Stellar Structure – New Format Exam 2008 – Sample Paper

Section A You should attempt all questions; 5 marks per question.

A1. A binary star system consists of two stars, one of which has twice the luminosity of the other. If the apparent bolometric magnitude of the *system* is 7.5, what are the apparent bolometric magnitudes of the individual stars?

The more luminous star is slightly evolved and has a radius twice that of its main-sequence companion, whose effective temperature is 10^4 K. What is the effective temperature of the evolved star?

A2. The Lane-Emden equation for a polytrope of index n is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,$$

in the usual notation. Show that for a polytrope of uniform density $(n = 0)$ the solution of this equation that satisfies the boundary condition $\theta = 1$ at $\xi = 0$ is

$$\theta = 1 - \frac{\xi^2}{6}.$$

What is the value of ξ at the surface of this polytrope?

A3. A fully convective low-mass star may be approximated by a polytrope of index 1.5, so that $P \propto \rho^{5/3}$, where P, ρ are pressure and density. Use a simple scaling argument to find a mass-radius relationship for objects of this kind.

A4. Explain briefly how a star evolves after the exhaustion of hydrogen at its centre, distinguishing between the cases of a radiative core and a convective core. Show that in either case the central core is isothermal.

Section B Answer 2 questions; each question is worth 15 marks.

B1. Derive the three structure equations

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dL}{dr} = 4\pi r^2 \rho\epsilon$$

for a spherical star, including appropriate sketches to illustrate your arguments and carefully defining all the symbols. [7]

Given that the fourth structure equation for a radiative star is

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{16\pi a c r^2 T^3},$$

where T is the temperature, κ is the opacity, a is the radiation density constant and c is the speed of light, write down four simple boundary conditions for these four differential equations. [2]

Under what circumstances is it necessary to improve the surface boundary conditions, and what form do the improved boundary conditions take? Your answer should include definitions of optical depth τ and effective temperature T_{eff}. [6]

B2. Make appropriate use of the Heisenberg Uncertainty Principle to define a quantum state for free particles and hence show that the number of quantum states in a volume V and with magnitude of momentum between p and $p + dp$ is $V 4\pi p^2 dp/h^3$. [3]

How many electrons can occupy a quantum state? Define a completely degenerate electron gas and use the above result to show that the electron density in such a gas is given by

$$n_{\text{e}} = \frac{8\pi p_0^3}{3h^3},$$

where p_0 is the Fermi momentum, which you should define. [5]

The pressure of an electron gas is given by kinetic theory as

$$P = \frac{1}{3}\int_0^\infty \frac{N(p)}{V} p v_{\text{p}} dp,$$

where $N(p)dp$ is the number of electrons in a volume V with speed v_{p} and with magnitude of momentum between p and $p + dp$. Hence show that the relations between P and n_{e} for a completely degenerate electron gas in the two extreme cases of non-relativistic electrons and ultra-relativistic electrons are, respectively,

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_{\text{e}}} n_{\text{e}}^{5/3}$$

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hc n_{\text{e}}^{4/3},$$

where m_{e} is the mass of the electron and c is the speed of light. [4]

Sketch the general relationship between radius and mass for a set of stars of different total mass whose internal pressures are given by non-relativistic degenerate electrons; what observed stars follow a relationship of this kind? Show that when the electrons are non-relativistic the mean density of such a star is proportional to the square of its total mass. [3]

B3. Why is radiation normally much more efficient at carrying energy through a star than conduction, and why is this at first sight surprising? Under what circumstances may conduction be important? [4]

By considering two surfaces at distances r and $r + \delta r$ from the centre of a star, where δr is equal to the photon mean free path l, show that the energy flux carried by radiation, F_{rad}, can be written as

$$F_{\mathrm{rad}} = -\frac{ac}{\kappa\rho} T^3 \frac{dT}{dr},$$

where a is the radiation density constant, c is the speed of light, κ is the opacity of the stellar material of density $\rho(r)$ and temperature $T(r)$. [6]

Now consider a small element of stellar material that is displaced upwards with a speed much less than the speed of sound. Assuming that it moves adiabatically, show by comparing conditions inside and outside the element after it has risen by a small amount δz that it will continue to move upwards if

$$\frac{\rho}{\gamma P}\frac{dP}{dz} < \frac{d\rho}{dz},$$

where P is the pressure and γ is the ratio of specific heats. [4]

Under what circumstances is this criterion likely to be satisfied, and whereabouts in a star are they likely to occur? [1]