To appear in: A.-V. Pietarinen et al. (Eds.): Diagrams 2020, LNAI 12169, pp. 93–109, 2020.
https://doi.org/10.1007/978-3-030-54249-8_8

A sketch of a theory and modelling notation for elucidating the structure of representations

Peter C-H. Cheng 0000-0002-0355-5955

Department of Informatics, University of Sussex, Brighton, UK.
p.c.h.cheng@sussex.ac.uk

Abstract. A structural theory of visual representations and an accompanying modelling notation are outlined. The theory identifies three types of fundamental representational components, specified as cognitive schemas, that span internal mental and external physical aspects of representations. The structure of diverse and complex example representations are analyzed. Twenty-three requirements that a general theory of representations must address are presented. The new theory satisfies the large majority of them. The theory and notation may provide a basis for future methods to predict the efficacy of representations.

Keywords: Representations, structural theory, compositional analysis, diagrams, notations, cognitive explanation.

1 Introduction

This sentence that your eyes are running over is a representation. Figures 1, 2 and 3, which are explained in section 2, are other representations. The set of numbers that index the section and subsection headings here is yet another. All these examples are different, but they are but a small sample of the vast diversity of existing visual representations. What do all representations have in common that makes them representations? How do they differ as representations irrespective of the domains they encode? Being able to answer these questions will allow us to study representations more systematically than is possible currently, and in the future to ask hard questions such as: How can we choose representations to suit individuals with different levels of domain knowledge and experience of representations, for specific problems, in particular domains [24]? How can we systematically invent novel representations [7]?

The nature of representations is an enduring and important subject for study. Classifications and taxonomies of representations have been proposed [18, 10, 11]. Accounts of representations have been given in terms of formal attributes using formal languages [1, 13, 20, 23, 27, 28], cognitive properties [4, 8, 14, 17, 19, 23], graphical or perceptual attributes [2, 8, 29], and properties of information [5, 10, 21, 32, 33, 34]. For sake of illustration each reference has been exclusively cited in just one of the informal categories above, but many of them span multiple categories. See [3] for a meta-taxonomy of representation classifications.

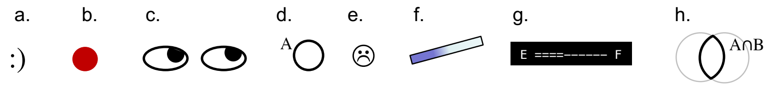

Fig. 1. Simple representations: (a) original smiley; (b) painting sold (c) "look there"; (d) set *A*; (e) modern emoticon; (f) litmus paper; (g) fuel gauge, (h) intersection of two sets.

Despite the numerous accounts of representations, none appears sufficient to address even the first set of challenging questions posed in the first paragraph. None have the scope to systematically analyze any representation, for any domain, and at any level of user competence. Thus, the first aim of this paper is to specify a set of characteristics for a general and operationalizable theory of representations; a set of stringent requirements that an adequate theory must satisfy (section 3). The second aim of the paper is to outline a cognitive theory with an accompanying modelling notation for the analysis of notations and visual representations (section 4). The theory posits three fundamental representational components from which all representations are built. An demonstration of the utility of the theory and notation is provided (section 4) by using them to analyze the structure of the diverse examples of representations to be introduced in the next section. The potential uses and limitations of the theory and notation are discussed at the end (section 6).

2 Sample representations

Here are some representations to serve as ongoing examples in the following sections.

$$x_{1,2} = -b \pm \frac{\sqrt{b^2 + 4ac}}{2a}$$

Fig. 2. Solutions to the quadratic equation.

Simplest representations. Fig. 1 shows some elementary representations. Fig. 1a is a smiley from early in the history of email before the idea of a meta-comment on some text was extended to a whole range of emoticons (Fig. 1e). Fig. 1b is a red dot on the frame of a painting in an art gallery indicating that it has been sold. Fig. 1c was printed on a flier to capture peoples' attention. Fig 1d is one set from a Venn diagram. Fig. 1f is a piece of litmus paper whose purple end registers a pH of 9. Fig. 1g is a fuel gauge whose segments show that the tank is 4/10ths full. Fig. 1h is an intersection of a two set Venn diagram.

Equation representation. Fig. 2 is the formula for the roots of the quadratic equation. It is primarily a sentential representation, a linear concatenation of symbols, that encodes mathematical meaning through the chosen symbols and syntactic rules. Note that it has multiple occurrences of some variables and that it encodes two solutions.

Thermodynamics graph. Fig. 3 is a "monster" representation that engineers use to understand how the Second Law of thermodynamics determines the efficiency of steam engines [12, 25]. It is a graph with axes for entropy, *s*, temperature, *T*, and pressure, *P*. Under different conditions water will be liquid, vapour or a mixture of the two. The bell curve, or *saturation dome*, marks the transition between these states. Under the saturation dome, the dryness fraction, *x*, gives the proportion of vapour to liquid. The operation of steam engines (running Rankine cycles) are shown by the dashed rectangles. Each side of the rectangle stands for a thermodynamic process: 1-2 – pressurizing the mixture so it all turns to liquid; 2-3 – heating the water until it is all vapour; 3-4 – heat in the vapour is transformed into mechanical energy by a turbine, which turns the vapour back in to a mixture; 4-1 – further heat is released in a condenser to turn most of the vapour back in to liquid, so the cycle can repeat. From the Second Law we can determine the efficiency of a cycle in two ways. First, the energy put into the water by heating is given by the area under line 2-3 (down to T=0 K) and the heat extracted in the condenser is the area under line 4-1, so the energy produced by the engine is the area of the rectangle, hence the efficiency is that area divided by the total heat input, the area under line 2-3. The second way to compute efficiency is to divide T_2-T_1 by T_2, which can be computed by comparing the length of line 1-2 with the altitude of T_2.

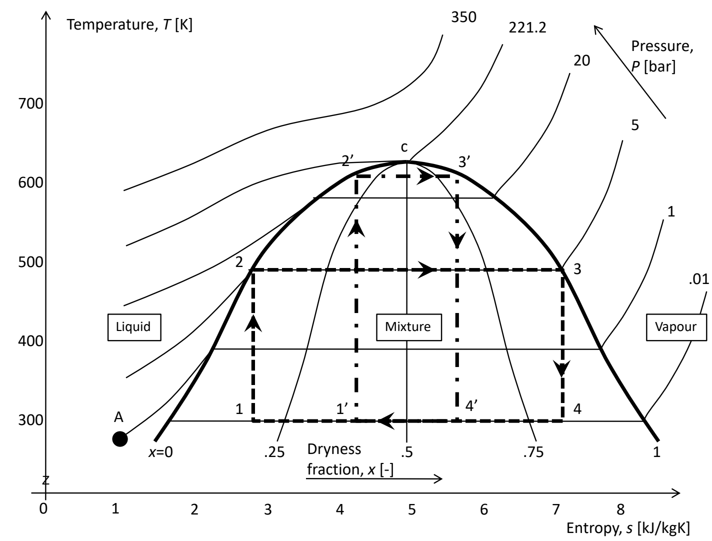

Fig. 3. A monster thermodynamics graph.

3 Criteria for a theory of representation structure

What should a general broad scope theory of representations encompass? Five groups of requirements that a structural theory should address are considered, ranging from fundamental properties of representations to pragmatic concerns about their utility. Some of these are implicit common assumptions or just basic tenets of good science, but where they are explicitly identified in the literature references are given.

3.1 Fundamental requirements

These aspects are essential things that a theory must encompass, such that in their absence we would not consider the theory to be an account of representations.

R1. Represented world: a representation encodes knowledge about some domain, a *represented world*, including general information about objects, properties, values and relations. [22]

R2. Representing world: a representation has graphical parts, including icons and words, that do the encoding of the things from the represented world. [22]

R3. Compositional structures: represented and representing worlds are typically rich compositional structures, which are often hierarchical. [16]

R4. Semantics: a representation represents, so knowing the encoding relations between the things in the represented and the representing worlds is critical. [26]

R5. Syntax: a representation has rules that govern the valid configurations of graphical parts that are potentially meaningful. [10]

R6. Supplementary structure. A representation may include graphical parts that are not intended to encode domain information, and not covered by the representation's syntax, but are an integral part of the representing world for practical reasons (e.g., the typeface of this text and locations of the line returns).

3.2 Interpretation requirements

This set of requirements concerns the rich ways in which a representation may encode meaning, and constraints on such encodings, that a theory must recognize.

R7. Parsimony: the theory should propose a small number of types of components that should be the same across all classes of representations.

R8. Multi-level interpretation: as representations are compositional, their meaning can be interpreted at multiple levels; from domain elements, represented by graphic elements, through to high level general abstractions, represented by complex expressions. [1, 6]

R9. Alternative interpretations: representations can support interpretations from different perspectives depending on the user's goals and knowledge [14]: e.g., one may view a representation as a composition of its components, or one may think of mutually interacting constraints among components.

R10. Alternative representations: a domain may be encoded in representations with different structures (e.g., 24 hour versus AM/PM time of day formats).

R11. Alternative domains: the same representation may be used to encode quite distinct domains (e.g., mathematics is a domain general representation).

R12. Cognitive theory compatibility: for human users of representations, a theory of representations must be compatible with our knowledge about human cognition in general, including perception, memory, thinking and learning.

R13. Theories about information and knowledge: a theory of representations should be compatible with accounts of the nature of particular kinds of information and knowledge; e.g., Steven's [31] analysis of quantity scales.

3.3 Scope of theory requirements

These are requirements about the coverage or scope of a theory of representations.

R14. Representation scope: a theory should cover all types of representations, although, below we focus on static visual notations and diagrams.

R15. Domain scope: a theory should address representations from any domain.
R16. Complexity: a theory should span representations of all degrees of complexity; e.g., from Fig. 1 to Fig. 3, and beyond.

3.4 Existing representational theory requirements

This single requirement recognizes that much has been discovered about the nature of representations, some empirically verified. So, a theory must either make equivalent predictions about previous findings or be able to interpret such existing accounts.

R17. Embody existing theories. Three illustrative examples: *Locational indexing*: Diagrams are (sometimes) superior to sentential representations, because they use spatial associations to establish relations among elements [16]. *Isomorphism*: prefer representations that use just one distinct symbol in a display to stand for one distinct domain concept (i.e., isomorphic) rather than one-to-many or many-to-one mappings [1, 13, 21, 32]. *Separable dimensions*: favour visual properties that are naturally *separable*, because they depend upon different perceptual processes and so demand less conscious effort to distinguish [33].

3.5 Utility requirements

This final set considers features expected of a valuable and effective theory.

R18. Explanatory: the theory should provide analysis of representations that predict their likely efficacy and explain why they are so or otherwise.
R19. Design patterns: the theory should identify general patterns of representational structures that serve similar functional roles, because they encode similar types of concepts in equivalent ways.
R20. Precise components: the components of the theory should be well-defined and clearly distinct from each other.
R21. Precise sub-components: similarly, subclasses of components should be well-defined and clearly distinct from each other.
R22. Analysis rules: clear operational rules to guide the analysis of representations should be provided.
R23. Functional components: the theory should readily identify those components of a representation that are core to its function as a representation, in contrast to superfluous decoration or "chart junk".

In what follows, numbers in curly brackets, e.g., {R23}, refer to specific requirements.

4 Structural theory of representations and modelling notation

The proposed theory draws its inspiration from molecular theory in chemistry that explains the diverse properties of countless substances in terms of structures composed of distinct elements. What are the representational elements – fundamental components – and how are they combined in representational molecules – representational structures {R3}? Three types of fundamental components are proposed {R7}.

4.1 Preliminaries

Before introducing the components, this sub-section gives some terminology.

A *topic* is some part of a larger knowledge domain pertaining to a task on which a user is working with a representation {R1}. The thermodynamic power cycles of a particular type of heat engine is the topic of Fig. 3. Not all the concepts of the topic are necessarily encoded in the representation.

A *display* is the external part of a representation in some physical medium, such as print on paper, pixels on a computer screen, or the raised lines on a tactile graphic {R2}.

A *concept* is an idea, fact, category, property, or value of a property from the topic {R1}. Ideas include things such as laws, classifications schemes, constraints, prototypical and extreme cases, which may be complex and hierarchically structured.

A *schema* is a mental cognitive structure that encodes categorial information as a collection of *slots*, variables, that contain *fillers*, values [26]. The set of slots defines the category and a specific set of fillers instantiates a particular instance of the category. Schemas are widely used in cognitive science to explain how the mind systematically stores and organizes knowledge [26]. Specialized schemas are used for reasoning in domains that combine both propositional and diagrammatic information [6, 15]. The present theory generalizes the idea of such schemas to all representations {R12}.

Graphic-objects and *properties* are visual entities that a viewer of a display takes as separate things or features of the display. Elementary graphic objects are established by our perceptual systems (visual or haptic). Parts of graphic objects may themselves be graphic objects {R7, R8}; e.g., the dryness axis is that part of the *P* curves that are under the bell curve in Fig. 3. Graphic objects may be composites; e.g., an axis comprises a scale, tick marks and labels. Similarly, different features of a graphical object may represent different concepts {R7, R8}; e.g., the height of a rectangle in Fig. 3 is a temperature difference and its area is an amount of heat. The relative height of the two rectangles, with conscious effort, may be interpreted as a composite graphic object.

4.2 Representational components

Tokens, *Representation-dimensions* (R-dimensions), and *Representation-schemes* (R-schemes) are the three proposed types of components {R7}. The fundamental function of these components is to encode and associate information about the target domain and about the display. Each is specified as a schema {R9, R12} and are represented, respectively, in the modelling notation by particular shapes, Figs. 4b, c, & d. The tablet shape icon stands for a whole representation; Fig. 4a. Again, the main purpose of all three components is to coordinate {R4} things from the represented world {R1} with things in the representing world {R2}. The form of the component icons reflects this fundamental idea: the top label of each icon names the encoded concept and the bottom label names the encoding graphic object (or property).

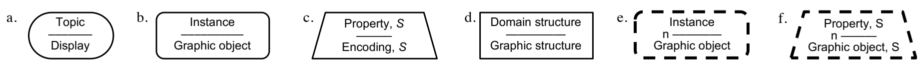

Fig. 4. Icons for (a) representations, (b) Tokens, (c) R-dimensions, (d) R-schemes, (e) multiple Tokens, (f) multiple R-dimensions.

A **Token** is a "fixed" component that pairs (A) one concept and (B) one graphic object. It's icon is lozenge shaped, Fig. 4b. Examples include: (i) Figs. 1a-d and their associated concepts; (ii) a letter and its domain variable (e.g., *T – temperature*); (iii) a rectangle and a cycle in Fig. 3; (iv) the whole equation in Fig. 2 and two quadratic equation solutions; (v) in Fig. 1h, the middle zone of the Venn diagram and an intersection.

Concept: 1 object, instance or value
Graphic: 1 graphic object
Function: *semantic, auxiliary, arbitrary*
Explicit: *yes*, *no*
Tokens: 0 or more
R-dimensions: 0 or more
R-schemes: 0 or more

Fig. 5. Token schema. Slot names in bold. Names of specific fillers in italic.

A Token schema has seven slots. One pair of slots is for the concept and the graphic object {R4}. Three slots hold Tokens, R-dimensions and R-schemes that are directly associated, children, of the Token. This captures local connections that on the large scale define the overall network structure of components {R3} (see below). The other slots differentiate further features of a token {R21}. The *function* slot specifies the role of the token in the representation. A *semantic* filler means that the Token encodes a domain concept. A Token whose function is to pragmatically aid the interpretation of the representation but is not itself semantic has *auxiliary* as a filler {R6}: e.g., commas grouping triplets of digits in long numbers; the size and position of circles in a Venn diagram (whereas an overlap of circles is semantic). An *arbitrary* filler indicates the Token is neither semantic nor auxiliary, so merely serves a decorative or aesthetic purpose. In an icon for a concept-less Token, the concept label is replaced by a '##' symbol. The *explicit* slot specifies whether a Token's graphic object is physically present in the display or is to be imagined by the user; e.g., envisage a new *P* curve in Fig. 3 between a pair of printed isobars. In icons for such imagined concepts, a '##' symbol is used in place of a label to denote the absence of a graphic object.

A Representation-dimension (R-dimension) is a component that deals with "variation" in a class of Tokens. It pairs (A) a concept that can take alternative values of one domain property with (B) some means of encoding alternative values {R4}. The R-dimension icon is a trapezium, Fig. 4c. R-dimension examples include: (i) pH values as colours; (ii) metered quantities as numbers of bars on a scale, Fig. 1g; (iii) alternative emotions depicted by different emoticons; (iv) the basic relations between sets encoded by spatial configurations of circles (separate, overlapping (Fig. 1h), embedded); (v) values of some property as a x-coordinate (horizontal) position in a graph; (vi) quantities of energy as areas of rectangles in Fig. 3; (vii) a list of some facts.

Concept: domain property
Concept-scale: *nominal* (N), *ordinal* (O), *interval* (I), *ratio* (R)
Concept-attributes: e.g., max, min, magnitude range, %
Graphic: graphic property, axis, sub-notation (see text)
Graphic-scale: *nominal* (N), *ordinal* (O), *interval* (I), *ratio* (R)
Graphic-attributes: e.g., graphic range, type (linear, logarithmic)
Scope: *global*, *local*
Function: *semantic, auxiliary, arbitrary*
Explicit: *yes*, *no*
R-dimension: 0 or more R-dimensions as sub-dimensions
Tokens: 1 or more

Fig. 6. R-dimension schema. Specific fillers values in italic.

The R-dimension schema has eleven slots. The first three *concept* slots name the concept, specify its type of quantity scale, and give selected attributes of the quantity {R1}.

Three matching *graphic* slots do the same for the means of encoding {R2}. For a *concept-scale* or *graphic-scale* slot, the fillers are one of Steven's [31] types of quantities: in the icon the letters following the concept and the graphical object names give the scale type (i.e., replace *S* in Fig. 4c, with *N*, *O*, *I*, or *R*). The scale types of the concept and graphical object may differ. Forms of encoding may be graphic properties and also more complex graphic structures, such as an axis of a Cartesian graph or a sub-notation (e.g., a numeration system or an indexing scheme) {R5}.

Three further slots are used to define types of R-dimensions {R21}. The *function* slot is the same as the function slot in the Token schema; for example, position in a simple unordered list is an *auxiliary* R-dimension, because positions merely differentiate Tokens but do not encode a domain concept. The function slot is important because it has a key role in distinguishing sentential from diagrammatic representations (see below). The *explicit* slot specifies whether the R-dimension is physically represented in the display or must be imagined; e.g., *no* fills the slot for the R-dimension standing for areas beneath curves in Fig. 3, because the origin of the *T* axis is below the *s* axis scale. The *scope* slot specifies whether the R-dimension covers the whole display or is more localized (e.g., in Fig. 3, *T*, *s*, and *P* are global, whereas *x* is local). The *Token* slot holds at least one token whose value is encompassed by the R-dimension. The *R-dimensions* slot identifies any subparts of an R-dimension that happen to be specifically meaningful (e.g., the excess revs red zone of a tachometer.)

A Representation-scheme (R-scheme) is a "structural" component that pairs (A) a conceptual structure of the domain with (B) a graphical structure that is more complex than an R-dimension. Examples of R-schemes include: (i) a graph co-ordinate system (e.g., in Fig. 3 T-s is Cartesian); (ii) a table's grid of rows and columns; (iii) Hindu-Arabic numbers that exploit digit position as power and digit shape as numerosity [33]; (iv) the coordination of labels and zones in a Venn diagram (e.g., Fig. 1h); (v) the combination of location and shapes of icons in a map; (vi) multi-dimensional index systems (e.g., Library of Congress book classification scheme).

The R-scheme icon is rectangular, Fig. 4d, and its schema has nine slots. The *concept-structure* and *graphic-structure* slots name their respective target structures {R4}. Again, the schema has *function, scope* and *explicit* slots that are equivalent to those slots in R-dimensions {R21}. The *Tokens*, *R-dimensions*, and *R-schemes* slots hold the constituents of the R-scheme. The relations among an R-scheme's components may be complex, so the names in the concept-structure and graphic-structure slots are pointers to descriptions of them (e.g., in the modelling notation or text). The *Organization* slot contains a description of how concept structures and graphic structures are related {R4}. A R-scheme cannot contain a single R-dimension alone; it would be an R-dimension. An ordered list is an R-scheme, because it combines a categorical R-dimension to identify different items with an ordinal R-dimension for the priority of the

Domain-structure: relations among domain concepts
Graphic-structure: graphical structure
Function: *semantic*, *auxiliary*, *arbitrary*
Scope: *global*, *local*
Explicit: *yes*, *no*
Tokens: 0 or more
R-dimensions: 2 or more, 1 if also ≥1 Tokens
R-schemes: 0 or more
Organization: specifies relations among components

Fig. 7. R-scheme schema.

items. The reading directions of text, such as top-to-bottom then right-to-left in traditional Chinese, is an auxiliary function R-scheme.

4.3 Modelling notation

The modelling notation supports the explication of the relations between Token, R-dimensions and R-schemes through the construction of network diagrams to show their organization {R3}. A tablet icon for the representation tops what is generally a tree-like structure. Below each icon for a schema, further schema icons are drawn for the contents of its Token, R-dimension and R-scheme slots. Imagine a simple a bar chart for income in three age bands: Fig. 8 shows one possible interpretation of its structure. Below the representation icon, Fig. 8 (a), it possesses an R-scheme (b) that is a bar chart assembly for data. The graph has a coordinate system R-scheme (c1) in which data values, drawn from a nominal scale R-dimension (c2), are plotted. The coordinate system has an ordinal scale R-dimension for age-bands (d1) and a ratio scale R-dimension for income (d2). The bars for three cases are the Tokens (e1-e3). Fig. 8 may be viewed as a design pattern for this class of representations {R19}.

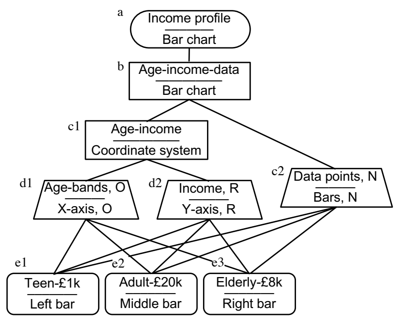

Fig. 8. A sample representation structure.

For compactness in the diagrams, multiple instances of Tokens or R-dimensions of the same kind may be represented with a dashed perimeter, Figs. 4e & 4f, and an expression (replacing *n*) indicates the number of instances. For complex domains, additional information, not defined in the schemas themselves, can be encoded using supplementary notational elements between icons as required (see below).

4.4 Analysis guidelines

From the experience of analyzing more than a dozen disparate representations, here are ten representation analysis guidelines: a step towards {R22}. (1) Consider a target user with a particular level of knowledge of the domain and a certain degree of experience of the class of representation in question. (2) From the perspective of the target user, catalogue domain concepts noting the levels of abstraction and granularity at which they occur, in particular what concepts are fixed and variable things, and what concepts are relations. (3) Select typical examples of displays appropriate to the competence of the target user. (4) Catalogue all the graphical parts of the display that have semantic and auxiliary functions, plus any other parts that may be arbitrary but important; for instance, because they are likely to be misconceived by target users. (5) Using the two catalogues, define schemas for Tokens, including those at higher levels of abstraction or granularity. (6) From the list of Tokens and the catalogues, specify R-dimensions for sets of similar tokens, paying special attention to *concept-scale* and *graphical-scale* slots of the schemas. Unless a token is solitary and stands for an unvarying concept, it will be an instance of some R-dimension. (7) Identify the R-schemes using two approaches. (a) Propose configurations of R-dimensions, perhaps with anchor Tokens, for the primary conceptual relations of the topic, such as its sets of underpinning laws. (b) Sets of R-dimensions, and Tokens, with strong conventional associations may constitute R-schemes. (8) Alternate between bottom-up composition from Tokens and top-down decomposition from R-schemes {R8}. (9) In a drawing package, construct a model with icons for each schema, whilst iteratively working through steps (1) to (8). (10) Revisit steps (1) and (3) in order to explore alternative interpretations of the representation. Consider the overall coherence and parsimony of the structures in order to judge the plausibility of particular interpretations {R9}.

The next section analyses the representations presented in section 2.

5 Examples analyzed

The structures of the examples are modelled in fine detail to demonstrate the rigour of the theory and utility of the notation. However, the reader need not track all the technicalities of the analyses in order to follow the discussion of the implications below.

5.1 Icons and indicator scales

The representations in Fig. 1 are simple. In the historical context of its first use as a meta-comment on some text, the smiley was just a Token without a R-dimension as no other types of this Token existed. The later coining of related emoticons created an R-dimension. In their typical use, "sold" dots (Fig. 1b) are also Tokens, because alternative colours are not used for "unsold" or "under offer". Figs 1c, d, f, & g have R-dimensions that, respectively, vary across (c) gaze direction and pupil position, (d) sets & labeled circles, (f) pH and colour, and (g) fuel level and number of bars. Fig. 1h is a Token with an R-dimension for set relations encoded by degrees of overlapping circles, and in turn is a part of a larger R-scheme for sets and Venn diagrams.

5.2 Equations

Fig. 9 is a representation structure model for Fig. 2, assuming a user who is competent in mathematics. The representation icon is located at C1 in the figure's coordinate system. Overall, the model encodes the idea that the equation is a sentential representation based on a linear concatenation of symbols. Nested R-schemes encode successive layers of expressions that includes R-dimensions and Tokens for mathematical operators, variables, numbers and signs. The overall equation consists of a left hand side (LHS) formula, the equals sign, and a right hand side (RHS) formula, which are encoded by a R-scheme (B3), a Token (C3) and another a R-scheme (F3), respectively. The arrow (D3) indicates the nodes below are ordered. The LHS formula is an elementary expression (*Elem exp*) (B3) consisting of a R-dimension for a sign (A4-5) and a R-dimension for a variable or number (*Var/num*) (B4-5), in this case x. An additional R-scheme for two subscripts that identify alternative solutions is anchored to the x Token (B6-C8). The comma between the numbers is an auxiliary Token (B7).

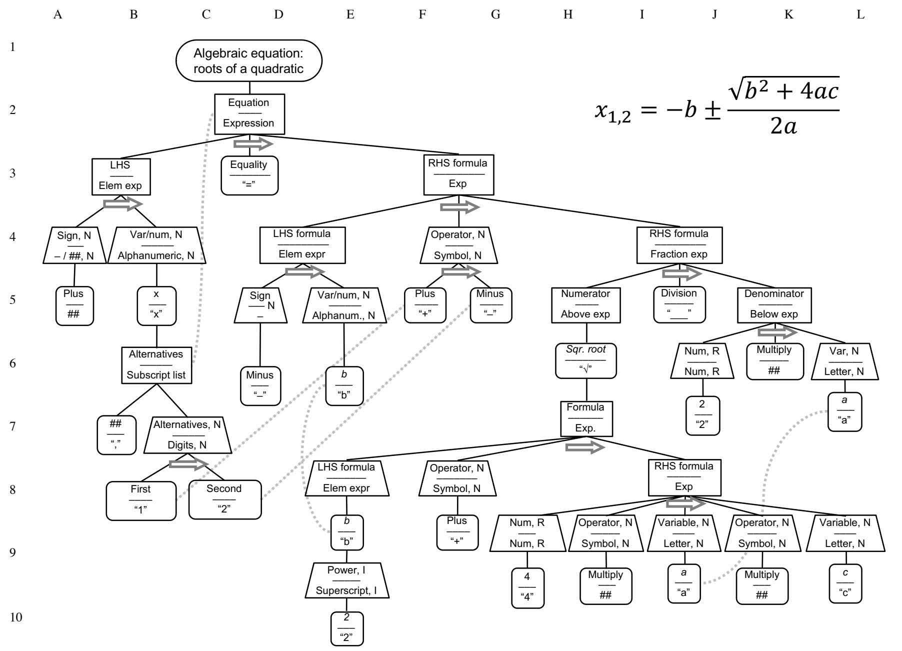

Fig. 9. Representation structure diagram for the equation in Fig. 2 (repeated top right).

The equation's RHS is a formula R-scheme (F3), whose LHS is another elementary expression R-scheme (D4-E6). Unusually, the operator of the RHS formula's is not a Token but a R-dimensions (F4) because it provides options of *plus* or *minus* Tokens (F5, G5). The RHS of the formula is a R-scheme (I4) and notably includes a Token for the square-root that anchors another formula R-scheme (H6). The rest of the decomposition follows in a similar fashion.

Some noteworthy features of the model include: (i) the large number of R-schemes; (ii) ##s note the absence of graphic objects for the multiplication (×) Tokens (e.g., K6); (iii) several concepts are represented multiple times as denoted by the dotted lines.

A key feature of the theory and notation is its ability to support alternative in-

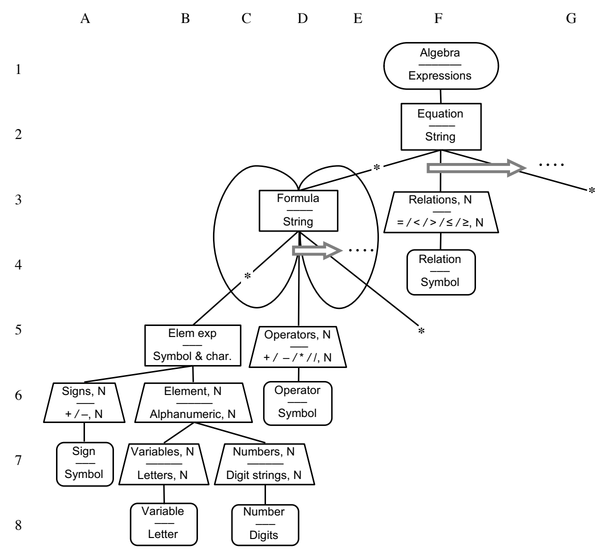

Fig. 10. Representational structure of equations.

terpretations of a representation {R9}. To illustrate this, Fig. 10 gives a representation structure diagram for algebraic equations that abstracts away from all the detail in Figure 9. The overall structure is a tree whose leaves are Tokens for variables, numbers, signs (+/-), or operators. Nodes are R-schemes for expressions or R-dimensions for those Tokens. The top level R-scheme (F2) encodes the relation between two formulas encoded as R-schemes (D3, G3). Various graphical devices are used to concisely encode expressions and recursive structure. A pair of *s on a left branch (E3) and on a corresponding right branch (G3) of a node indicates that the structure on the left is repeated on the right (similarly, C4–F5). The loops from the bottom of a R-scheme to its top (C3-4, E3-4) signify the potential for building expressions recursively. The possible relations between the equation's formulas are held in a nominal R-dimension (F3) and one is the chosen Token (F4). The description of the rest of the model is similar to the lower parts of Fig. 9. Overall, the model captures the idea that equations may be described by a generative set of syntactic rules {R5} .

5.3 Thermodynamics graph

Fig. 11 is the model for Fig. 3. It assumes a user whose is familiar with both thermodynamics and power cycle property graphs. The primary R-scheme (C2) is a x-y-z graph coordinate system, consisting of ratio scale R-dimensions for variables *T*, *s* and *P* (B3, D3, E3), in to which data point Tokens are plotted from a nominal scale R-dimension (A3). The *P* z-axis is encoded by the isobar curves. Vast numbers of Tokens for data points are depicted in Fig. 3 (A4) but just two are specifically noted (C4, D4). The saturation curve, bell dome, is a Token (G4) composed of many data points. The

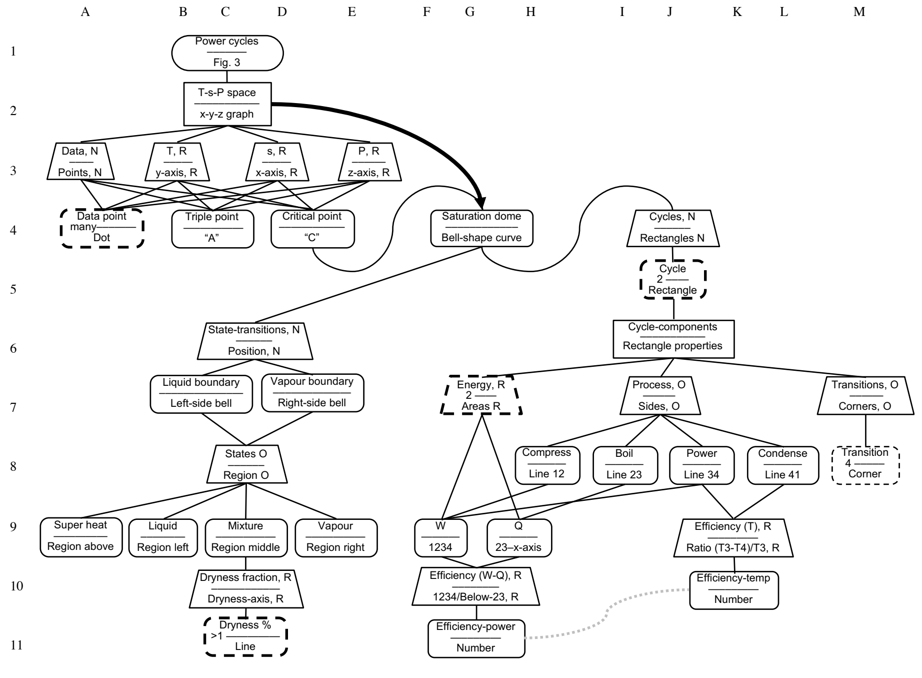

Fig. 11. Representational structure of the thermodynamics graph in Fig. 3.

thick arrow to that Token specifies that it, and subsequent components, inherit all the structure of the *T-s-P* R-scheme. The dome (G4) and critical point (D4) are anchor Tokens for two local nominal R-dimensions (C6, J4). The first of these R-dimensions (C6) encodes state transitions and has two Tokens for the liquid (B7) and the vapour boundaries (D7). In turn, these Tokens serve as anchors for a nominal scale R-scheme (C8), which has four tokens for the states themselves (A9-E9). One of those Tokens, the mixture Token (C9) is an anchor for the local ratio scale dryness fraction R-dimension (C10) and its many possible values (BC11).

The second R-scheme (J4), anchored by the saturation dome, encodes the two power cycle Tokens (J5). Two cycles are presented so that efficiencies can be compared. An R-scheme (J6) is attached to both cycles, which has three R-dimensions for quantities of energy (G7), process stages (J7) and transitions between those stages (M7). The process stage R-scheme provides four Tokens (H8-L8). The ordinal scale transitions R-scheme has four Tokens (I8). Areas defined by the energy ratio scale R-dimension (G7) and certain process Tokens definer quantities of power, *W* (F9), and heat, *Q* (H9). From these an efficiency ratio scale R-dimension (G10) is defined and Tokens for efficiency are given (G11). Further, from just two of the process Tokens (J8, L8) another efficiency R-dimension (K9) is defined on the basis of temperature differences. The two efficiency values (G11, K10) are equal, shown by the grey dotted line.

Fig. 9 is complex because Fig. 3 is complex, but the model reveals interesting things about that complexity. First, Fig. 3 is actually relatively simple as it has only two R-schemes compared to the quadratic solution's nine (Fig. 9). Second, it is a relatively coherent representation: the global coordinate system permeates nearly all aspects of the representation, with all graphical objects interpretable in terms of *T*, s and *P*. Further, the second, local, R-scheme is fully embedded within that main coordinate system. The diagram has multiple R-dimensions for specific aspects of the topic but mappings between the concepts and graphic elements is isomorphic. A seeming exception is the efficiency of the cycles, the dotted line between two Tokens (G11-K10). However, they and their R-dimensions refer to two quite different ways to compute thermodynamic efficiency from the Second Law.

6 Discussion

A theory of representational structure has been proposed with an accompanying modelling notation. The theory is novel in various respects. First, it proposes that all representations are built from just three core components. The fundamental function of these components is *representational*: they serve to integrate domain concepts with graphical structures. Second, the theory focuses on modelling the multi-level structure of representations as relations among Tokens, R-dimensions and R-schemes, rather than positing properties of whole representations as has been common in the literature [e.g., 18, 27]. The guidelines in section 4.4 suggest the possibility of developing systematic approaches to the analysis of representations supported by software tools. Third, in contrast to other approaches that focus on the formal or computational analysis of representations in terms of the composition of basic graphic elements [e.g., 20, 28], the components proposed by the present theory are cognitive schemas posited as psychological entities, whose existence and impact might be empirically tested [e.g., 19, 15, 23]. Further, the present theory occupies the middle ground between descriptive classifications and formal accounts, whilst being usable without extensive formal training. Fourth, the scope of the theory is intended to cover all representations of any complexity {R16}, from any knowledge domain {R15}, and in any type of format or display {R14}. Fifth, the theory can be used to model alternative interpretations of representations for users with differing levels of domain knowledge and familiarity in specific graphical formats, rather than providing a single canonical characterization {R9}.

The theory and notation appear to satisfy most of the 23 requirements given in section 3, so it might be superior to previous accounts. A definite assessment demands a systematic review of the previous theories in terms of the requirements. The satisfaction of the requirements by the theory may partially be attributed to the compositional structural approach using just three core cognitive components that are fundamentally defined as things that *represent*.

The examples above suggest that the theory and notation may provide a means to systematically contrast disparate representations across classes of format. For instance, the above examples allowed us to compare the relative complexity of a diagram and sentential representation that are not informationally equivalent (cf., [16]). Although the formula is simpler graphically than the thermodynamics graph, the latter is simpler and more coherent in various ways. (1) It employs just two R-schemes rather than the equation's nine. (2) All of its concepts are explicitly represented, whereas some of the equation's graphic objects are absent from its Tokens. (3) The equation has multiple Tokens for some concepts, whereas the graph is desirably isomorphic {R17}.

The author has applied the theory and notation to over a dozen other representations, with interesting results. For example, revisiting Larkin and Simon's [16] seminal work by modelling their alternative representations for the pulley system problem, reveals that although the depth of the sentential representation's structure is similar to that of the diagram, it is branchier and composed of more R-schemes. This observation might yield a complementary explanation to the benefit of diagrammatic representations: they demand fewer R-schemes than sentential representations. This mirrors the observed contrast between the quadratic solution equation and the thermodynamics graph.

The representation structure diagram reveals the hidden complexity of Fig. 2, of which someone proficient in algebra may no longer be consciously aware. Fig. 9 could serve as a guide to an instructor of all the features of the equation that must be explained to learners. Similarly, Fig. 11 might be used to guide instruction on Fig. 3.

Analysis of further representations by others is required to fully evaluate the utility of the theory and notation. In particular, are the three proposed components sufficient and are their slots necessary and sufficient? The "sketch" in the title acknowledges that the theory has only been outlined here: some aspects of the theory and notation need further development. The compatibility of the theory with existing theories about the efficacy of representations must be established {R17}. For example, considerations of isomorphism [13, 21] can be examined through occurrence of repeated Tokens and by the presence of components with concepts but no graphic objects, and vice versa.

Finally, it is noted that the theory and notation may be able to address theoretical implications about the *cognitive cost* in representation choice {23}, or might be used to investigate how alternative representations might impact learning. The relative number of R-dimensions and R-schemes, whether models are simple hierarchies rather than more complex networks, and the extent use of auxiliary R-schemes and R-dimensions are potential avenues for exploration {R6}.

7 Acknowledgements

My thanks go to members of the Representational Systems lab and Rep2Rep team, and the three anonymous *Diagrams 2020* reviewers. This work was supported by the UK EPSRC: grants EP/R030642/1 and EP/T019034/1.

8 References

1. Barwise, J., & Etchemendy, J. (1995). Heterogenous logic. In J. Glasgow, N. H. Narayanan, & B. Chandrasekaran (Eds.), *Diagrammatic Reasoning: Cognitive and Computational Perspectives* (pp. 211-234). Menlo Park, CA: AAAI Press.
2. Bertin, J. (1983). *Seminology of Graphics: Diagrams, Networks, Maps*. Wisconsin: University of Wisconsin Press.
3. Blackwell, A., & Engelhardt, Y. (2002). A Meta-Taxonomy for Diagram Research. In M. Anderson, B. Meyer, & P. Olivier (Eds.), *Diagrammatic Representation and Reasoning* (pp. 47-64). London: Springer London.
4. Blackwell, A. F., Britton, C., Cox, A., Green, T. R. G., et al. (2001). Cognitive Dimensions of Notations: Design Tools for Cognitive Technology. In M. Beynon, C. L. Nehaniv, & K. Dautenhahn (Eds.), *Cognitive Technology: Instruments of Mind* (pp. 325-341). Berlin, Heidelberg: Springer Berlin Heidelberg.
5. Card, S., MacKinlay, J., & Shneiderman, B. (1999). Information Visualization. In S. Card, J. MacKinlay, & B. Shneiderman (Eds.), *Readings in Information Visualization: Using Vision to Think* (pp. 1-34). Mahwah, N.J.: Lawrence Erlbaum Associates.
6. Cheng, P. C.-H. (1999). Networks of Law Encoding Diagrams for understanding science. *European Journal of Psychology of Education, 14*(2), 167-184.
7. Cheng, P. C.-H. (2016). What Constitutes An Effective Representation? In J. Mateja, U. Yuri, & S. Stephanie (Eds.), *Diagrammatic Representation and Inference: 9th International Conference, Diagrams 2016* (9781 ed., pp. 17-31). Heidelberg, Germany: Springer.
8. Cheng, P. C.-H., Lowe, R. K., & Scaife, M. (2001). Cognitive science approaches to diagrammatic representations. *Artificial Intelligence Review, 15*(1-2), 79-94.
9. Cleveland, W. S., & McGill, R. (1985). Graphical perception and graphical methods for analyzing scientific data. *Science, 229*, 828-833.
10. Engelhardt, J. (2002). *Language of Graphics*. Amsterdam ILLC, University of Amsterdam
11. Engelhardt, Y., & Richards, C. (2018). A Framework for Analyzing and Designing Diagrams and Graphics. In P. Chapman, G. Stapleton, A. Moktefi, et al. (Eds.), *Diagrammatic Representation and Inference* (pp. 201-209): Springer International Publishing.
12. Ewing, J. A. (1926). *The Steam-Engine And Other Heat-Engines* (4th ed.). Cambridge: CUP.
13. Gurr, C. A. (1998). On the isomorphism, or lack of it, of representations In K. Marriott & B. Meyer (Eds.), *Visual Language Theory* (pp. 293-306). New York, NY: Springer-Verlag.

14. Hegarty, M. (2011). The Cognitive science of visual-spatial displays: Implications for design. *Topics in Cognitive Science, 3*, 446-474.
15. Koedinger, K. R., & Anderson, J. R. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science, 14*, 511-550.
16. Larkin, J. H., & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11*, 65-99.
17. Lohse, G. L. (1993). A cognitive model for understanding graphical perception. *Human-Computer Interaction, 8*, 353-388.
18. Lohse, G., Biolsi, K., Walker, N., & Rueter, H. (1994). A classification of visual representations. *Communications of the ACM, 37*(12), 36-49.
19. Markman, A. B. (1999). *Knowledge Representation*. Mahwah: NJ.: Lawrence Erlbaum.
20. Marriott, K., Meyer, B., & Wittenburg, K. B. (1998). A survey of visual language specification and recognition. In K. Marriott & B. Meyer (Eds.), *Visual Language Theory* (pp. 5-86). New York, NY: Springer-Verlag.
21. Moody, D. L. (2009). The "physics" of notations: toward a scientific basis for constructing visual notations in software engineering. *Software Engineering, IEEE Transactions on, 35*(6), 756-779.
22. Palmer, S. E. (1978). Fundamental aspects of cognitive representation. In E. Rosch & B. B. Lloyd (Eds.), *Cognition and Catergorization.* (pp. 259-303). Hillsdale, N.J.: Erlbaum.
23. Peebles, D. J., & Cheng, P. C.-H. (2003). Modelling the effect of task and graphical representations on response latencies in a graph-reading task. *Human factors, 45*(1), 28-45.
24. Raggi, D., Stockdill, A., Jamnik, M., Garcia Garcia, G., Sutherland, H. E. A., & Cheng, P. C.-H. (2019). Inspection and selection of representations. In C. Kaliszyk, E. Brady, A. Kohlhase, & C. Sacerdoti Coen (Eds.), *Intelligent Computer Mathematics - CICM 2019, Lecture Notes in Computer Science, vol 11617.* (pp. 227-242). Berlin: Springer.
25. Rogers, G. F. C., & Mayhew, Y. R. (1992). *Engineering Thermodynamics* (4th ed.). Harlow, UK: Longman.
26. Schank, R. C., & Abelson, R. P. (1977). Scripts, plans, goals, and understanding : an enquiry into human knowledge structures. Mahwah, NJ: Erlbaum.
27. Shimojima, A. (2015). Semantic properties of diagrams and their cognitive potentials. Stanford, CA: CSLI Press.
28. Shimojima, A., & Barker-Plummer, D. (2018). Operations on single feature indicator systems. In P. Chapman, G. Stapleton, A. Moktefi, et al. (Eds.), *Diagrammatic Representation and Inference* (pp. 296-312): Springer International Publishing.
29. Simkin, D., & Hastie, R. (1987). An information-processing analysis of graph perception. *Journal of hte American Statistical Association, 82*(398), 454-645.
30. Stenning, K., & Oberlander, J. (1995). A cognitive theory of graphical and linguistic reasoning: logic and implementation. *Cognitive Science, 19*(1), 97-140.
31. Stevens, S. S. (1946). On the theory of scales of measurement. *Science, 103*(2684), 677–680.
32. Zhang, J. (1996). A representational analysis of relational information displays. *International Journal of Human Computer Studies, 45*, 59-74.
33. Zhang, J., & Norman, D. A. (1993). A cognitive taxonomy of numeration systems. In M. Polson (Ed.), *Proceedings of the Fifteenth Annual Conference of the Cognitive Science Society* (pp. 1098-1103). Hillsdale, N.J.: Lawrence Erlbaum.