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REGULARLY VARYING PROBABILITY DENSITIES

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Abstract. The convolution of regularly varying probability densities is proved asymptotic to their sum, and hence is also regularly varying. Extensions to rapid variation, O-regular variation, and other types of asymptotic decay are also given.

Regularly varying distribution functions have long been used in probability theory; see e.g. Feller [**7**, VIII.8], Bingham, Goldie and Teugels [**5**, Ch. 8]. This note addresses some questions on regularly varying probability densities that seem—surprisingly—to have been overlooked.

1. Convolution of regularly varying densities

Theorem 1.1. *If f and g are probability densities on $\mathbb{R}$, both regularly varying at ∞, then their convolution has the property*

$$\frac{f * g(x)}{f(x)+g(x)} \to 1 \qquad (x \to \infty), \tag{1.1}$$

and so is regularly varying (with index the maximum of the index of f and the index of g).

Proof. We have

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(u)g(x-u)\,du \\ &= \int_{-\infty}^{x/2} f(u)g(x-u)\,du + \int_{x/2}^{\infty} f(u)g(x-u)\,du \\ &= \int_{-\infty}^{x/2} f(u)g(x-u)\,du + \int_{-\infty}^{x/2} g(u)f(x-u)\,du \\ &=: B(x) + A(x). \end{aligned}$$

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We can write

$$\frac{A(x)}{f(x)} = \int_{-\infty}^{\infty} h_x(u)\,du \tag{1.2}$$

where

$$h_x(u) := g(u)\frac{f(x-u)}{f(x)}\mathbf{1}_{u<x/2}.$$

Now f is regularly varying with index $-\rho$, notated $f \in R_{-\rho}$, where $\rho \geqslant 1$ because $f \in L^1$. The Potter bound of [**5**, Theorem 1.5.6(iii)], with $A = 2$, $\delta = \rho/2$, gives the existence of x_0 such that

$$\frac{f(y)}{f(x)} \leqslant 2\max\left(\left(\frac{y}{x}\right)^{-\rho/2}, \left(\frac{y}{x}\right)^{-3\rho/2}\right) \qquad (x \geqslant x_0),\ y \geqslant x_0).$$

So for $x \geqslant 2x_0$ we find that

$$\text{for } 0 < u < \frac{x}{2},\quad \frac{1}{2} < \frac{x-u}{x} < 1 \quad \text{so} \quad \frac{f(x-u)}{f(x)} \leqslant 2\left(\frac{x-u}{x}\right)^{-3\rho/2} < 2^{1+3\rho/2}, \tag{1.3}$$

and

$$\text{for } -\infty < u \leqslant 0,\quad 1 \leqslant \frac{x-u}{x} < \infty \quad \text{so} \quad \frac{f(x-u)}{f(x)} \leqslant 2\left(\frac{x-u}{x}\right)^{-\rho/2} \leqslant 2. \tag{1.4}$$

Therefore

$$0 \leqslant h_x(u) \leqslant 2^{1+3\rho/2} g(u) \quad \text{for all } x \text{ and } u,$$

and as $g \in L^1$ we can use the Dominated Convergence Theorem to obtain the limit of the integral in (1.2). For each fixed u, $h_x(u) \to g(u)$ as $x \to \infty$, so

$$\frac{A(x)}{f(x)} \to \int_{-\infty}^{\infty} g(u)\,du = 1.$$

We have also $B(x)/g(x) \to 1$ by interchange of notation, so the result (1.1) follows. Regular variation of $f + g$, with index the maximum of the index of f and the index of g, follows by an elementary closure property of regular variation ([**5**, Prop. 1.5.7(iii)]). Regular variation of $f * g$, with the same index, then follows from (1.1). □

To set the above result in context we need to define the following class.

Definition 1.2. A function $f : \mathbb{R} \to \mathbb{R}$ is in the class L if $f \circ \ln$ is slowly varying; that is, if f is measurable, eventually positive, and such that

$$\lim_{x\to\infty} \frac{f(x+y)}{f(x)} = 1 \quad \text{for every } y \in \mathbb{R}.$$

We then have the following result.

Proposition 1.3. *If f and g are probability densities on $\mathbb{R}$, both in L, then*

$$\liminf_{x\to\infty} \frac{f * g(x)}{f(x) + g(x)} \geqslant 1.$$

PROOF. We decompose $f*g(x)$ as in the previous proof. Application of Fatou's Lemma to (1.2) gives that

$$\liminf_{x\to\infty} \frac{A(x)}{f(x)} \geqslant \int_{-\infty}^{\infty} g(u)\,du = 1.$$

Similarly $\liminf_{x\to\infty} B(x)/g(x) \geqslant 1$. The result follows. □

This enables us to see Theorem 1.1 in the following light: we have f, $g \in R \subset L$. The L property gives a sharp asymptotic lower bound on $f * g(x)/(f(x) + g(x))$, and regular variation allows use of the Dominated Convergence Theorem to show that the lower bound is in fact the limit.

2. Convolution of densities, one dominating the other

When one density dominates the other we can drop all further assumptions on the dominated density, as in the following result. A further result involving rapid variation (Corollary 2.3) will ensue.

THEOREM 2.1. *If f and g are probability densities on $\mathbb{R}$, with f regularly varying and $g(x) = \mathrm{o}(f(x))$ as $x \to \infty$, then their convolution has the property*

$$\frac{f * g(x)}{f(x)} \to 1 \qquad (x \to \infty), \tag{2.1}$$

equivalent to (1.1), *and is regularly varying with index the same as that of f.*

PROOF. We use the same decomposition of $f * g(x)$ as in the proof of (1.1), and the proof that $A(x)/f(x) \to 1$ as $x \to \infty$ remains valid. Indeed, it further remains valid if we replace g by f; that is, we have

$$\frac{1}{f(x)} \int_{-\infty}^{x/2} f(u)f(x-u)\,du \to \int_{-\infty}^{\infty} f(u)\,du = 1.$$

Given $\epsilon > 0$ we may find x_1 such that $g(x) \leqslant \epsilon f(x)$ for all $x \geqslant x_1$. Then for $x \geqslant 2x_1$,

$$0 \leqslant \frac{B(x)}{f(x)} \leqslant \frac{\epsilon}{f(x)} \int_{-\infty}^{x/2} f(u)f(x-u)\,du = \epsilon(1 + \mathrm{o}(1)).$$

The conclusion (2.1) follows.

(2.1) is equivalent to (1.1) in this case because of the assumption $g = \mathrm{o}(f)$. Finally, (2.1) makes $f * g$ inherit regular variation, with the same index, from f. □

The extension to rapid variation will be immediate given the following general result on rapidly varying functions. As usually defined ([**9**]; [**5**, §2.4]), rapidly varying functions are assumed measurable.

LEMMA 2.2. *Let $g \in R_{-\infty}$. Then given any $r > 0$ there exists a constant X, depending on r, such that $g(x) \leqslant x^{-r}$ for all $x \geqslant X$.*

As a consequence, if f is regularly varying then $g(x)/f(x) \to 0$ as $x \to \infty$.

Proof. By the 'Uniform Convergence Theorem for Rapid Variation' (due to Heiberg: see [**5**, Theorem 2.4.1]), there exists Y such that $g(y)/g(x) \leqslant e^{-r-1}$ for all $y \geqslant ex$ and $x \geqslant Y/e$. Fix $y \geqslant Y$ and let n be the non-negative integer such that $Ye^n \leqslant y < Ye^{n+1}$. Then

$$\frac{g(y)}{g(Y/e)} = \frac{g(Y)}{g(Y/e)}\,\frac{g(Ye)}{g(Y)} \cdots \frac{g(Ye^{n-1})}{g(Ye^{n-2})}\,\frac{g(y)}{g(Ye^{n-1})} \leqslant e^{-(n+1)(r+1)}.$$

Because $n+1 \geqslant \ln y - \ln Y$ the right-hand side is at most $e^{-(\ln y - \ln Y)(r+1)}$, which is $Y^{r+1}y^{-r-1}$. Thus $g(y) \leqslant Ay^{-r-1}$ where $A := Y^{r+1}g(Y/e)$. If we set $X := \max(Y, A)$ then $Ay^{-r-1} \leqslant y^{-r}$ for all $y \geqslant X$, so we have the result. □

Corollary 2.3. *If f and g are probability densities on $\mathbb{R}$, with f regularly varying and g rapidly varying at ∞, then the conclusions of Theorem 2.1 hold for their convolution $f * g$.*

3. Bounds instead of limits

For functions $a(\cdot)$, $b(\cdot)$ on $\mathbb{R}$, both eventually positive, we use the notation $a(x) \asymp b(x)$ to mean that

$$0 < \liminf_{x\to\infty} \frac{a(x)}{b(x)} \leqslant \limsup_{x\to\infty} \frac{a(x)}{b(x)} < \infty.$$

Recall from [**5**, §§2.1–2.2], that an eventually positive function f on $\mathbb{R}$ is called *almost decreasing*, notation $f \in AD$, if

$$f(x) \asymp \sup_{y\geqslant x} f(y),$$

and is said to have *bounded decrease*, notation $f \in BD$, if

$$\text{for some } \lambda > 1, \quad \liminf_{x\to\infty} \inf_{\mu\in[1,\lambda]} \frac{f(\mu x)}{f(x)} > 0.$$

Recall also, from [**5**, Ch. 2], the class OR of O-regularly varying functions, which includes the classes R and ER of regularly varying and extended regularly varying functions; $R \subset ER \subset OR$. A function f on $\mathbb{R}$ is in OR if it is measurable, eventually positive, and has the property that

$$0 < f_*(\lambda) \leqslant f^*(\lambda) < \infty \quad \text{for all } \lambda > 1,$$

where

$$f_*(\lambda) = \liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)}, \quad f^*(\lambda) = \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)}.$$

For $f \in OR$ the upper and lower *global* or *Matuszewska* indices $\alpha(f)$ and $\beta(f)$ are given respectively by

$$\alpha(f) = \lim_{\lambda\to\infty} \frac{\log f^*(\lambda)}{\log\lambda} = \inf_{\lambda>1} \frac{\log f^*(\lambda)}{\log\lambda}, \quad \beta(f) = \lim_{\lambda\to\infty} \frac{\log f_*(\lambda)}{\log\lambda} = \sup_{\lambda>1} \frac{\log f_*(\lambda)}{\log\lambda};$$

both are thus finite.

By [**5**, Theorem 2.1.7], $f \in BD$ implies $\beta(f) > -\infty$. By the 'Almost-Monotonicity Theorem' of Aljančić and Arandelović ([**1**]; [**5**, Theorem 2.2.2]), $f \in AD$ implies $\alpha(f) \leqslant 0$ and is implied by $\alpha(f) < 0$. We also have, again by [**5**, Theorem

2.1.7], that $OR \subset BD$, and conversely the set of measurable $f \in AD \cap BD$ is a subset of OR. Thus for measurable f, and in particular for a probability density, the assertion $f \in AD \cap BD$ is equivalent to $f \in AD \cap OR$.

With all these definitions we may now give a 'boundedness' extension of Theorem 1.1.

Theorem 3.1. *If f and g are probability densities on $\mathbb{R}$, both almost decreasing and of bounded decrease, then their convolution $f * g$ has the property*

$$f * g(x) \asymp f(x) + g(x). \tag{3.1}$$

The convolution is then also almost decreasing and of bounded decrease, and its global indices are bounded by those of its constituent parts in that

$$-\infty < \min(\beta(f), \beta(g)) \leqslant \beta(f * g) \leqslant \alpha(f * g) \leqslant \max(\alpha(f), \alpha(g)) \leqslant 0.$$

Proof. The method of proof of Theorem 1.1 again applies. Bounded decrease of f gives the bound [**5**, (2.2.1′)], which leads to (1.3) being replaced by $f(x - u)/f(x) \leqslant (C'2^{\beta})^{-1}$. The assumption $f \in AD$ is equivalent to the existence of a constant $M < \infty$ such that $f(y) \leqslant Mf(x)$ for all $y \geqslant x \geqslant x_0$, and that implies (1.4) can be replaced by $f(x - u)/f(x) \leqslant M$. We conclude that the functions $h_x(u)$ are bounded above by a constant multiple of $g(u)$, as before. In place of $h_x(u) \to g(u)$ we have the existence of $0 < c \leqslant C < \infty$ such that for each fixed $u \in \mathbb{R}$, $cg(u) \leqslant h_x(u) \leqslant Cg(u)$ for all $x \geqslant x_0(u)$. We may then employ a limsup-and-liminf form of Lebesgue's Dominated Convergence theorem, e.g. [**10**, (12.24)], to conclude that $A(x) \asymp f(x)$ as $x \to \infty$. In combination with $B(x) \asymp g(x)$, obtained as before by interchange of notation, this yields $A(x)+B(x) \asymp f(x)+g(x)$, that is, (3.1).

For the remaining claims, note first that if $f(y) \leqslant Mf(x)$ for all $y \geqslant x \geqslant x_0$, and $g(y) \leqslant M'g(x)$ for all $y \geqslant x \geqslant x_0'$, then $f(y)+g(y) \leqslant \max(M, M')(f(x)+g(x))$ for all $y \geqslant x \geqslant \max(x_0, x_0')$, so $f + g$ inherits the almost-decreasing property from f and g. We also clearly have

$$(f + g)_*(\lambda) \geqslant \min(f_*(\lambda), g_*(\lambda)) \quad \text{and} \quad (f + g)^*(\lambda) \leqslant \max(f^*(\lambda), g^*(\lambda))$$

for all $\lambda > 1$, so that $f + g \in OR$ and

$$-\infty < \min(\beta(f), \beta(g)) \leqslant \beta(f + g) \leqslant \alpha(f + g) \leqslant \max(\alpha(f), \alpha(g)) \leqslant 0.$$

Finally, all these properties are immediately inherited by $f * g$ from $f + g$ via (3.1). □

4. Limits restored

In §1 we assumed f, $g \in L$ whereas in §3 we did not. With the assumption that f, $g \in L$ we gained the limit conclusion (1.1); without it, only the asymptotic comparability conclusion (3.1). Let us restore the assumption; then we may weaken the regular variation assumed in §1 to conditions of the type employed in §3, as follows.

THEOREM 4.1. *If f and g are probability densities on $\mathbb{R}$, both in $AD \cap BD \cap L$, then their convolution has the property* (1.1). *The convolution is then also in $AD \cap BD \cap L$.*

PROOF. A further variant of the proof of Theorem 1.1. The conditions of Theorem 3.1 are satisfied, so the functions $h_x(u)$ are bounded above by a constant multiple of $g(u)$, as in the proof of Theorem 3.1. Our assumption that $f \in L$ brings us back to $h_x(u) \to g(u)$ as $x \to \infty$ for each fixed u, as in the proof of Theorem 1.1, so we may again use the standard Dominated Convergence Theorem to conclude $A(x)/f(x) \to 1$ and hence (1.1). The further conclusions about $f * g$ are as already shown in Theorem 3.1, plus the immediate conclusion that $f * g \in L$. $\square$

There are many densities satisfying the conditions of this result but not those of Theorem 1.1. For instance, if $f(x) := c\mathbf{1}_{x>1}x^{-2}(2 + \sin(\ln x))$ where c is such as to make $\int f = 1$, then f is in $AD \cap BD \cap L$, but it is not regularly varying.

Is there an extension of the index conclusions of Theorem 3.1 to the above setting? Yes. Recall that the class ER mentioned at the beginning of §3 is defined in [**5**, §2.0.2] as the set of measurable, eventually positive f on $\mathbb{R}$ for which there exist constants c, d such that

$$\lambda^d \leqslant f_*(\lambda) \leqslant f^*(\lambda) \leqslant \lambda^c \quad \text{for all } \lambda > 1.$$

For $f \in ER$ the upper and lower *local* or *Karamata* indices $c(f)$ and $d(f)$ are given respectively by

$$c(f) = \lim_{\lambda \downarrow 1} \frac{\log f^*(\lambda)}{\log \lambda} = \sup_{\lambda > 1} \frac{\log f^*(\lambda)}{\log \lambda}, \quad d(f) = \lim_{\lambda \downarrow 1} \frac{\log f_*(\lambda)}{\log \lambda} = \inf_{\lambda > 1} \frac{\log f_*(\lambda)}{\log \lambda};$$

both are thus finite. (Their equality would bring us back to regular variation.)

COROLLARY 4.2. *If f and g are probability densities on $\mathbb{R}$, both in $AD \cap ER$, then their convolution $f * g$ has the property* (1.1). *The convolution is then also in $AD \cap ER$, and its local indices are bounded by those of its constituent parts in that*

$$-\infty < \min(d(f), d(g)) \leqslant d(f * g) \leqslant c(f * g) \leqslant \max(c(f), c(g)) < \infty.$$

PROOF. From (1.1) it is straightforward to show that $f + g \in ER$ and

$$-\infty < \min(d(f), d(g)) \leqslant d(f + g) \leqslant c(f + g) \leqslant \max(c(f), c(g)) < \infty.$$

The final conclusions concerning $f * g$ then follow via (1.1). $\square$

5. Alternatives to the almost-decreasing property

In Theorem 3.1, suppose that g is a density on the positive half-line. Then in the proof we may re-define the function h_x as

$$h_x(u) := g(u)\frac{f(x-u)}{f(x)}\mathbf{1}_{0<u<x/2},$$

and we find that in the lines following we do not employ our assumption $f \in AD$ either for the conditions of the Dominated Convergence Theorem or for its application. We gain the conclusion (3.1) without using the assumption $f \in AD$.

Similarly, mutatis mutandis, with f and g interchanged. If both f and g are densities on the positive half-line then we gain the conclusion (3.1) without assuming either $f \in AD$ or $g \in AD$.

Similarly, in Theorem 4.1, if g is a density on the positive half-line we gain the conclusion (1.1) without using the assumption $f \in AD$. If f is a density on the positive half-line then we gain the same conclusion without using the assumption $g \in AD$, while if both f and g are densities on the positive half-line then we conclude (1.1) without assuming either $f \in AD$ or $g \in AD$.

Another way to avoid assuming the almost-decrease property on one or other component of the convolution is to compensate for its lack by a moment assumption on the *other* component. Take the case of Theorem 3.1 (without any longer assuming either density is restricted to a half-line). If we assume $f \in OR$ rather than the more restrictive $f \in AD \cap BD$ then for any $r > \alpha(f)$ there exist positive constants C and x_0 such that

$$\frac{f(y)}{f(x)} \leqslant C\left(\frac{y}{x}\right)^r \quad \text{for all } y \geqslant x \geqslant x_0$$

([**5**, Prop. 2.2.1]). If we use this in place of (1.4) in the proof we will gain a suitable bound for dominated convergence if we also assume $\int_{-\infty}^0 (-u)^r g(u)\,du < \infty$. Similarly, if we assume $g \in OR$ rather than $g \in AD \cap BD$ then we may retrieve dominated convergence by a moment assumption on the left tail of f. We formulate this precisely for the case where the almost-decreasing assumption is removed for both f and g, leaving to the reader the variants where only one such assumption is removed.

In the following we let X and Y be random variables with densities f and g respectively. We use the notation $x^- := (-x)\mathbf{1}_{x<0}$.

THEOREM 5.1. *Let f and g be probability densities on $\mathbb{R}$, both in the class OR. If, for some $r > \alpha(f)$ and $s > \alpha(g)$, $E((X^-)^s + (Y^-)^r) < \infty$, then* (3.1) *holds. The convolution $f * g$ is then also in OR and we have*

$$-\infty < \min(\beta(f), \beta(g)) \leqslant \beta(f * g) \leqslant \alpha(f * g) \leqslant \max(\alpha(f), \alpha(g)) < \infty.$$

PROOF. In place of (1.4) we have that for $x \geqslant 2x_0$,

$$\text{for } -\infty < u \leqslant 0, \quad 1 \leqslant \frac{x-u}{x} < \infty \quad \text{so} \quad \frac{f(x-u)}{f(x)} \leqslant C\left(1 + \frac{-u}{x}\right)^r.$$

If we ensure $x_0 \geqslant 1$ and then set $C' := C(1 + x_0^{-1})^r$, the right-hand side is at most $C' \max(1, (-u)^r)$ for all $x \geqslant 2x_0$ and $u < 0$. When multiplied by $g(u)$ this bound, by assumption, is integrable over $-\infty < u < 0$. Hence, as in the proof of Theorem 3.1, we have the condition of the Dominated Convergence Theorem, and may use it again to conclude $A(x) \asymp f(x)$. With $B(x) \asymp g(x)$, similarly derived, this leads to (3.1) as before.

The remaining conclusions are straightforward. □

In each of the results of §4 we may similarly weaken membership of $AD \cap BD$ for either component to membership of OR, if we also impose a suitable moment condition on the left tail of the other component. We give the (double) extension of Theorem 4.1 only, leaving the proof and other possibilities to the reader.

THEOREM 5.2. *Let f and g be probability densities on $\mathbb{R}$, both in $OR \cap L$. If, for some $r > \alpha(f)$ and $s > \alpha(g)$, $E((X^-)^s + (Y^-)^r) < \infty$, then* (1.1) *holds. The convolution $f * g$ is then also in $OR \cap L$.*

6. Convolutions of distributions

The situation for probability *distribution* functions is much better known. The result below is Theorem 6.1 in Applebaum [**2**], where it is attributed to G. Samorodnitsky.

THEOREM. *If F, G are distribution functions with regularly varying tails—that is, if $\overline{F} := 1 - F$ and $\overline{G} := 1 - G$ are regularly varying—then their convolution $F * G$ satisfies*

$$\frac{\overline{(F * G)}(x)}{\overline{F}(x) + \overline{G}(x)} \to 1 \qquad (x \to \infty). \tag{6.1}$$

The proof (which is probabilistic in the reference cited) is given also in Embrechts, Klüppelberg and Mikosch [**6**, Lemma 1.3.1, p. 37], Feller [**7**, VIII.8, Proposition, pp. 278–279] and Resnick [**14**, Prop. 4.1], in the case when the indices of regular variation of $\overline{F}$ and $\overline{G}$ are the same. The argument holds in the general case also.

We may extend this result using a similar method to those for densities. Note that for non-increasing functions (such as distribution tails $\overline{F}$, $\overline{G}$, ...), membership of BD is equivalent to membership of OR.

THEOREM 6.1. *If F, G are distribution functions with $\overline{F}$, $\overline{G} \in OR \cap L$ then their convolution $F * G$ satisfies* (6.1).

PROOF. Let X, Y be independent with distributions F, G respectively. For $z \in \mathbb{R}$,

$$P(X + Y > z) \leqslant$$
$$P(X + Y > z,\ X \leqslant z/2) + P(X + Y > z,\ Y \leqslant z/2) + P(X > z/2,\ Y > z/2),$$

so

$$\overline{F * G}(z) = B(z) + A(z) + \overline{F}(z/2)\overline{G}(z/2)$$

where

$$A(z) := \int_{(-\infty, z/2]} \overline{F}(z - y)\, dG(y) \quad \text{and} \quad B(z) := \int_{(-\infty, z/2]} \overline{G}(z - x)\, dF(x).$$

Now in

$$\frac{A(z)}{\overline{F}(z)} = \int_{-\infty}^{\infty} \mathbf{1}_{y \leqslant z/2} \frac{\overline{F}(z - y)}{\overline{F}(z)}\, dG(y) \tag{6.2}$$

we have that

$$\text{for } 0 < y < \frac{z}{2}, \quad \frac{1}{2} < \frac{z - y}{z} < 1 \quad \text{so} \quad \frac{\overline{F}(z - y)}{\overline{F}(z)} \leqslant C < \infty$$

since $\overline{F} \in OR$, and

$$\text{for } -\infty < y \leqslant 0, \quad 1 \leqslant \frac{z-y}{z} < \infty \quad \text{so} \quad \frac{\overline{F}(z-y)}{\overline{F}(z)} \leqslant 1$$

since $\overline{F}$ is non-increasing. We may thus apply the Dominated Convergence Theorem. In (6.2) the integrand converges pointwise to 1, hence $A(z)/\overline{F}(z) \to \int_{-\infty}^{\infty} dG = 1$. Similarly $B(z)/\overline{G}(z) \to 1$. For the final term we have for instance that

$$\frac{\overline{F}(z/2)\overline{G}(z/2)}{\overline{F}(z)} = \mathrm{O}(\overline{G}(z/2))$$

since $\overline{F} \in OR$, and thus $\overline{F}(z/2)\overline{G}(z/2)/\overline{F}(z) = \mathrm{o}(1)$. The result follows. □

Analogues for distribution functions of a number of our results above for densities are possible, but we shall not pursue that topic.

Since addition of independent random variables corresponds to convolution of their distribution functions, the theorems in this section have the pleasing interpretation that, asymptotically, tails add over independent summands; similarly for the results of previous sections in terms of densities.

7. Complements

7.1. One density. Theorem 4.1 is the 'two-densities' result corresponding to the 'no-densities' result Theorem 6.1. Note however that if one of F, G has a density, so does their convolution. Indeed, if F has density f, then $H := F * G$ has density h given by

$$h(x) = \int_{-\infty}^{\infty} f(x-y)\, dG(y)$$

([**7**, V.4, Theorem 4]). One may thus ask for a 'one-density' version, but we have no result here. One problem is the loss of symmetry (the proofs of Theorems 4.1 and 6.1 are both symmetric between F and G). Another is that, while the Monotone Density Theorem ([**5**, Th. 1.7.2]) allows us to 'differentiate asymptotic relations', and the weakest possible (Tauberian) condition relaxing monotonicity is known ([**5**, Th. 1.7.5]), conditions of that type are awkward to handle in practice.

7.2. One density dominates. Suppose in Theorem 3.1 that the closed intervals $[\beta(f), \alpha(f)]$ and $[\beta(g), \alpha(g)]$ are disjoint. Without loss of generality we may take it that $\alpha(g) < \beta(f)$. Then $g(x) = \mathrm{o}(f(x))$ as $x \to \infty$, so that (3.1) may be refined to $f * g(x) \asymp f(x)$, and the final conclusion of Theorem 3.1 becomes simply that $\beta(f * g) = \beta(f)$ and $\alpha(f * g) = \alpha(f)$. This is the analogue of the phenomenon that in Theorem 1.1, when the indices of regular variation are unequal the component of the convolution with the smaller index is negligible in comparison with the other component, so that the convolution is asymptotically equivalent to the larger component.

If the intervals $[\beta(f), \alpha(f)]$ and $[\beta(g), \alpha(g)]$ are not disjoint, neither f nor g necessarily dominates the other, so no refinement along these lines is generally available. Similar remarks apply in the case of local indices in Corollary 4.2.

7.3. Subexponentiality. The subexponential class—the class S of subexponential distributions—often occurs as a generalisation of the class R of regularly varying distributions. This class is known not to be closed under convolution (Leslie [**11**]; cf. [**5**, Appendix 4], [**6**, Appendix A3.2]). Thus (in an obvious notation) $S * S \subset S$ is false, while Theorems 1.1 and 2 may be written $D * D \subset D$ and $R * R \subset R$. One may ask whether $R * S \subset S$, for instance, but we leave such questions open.

For a subexponential distribution on $[0, \infty)$ it was proved by H. Kesten (see [**3**, IV]) that for all $\epsilon > 0$ there exists $K(\epsilon)$ with

$$\overline{F^{n*}}(x) \leqslant K(\epsilon)(1+\epsilon)^n \overline{F}(x) \quad \text{for all } n = 1, 2, \ldots \text{ and } x \geqslant 0$$

($(\cdot)^{n*}$ denotes n^{th} convolution power). One might conjecture that for a regularly varying distribution tail $\overline{F}$ or density f, a bound of the form

$$\overline{F^{n*}}(x) \leqslant K n^{\delta} \overline{F}(x) \quad \text{for all } n = 1, 2, \ldots \text{ and } x \geqslant 0,$$

or respectively

$$f^{n*}(x) \leqslant K n^{\delta} f(x) \quad \text{for all } n = 1, 2, \ldots \text{ and } x \geqslant 0,$$

might hold.

7.4. Credence. O'Hagan ([**12**]; cf. [**13**]) calls a random variable of *credence* c if its density decays like $|x|^{-c}$. In Bayesian statistics, one has two sources of information—one's prior beliefs (which summarise one's information before sampling), and the information in the sample. Problems arise where these two sources of information may conflict. He obtains results of two kinds, for elliptically contoured distributions. First, for non-conflicting sources of information (that is, where the centres agree), credences add when sources of information are combined. This simply corresponds to the power law $x^{a+b} = x^a \cdot x^b$. Next, for conflicting sources of information,

(a) the centre of the source with higher credence dominates;
(b) the tail of the source with lower credence dominates.

Part (b) is in keeping with our results, and with the coarse-averaging (or tower, or iterated conditional expectations) property of conditional expectations (see e.g. [**15**, 9.7(i)]).

7.5. Logarithmic derivatives. Berman [**4**] obtains results complementary to ours, where the focus is on the logarithmic derivatives of densities rather than densities themselves. Again, the heavier tail predominates. His work is motivated by applications to HIV-latency times.

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