The Equation of Radiative Transport

The intensity of radiation $I_\nu(\mathbf{x}, \mathbf{k})$ at a point $\mathbf{x}$ in a star in equilibrium is defined by

$$dE = I_\nu(\mathbf{x}, \mathbf{k})\, dS\, d\omega\, d\nu\, dt \tag{3.5}$$

where dE is the energy crossing an area dS perpendicular to the vector $\mathbf{k}$, in solid angle $d\omega$ about $\mathbf{k}$, in frequency range $d\nu$, in time dt. This intensity function is almost isotropic and almost equal to the Planck function B_ν (equation (1.2)). We can therefore write

$$I_\nu(\mathbf{x}, \mathbf{k}) = B_\nu(\mathbf{x}) + \delta_\nu(\mathbf{x}, \mathbf{k}) \tag{3.6}$$

where $\delta_\nu (\ll B_\nu)$ is the slight departure from isotropy that leads to a net flow of energy through the star.

As the radiation moves a distance ds along $\mathbf{k}$, I_ν changes, for three reasons:

1. Radiation is absorbed.

   We define a mass absorption coefficient $\kappa_\nu(\mathbf{x})$ by

   $$(dI_\nu)_{abs} = -\kappa_\nu \rho I_\nu\, ds; \tag{3.7}$$

   κ_ν is independent of $\mathbf{k}$, so long as the absorbing atoms have a random distribution of orientations.

2. Radiation is scattered, both into and out of direction $\mathbf{k}$.

   If we introduce a mass scattering coefficient $\sigma_\nu(\mathbf{x})$, we can write

   $$(dI_\nu)_{scatt} = -\sigma_\nu \rho I_\nu ds + \sigma_\nu \rho \left[\int I_\nu(\mathbf{k}') p\left(\mathbf{k}, \mathbf{k}'\right) d\omega' \right] ds \tag{3.8}$$

   where $p\left(\mathbf{k}, \mathbf{k}'\right)$ is the probability that radiation scattered from direction $\mathbf{k}'$ goes in direction $\mathbf{k}$ and $d\omega'$ is an element of solid angle about $\mathbf{k}'$.

3. Radiation is emitted.

   We define $j_\nu(\mathbf{x}, \mathbf{k})$ to be the energy emitted per unit mass per second per unit frequency range per unit solid angle about $\mathbf{k}$. Then

   $$(dI_\nu)_{em} = j_\nu \rho\, ds\,. \tag{3.9}$$

   With this definition, j_ν is not isotropic. However, it has two parts:

   (a) spontaneous emission, which *is* isotropic, and

   (b) stimulated emission, which follows absorption and is in the same direction as the absorbed radiation.

In conditions close to thermodynamic equilibrium, we have (see appendix)

$$(j_\nu)_{stim} = \kappa_\nu I_\nu \exp(-h\nu/kT) \tag{3.10}$$

and so stimulated emission may be formally regarded as a negative absorption. It is therefore convenient to introduce the isotropic emission coefficient

$$j'_\nu = j_\nu - (j_\nu)_{stim} \tag{3.11}$$

and to write

$$\kappa'_\nu = \kappa_\nu \left(1 - \exp(-h\nu/kT)\right). \tag{3.12}$$

The total change in I_ν can then be written (using equations (3.7), (3.8) and (3.9)) as:

$$dI_\nu = j'_\nu \rho ds - \kappa'_\nu \rho I_\nu ds - \sigma_\nu \rho I_\nu ds + \sigma_\nu \rho \left[\int I_\nu(\mathbf{k}') p\left(\mathbf{k}, \mathbf{k}'\right) d\omega'\right] ds. \tag{3.13}$$

In thermodynamic equilibrium, $I_\nu = B_\nu$ and j'_ν and κ'_ν satisfy Kirchoff's law $j'_\nu = \kappa'_\nu B_\nu$. Since conditions in stellar interiors are close to equilibrium, we can write

$$j'_\nu = \kappa'_\nu B_\nu + \delta'_\nu(\mathbf{x}) \tag{3.14}$$

where $\delta'_\nu \ll \kappa'_\nu B_\nu$.

We now substitute the expressions (3.14) and (3.6) for j'_ν and I_ν into equation (3.13) and obtain approximately (after some cancellation)

$$\frac{dB_\nu}{ds} = \delta'_\nu \rho - \kappa'_\nu \rho \delta_\nu - \sigma_\nu \rho \delta_\nu + \sigma_\nu \rho \int \delta_\nu(\mathbf{k}') p\left(\mathbf{k}, \mathbf{k}'\right) d\omega' \tag{3.15}$$

where we have used the facts that B_ν is isotropic and that $\int p\left(\mathbf{k}, \mathbf{k}'\right) d\omega' = 1$, and have neglected $d\delta_\nu/ds$. (The latter approximation is justified by noting that, in equilibrium (where $\delta_\nu \equiv 0 \equiv \delta'_\nu$), $dB_\nu/ds = 0$ (i.e. the temperature is the same everywhere). This means that dB_ν/ds is of first order of smallness in the actual solution and so $d\delta_\nu/ds$ is of higher order and can be neglected.)

If there is symmetry between forward and backward scattering (in other words, $p\left(\mathbf{k}, \mathbf{k}'\right) = p\left(\mathbf{k}, -\mathbf{k}'\right)$; but $(dB_\nu/ds')_{\mathbf{k}'} = -(dB_\nu/ds')_{-\mathbf{k}'}$), it can be verified that

$$\delta_\nu = -\frac{1}{(\kappa'_\nu + \sigma_\nu)\rho} \frac{dB_\nu}{ds} + \frac{\delta'_\nu}{\kappa'_\nu} \tag{3.16}$$

is a solution of equation (3.15). (Given that δ'_ν, κ'_ν and σ_ν are isotropic, the only term which does not obviously cancel or vanish is

$$-\frac{\sigma_\nu}{(\kappa'_\nu + \sigma_\nu)} \int \left(\frac{dB_\nu}{ds'}\right)_{\mathbf{k}'} p\left(\mathbf{k},\ \mathbf{k}'\right) d\omega'$$

and this vanishes if there is symmetry between forward and backward scattering, which is true for the most important scattering process in stellar interiors, scattering by free electrons.)

Thus we have

$$I_\nu(\mathbf{x},\mathbf{k}) = B_\nu(\mathbf{x}) - \frac{1}{(\kappa'_\nu + \sigma_\nu)\rho}\frac{dB_\nu}{ds} + \frac{\delta'_\nu(\mathbf{x})}{\kappa'_\nu}. \tag{3.17}$$

Only the second of these terms is not isotropic and contributes to the net flow of radiation.

In spherical symmetry

$$\frac{dB_\nu}{ds} = \cos\theta\frac{dB_\nu}{dr} = \cos\theta\frac{dB_\nu}{dT}\frac{dT}{dr}.$$

where θ is the angle between the direction of $\mathbf{k}$ and the outward radial direction: $dr = ds\cos\theta$ where ds is an element of length along the direction $\mathbf{k}$. Also the net flow of radiation, at all frequencies, across a spherical surface of radius r is

$$L(r) = \int_0^\infty L_\nu(r)\,d\nu = \int_0^\infty \left[4\pi r^2 \int I_\nu(r,\mathbf{k})\cos\theta\,d\omega\right] d\nu. \tag{3.18}$$

Hence, using

$$\int \cos\theta\,d\omega = \int_0^{2\pi}\int_0^{\pi} \cos\theta\sin\theta\,d\theta d\phi = 0$$

$$\int \cos^2\theta\,d\omega = \frac{4\pi}{3}$$

and the expression for I_ν, we have

$$L(r) = -\frac{16\pi^2 r^2}{3\rho}\frac{dT}{dr}\int_0^\infty \frac{dB_\nu}{dT}\frac{d\nu}{(\kappa'_\nu + \sigma_\nu)}. \tag{3.19}$$

Since

$$\int_0^\infty \frac{dB_\nu}{dT}\,d\nu = \frac{d}{dT}\int_0^\infty B_\nu\,d\nu$$

and

$$\int_0^\infty B_\nu\,d\nu = \frac{\sigma T^4}{\pi}$$

where $\sigma\,(= ac/4)$ is the Stefan-Boltzmann constant (a being the radiation density constant), we can write this as

$$L(r) = -\frac{16\pi acr^2T^3}{3\kappa\rho}\frac{dT}{dr} \tag{3.20}$$

where the opacity κ is defined by

$$\frac{1}{\kappa} = \int_0^\infty \frac{dB_\nu}{dT}\frac{d\nu}{(\kappa'_\nu + \sigma_\nu)} \Bigg/ \int_0^\infty \frac{dB_\nu}{dT}\,d\nu \tag{3.21}$$

and is known as the *Rosseland mean opacity*, κ_R. It is essentially an average of the frequency dependent conductivity, weighted with B_ν (the number of photons with frequency ν). The d/dT arises because energy is flowing down a temperature gradient. Energy transport by conduction can be formally included in the expression (3.20) if we write

$$\frac{1}{\kappa} = \frac{1}{\kappa_R} + \frac{3\rho\lambda_{cond}}{4acT^3}\,. \tag{3.22}$$

The expression (3.20) is more usually written as a differential equation for the temperature gradient:

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{16\pi acr^2T^3}\,. \tag{3.23}$$

Appendix: Expression for $(j_\nu)_{stim}$

In general

$$\frac{d(I_\nu)_{stim}}{d(I_\nu)_{abs}} = \frac{(j_\nu)_{stim}\rho ds}{\kappa_\nu I_\nu \rho ds} = \frac{(j_\nu)_{stim}}{\kappa_\nu I_\nu}.$$

Close to thermodynamic equilibrium, we also have

$$\frac{d(I_\nu)_{stim}}{d(I_\nu)_{abs}} = \frac{N_2B_{21}}{N_1B_{12}}$$

where N_1 and N_2 are the populations of the lower and upper energy levels of the transition, and B_{12} and B_{21} are respectively the Einstein coefficients for absorption and for stimulated emission (this equation assumes that the line shapes are the same for emission and absorption; i.e. the transition probabilities have the same dependence on frequency). If g_1 and g_2 are the statistical weights of the energy levels, and their energies are E_1 and E_2, we have

$$\frac{B_{21}}{B_{12}} = \frac{g_1}{g_2}$$

and

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}\exp(-(E_2 - E_1)/kT) = \frac{g_2}{g_1}\exp(-h\nu/kT)$$

where ν is the frequency of the transition. Hence equation (3.10).