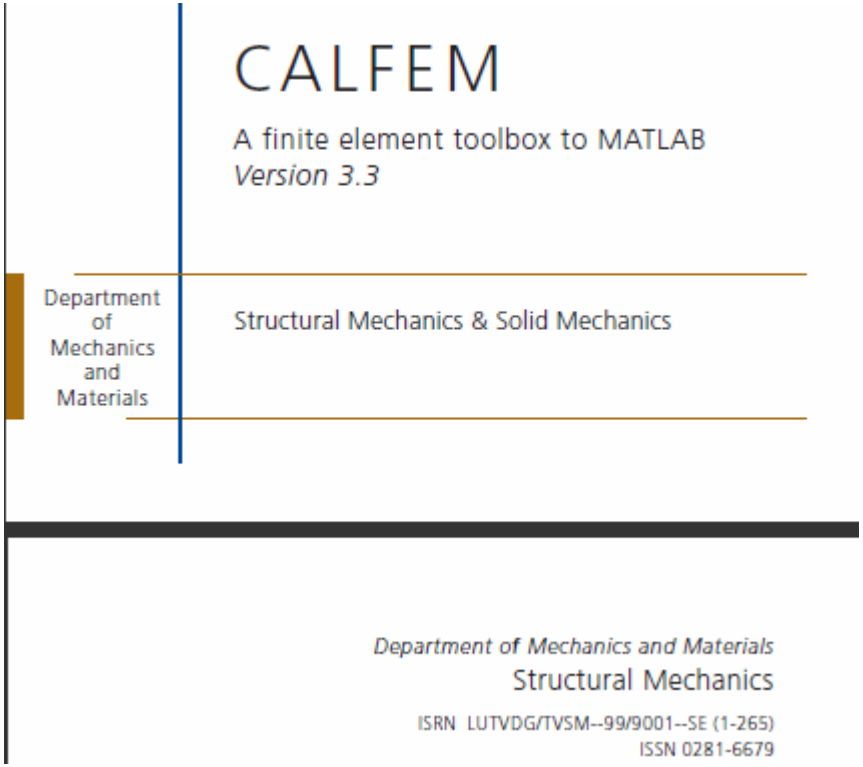




Define Material Model Behavior
Material
Edit
Favorite
Help
Material Models Defined
Material Model Number 1
Material Models Available
Favorites
Structural
Thermal
CFD
Electromagnetics
Acoustics
Fluids
Piezoelectrics
Piezoresistivity

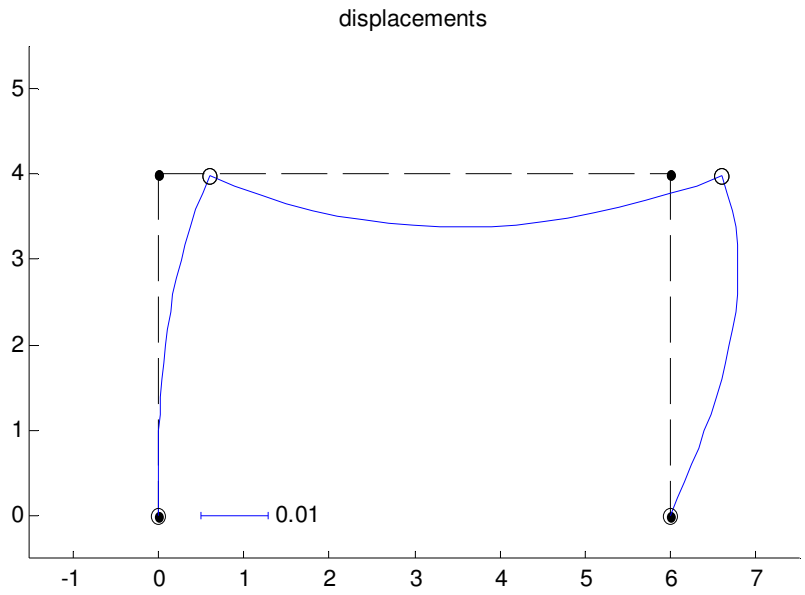

Define Material Model Behavior
Material
Edit
Favorite
Help
Material Models Defined
Material Model Number 1
Material Models Available
Favorites
Structural
Linear
Elastic
Isotropic
Orthotropic
Anisotropic
Nonlinear
Density
Thermal Expansion
Damping

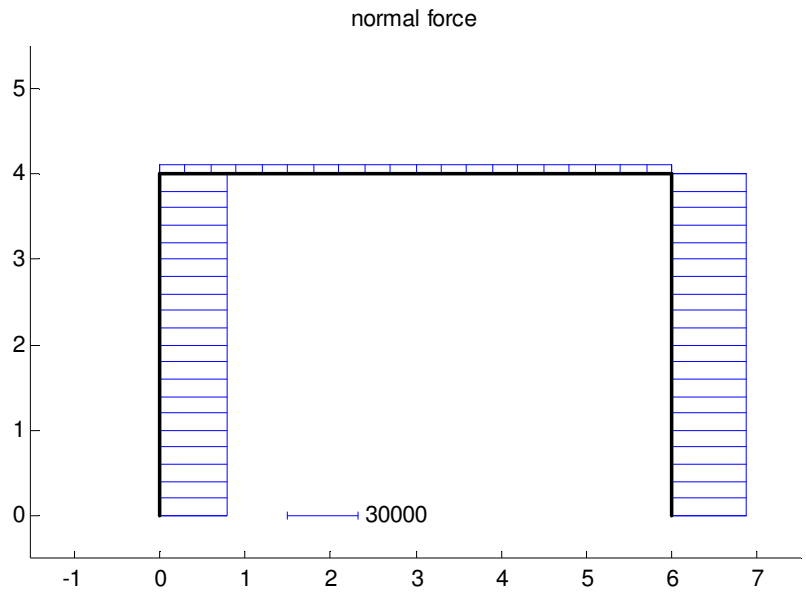

Linear Isotropic Properties for Material Numb...
Linear Isotropic Material Properties for Material Number 1
T1
Temperatures
EX
PRXY
Add Temperature
Delete Temperature
Graph
OK
Cancel
Help
Linear Isotropic Properties for Material Numb...
Linear Isotropic Material Properties for Material Number 1
T1
T2
T3
Temperatures
EX
PRXY
Add Temperature
Delete Temperature
Graph
OK
Cancel
Help

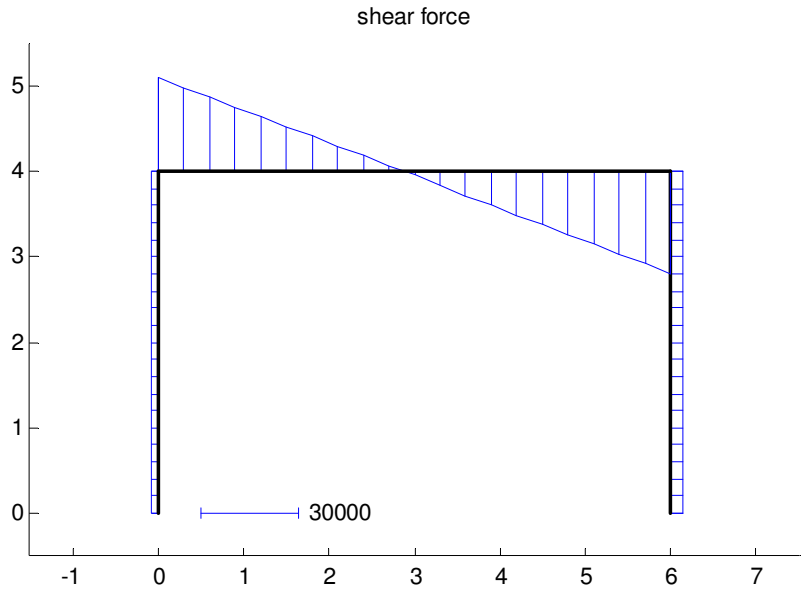
# MATLAB CALFEM toolbox

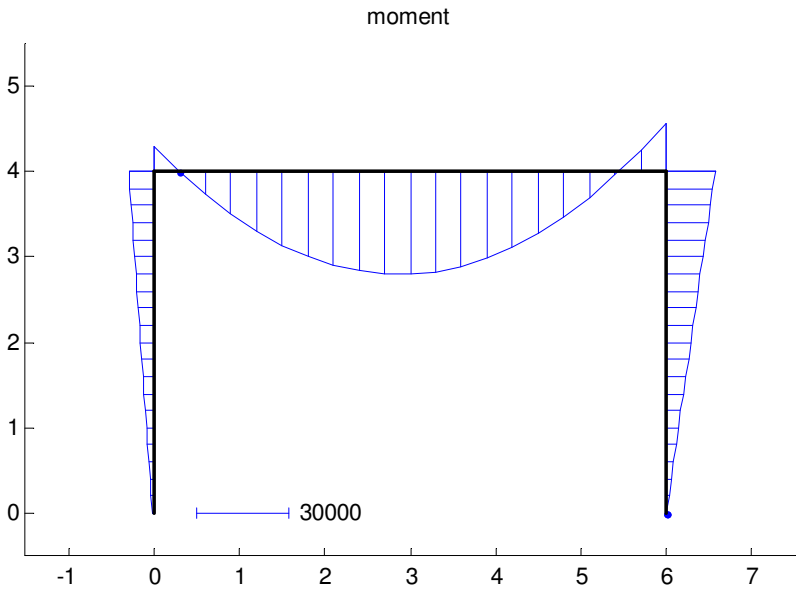


```
% example exs6
 % ------------------
 %  PURPOSE
 % Analysis of a plane frame.
 % ---------------
```




shear force
5
4
3
2
1
0
30000
-1
0
1
2
3
4
5
6
7


moment
5
4
3
2
1
0
30000
-1
0
1
2
3
4
5
6
7

```
% calfem
% example exs6
%-------------------------------------------------------------
% PURPOSE     Analysis of a plane frame.
%-------------------------------------------------------------
 Edof=[1  4  5  6 1  2  3;
       2  7  8  9 10 11 12;
       3  4  5  6  7  8  9];
%----- Stiffness matrix K and load vector f ---------------------
 K=zeros(12);   f=zeros(12,1);    f(4)=2e+3;
%----- Element stiffness and element load matrices  -------------
 E=200e9;
 A1=2e-3;     A2=6e-3;
 I1=1.6e-5;   I2=5.4e-5;
 ep1=[E A1 I1];  ep3=[E A2 I2];
 ex1=[0 0];      ex2=[6 6];      ex3=[0 6];
 ey1=[4 0];      ey2=[4 0];      ey3=[4 4];
 eq1=[0 0];

 eq2=[0 0];  eq3=[0 -10e+3];


 Ke1=beam2e(ex1,ey1,ep1);
 Ke2=beam2e(ex2,ey2,ep1);
 [Ke3,fe3]=beam2e(ex3,ey3,ep3,eq3);
%----- Assemble Ke into K ---------------------------------------
 K=assem(Edof(1,:),K,Ke1);
 K=assem(Edof(2,:),K,Ke2);
 [K,f]=assem(Edof(3,:),K,Ke3,f,fe3);
%----- Solve the system of equations and compute reactions ------
 bc=[1 0; 2 0; 3 0; 10 0; 11 0];
 [a,r]=solveq(K,f,bc)
```

```
%----- Section forces --------------------------------------------
 Ed=extract(Edof,a);
 es1=beam2s(ex1,ey1,ep1,Ed(1,:),eq1,21)
 es2=beam2s(ex2,ey2,ep1,Ed(2,:),eq2,21)
 es3=beam2s(ex3,ey3,ep3,Ed(3,:),eq3,21)

  %----- Draw deformed frame ---------------------------------------
 figure(1)
 plotpar=[2 1 0];
 eldraw2(ex1,ey1,plotpar);
 eldraw2(ex2,ey2,plotpar);
 eldraw2(ex3,ey3,plotpar);
 sfac=scalfact2(ex3,ey3,Ed(3,:),0.1);
 plotpar=[1 2 1];
 eldisp2(ex1,ey1,Ed(1,:),plotpar,sfac);
 eldisp2(ex2,ey2,Ed(2,:),plotpar,sfac);
 eldisp2(ex3,ey3,Ed(3,:),plotpar,sfac);
 axis([-1.5 7.5 -0.5 5.5]);
 pltscalb2(sfac,[1e-2 0.5 0]);
 axis([-1.5 7.5 -0.5 5.5]);
 title('displacements')

%----- Draw normal force diagram -------------------------------

 figure(2)
 plotpar=[2 1];
 sfac=scalfact2(ex1,ey1,es1(:,1),0.2);
 eldia2(ex1,ey1,es1(:,1),plotpar,sfac);
 eldia2(ex2,ey2,es2(:,1),plotpar,sfac);
 eldia2(ex3,ey3,es3(:,1),plotpar,sfac);
 axis([-1.5 7.5 -0.5 5.5]);
 pltscalb2(sfac,[3e4 1.5 0]);
 title('normal force')
```