A Point-process Approach to Almost-sure Behaviour of Record Values and Order Statistics

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Dedicated to the memory of Prof. Bill Pruitt

Abstract
Point-process and other techniques are used to make a comprehensive investigation of the almost-sure behaviour of partial maxima $M_n^{(r)}$ (the r^{th} largest among a sample of n i.i.d. random variables), partial record values $R_n^{(r)}$, and differences and quotients involving them. In particular, we obtain characterizations of such asymptotic properties as $\limsup_{n\to\infty}(M_n^{(r)} - M_n^{(r+s)}) \le c$ a.s. for some finite constant c, or $R_{n+s}^{(r)} - R_n^{(r)} \to c$ a.s. for some constant c in $[0,\infty]$, which tell us, in various ways, how quickly the sequences increase. These characterizations take the form of integral conditions on the tail of F, which furthermore characterize such properties as stability and relative stability of the sequence of maxima. We also develop their relation to the large-sample behaviour of trimmed sums, and discuss some statistical applications.

ALMOST-SURE BOUNDS; HEAVY TAILS; LIGHT TAILS; ORDER STATISTICS; OUTLIER PRONE; OUTLIER RESISTANT; RECORD VALUE; RELATIVE STABILITY; STABILITY; TRIMMED SUM

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1. Introduction

While it has been amply demonstrated in recent years that a point-process approach simplifies, unifies and extends many aspects of extreme-value theory, so far this theory has been applied mainly to obtain weak convergence results for extremes, record values, and record times. The aim of the present paper is to demonstrate that this powerful theory can also be used to obtain almost sure (a.s.) limit theorems for ratios and differences of order statistics and partial record values. In addition, we can use it to obtain complete characterizations of the a.s. stability and relative stability of maxima.

Our basic structure will be a sequence of independent random variables (r.v.s) $X_1, X_2, \ldots$, having right-continuous distribution function (d.f.) F satisfying $F(x) < 1$ for $x \in \mathbb{R}$. Let $M_n^{(1)} \ge \cdots \ge M_n^{(n)}$ denote the *order statistics* of a sample $X_1, \ldots, X_n$ from F, i.e., the values of $X_1, \ldots, X_n$ sorted into decreasing order. Consider, for $r \in \mathbb{N}$, the vector $\mathbf{M}_n^{(r)} = (M_n^{(1)}, \ldots, M_n^{(r)})$ of the first r order statistics. This first becomes well defined at time $n = r$. The *r-record times* are defined thus: $L_1^{(r)} := r$, and $L_2^{(r)}$, $L_3^{(r)}$, ... are the subsequent times at which $\mathbf{M}_n^{(r)}$ changes value. Since $F(x) < 1$ for all $x < \infty$, the $L_n^{(r)}$ are all finite a.s., but $L_n^{(r)} \uparrow \infty$ a.s. as $n \to \infty$.

The *initial rank* I_n of the observation X_n is defined to be r if exactly $r - 1$ of the preceding observations exceed or equal X_n; formally, $I_n := \sum_{i=1}^n \mathbf{1}\{X_i \geq X_n\}$ where (throughout) $\mathbf{1}_A$ or $\mathbf{1}\{A\}$ denotes the indicator function which takes the value 1 if statement A holds and 0 otherwise. The r-record times $L_1^{(r)} < L_2^{(r)} < \cdots$ consist of those times from r onwards at which observations of initial rank at most r arrive. The $L_n^{(r)}$ should be distinguished from the times at which observations of initial rank *exactly* r arrive; those were denoted by T_n^r in Goldie and Rogers (1984).

Besides the high order statistics, we shall be concerned with the *r-record value* sequence defined by

$$R_n^{(r)} := M_{L_n^{(r)}}^{(r)} \qquad (n = 1, 2, \ldots). \tag{1.1}$$

The sequence $(R_n^{(r)})_{n=1}^\infty$ is the *trace* of the observations of initial rank at most r, that is, it records just the values taken, without information on their order in time. It is also the trace of the r^{th} order-statistic sequence, i.e. the sequence of values passed through, without information on sojourn times. Since $L_n^{(r)} \uparrow \infty$ a.s., we see that $R_n^{(r)} \uparrow \infty$ a.s. as $n \to \infty$. When F is continuous we have, almost surely, that $M_n^{(1)} > M_n^{(2)} > \cdots > M_n^{(n)}$ and $R_1^{(r)} < R_2^{(r)} < \cdots$, while $R_n^{(r)}$ is the sequence of distinct values taken by $M_n^{(r)}$; but this is not so for general F. Note however the special case $r = 1$, for which, omitting the superscript 1, we always have $L_1 < L_2 < \cdots$ as the strict (upper) record times, and $R_1 < R_2 < \cdots$ as the strict (upper) record values; and $R_n = X_{L_n}$ for $n \in \mathbb{N}$.

Current theory for these topics stems on one side from work of de Haan, Pickands and Resnick (see de Haan and Resnick (1977), Pickands (1971), Resnick (1986, 1987)), where the structural and asymptotic theory of extremes is developed, while others (Vervaat (1977), Engelen *et al.* (1988), Goldie and Resnick (1989)) have provided a structural theory for partial record values and their asymptotics. Deheuvels (1981, 1983) has developed an almost-sure approximation theory related to the point-process approach.

The fundamental relation (1.1) allows us to transfer results from the record-value sequence to the sequence of maxima, and vice versa. The power of this technique lies in the fact that the point process whose points are $R_n^{(r)}$ has a surprisingly simple structure. To specify it, let $x \in \mathbb{R}$ and let δ_x denote the measure consisting of unit mass at the point x. Denote by $\zeta_r := \sum_{n=1}^\infty \delta_{R_n^{(r)}}$ the point process on $\mathbb{R}$ whose points are $R_n^{(r)}$; thus for any B in $\mathcal{B}$, the Borel σ-algebra in $\mathbb{R}$, $\zeta_r(B) = \sum_{n=1}^\infty \mathbf{1}\{R_n^{(r)} \in B\}$ is the number of $R_n^{(r)}$ in B. The result sometimes known as 'Ignatov's Theorem' (see Engelen *et al.* (1988)) states that ζ_r is the sum of independent and identically distributed (i.i.d.) point processes $\rho_1, \ldots, \rho_r$, the points of ρ_i being the i^{th} partial records, i.e., the values of the observations of initial rank i. Furthermore the process is *completely random*, i.e., $\zeta_r(B_1), \ldots, \zeta_r(B_k)$ are independent r.v.s for any disjoint Borel sets $B_1, \ldots, B_k$. In the nomenclature introduced by Goldie and Rogers (1984), each ρ_i is a *Shorrock Process*, a simple point process (no multiple points) characterized by its 'avoidance function'

$$P(\rho_i(B) = 0) = e^{-\eta(B)} \qquad (B \in \mathcal{B}). \tag{1.2}$$

Here η is the *tail measure* of F defined by

$$e^{-\eta(-\infty,x]} = P(X > x) \qquad (x \in \mathbb{R}). \tag{1.3}$$

We will use this structure to characterize such asymptotic properties of the r-record value sequence as $\limsup_{n\to\infty}(R^{(r)}_{n+s} - R^{(r)}_n) \le c$ a.s. for some finite constant c, or $R^{(r)}_{n+s} - R^{(r)}_n \to c$ a.s. for some constant c in $[0,\infty]$, which tell us, in various ways, how quickly the sequence increases. These characterizations take the form of integral conditions on the tail of F, with remarkably simple proofs hingeing on the complete randomness of the point process ζ_r. (1.1) then allows us to translate these conditions to corresponding properties of the sequence $M^{(r)}_n$, or more precisely, to the subsequence $M^{(r)}_{L^{(r)}_n}$. But this procedure is well adapted to the approach of Maller and Resnick (1984) and Pruitt (1987) of inferring the a.s. behaviour of the sequence of maxima from its behaviour on the subsequence of record times. In this way we build up, in the spirit of the modern point process approach to extreme value theory, a comprehensive picture of the a.s. behaviour of $R^{(r)}_n$ and $M^{(r)}_n$. One striking result obtained is that the a.s. stability (resp., a.s. relative stability) of $M^{(r)}_n$ for $r > 1$ is equivalent to that of $M^{(1)}_n$.

In the next section we introduce further notation and present the main results. Some applications of these in Statistics are discussed at the end of §2. In the final section, §3, we give the proofs and some further comments and related results.

2. Results

We use the symbol F to denote both the d.f. of the X_i and the probability measure associated with it; furthermore we abbreviate $F\{(x,\infty)\} = F(x,\infty) = 1 - F(x)$, and write, for example,

$$F[x,\infty) = \int_{[x,\infty)} dF(y) = 1 - F(x-) = P(X_1 \ge x).$$

We shall also write $F^r[x,\infty)$ for $(F[x,\infty))^r$ and $F^r(a,b]$ for $(F(a,b])^r$.

To formulate the results we need the function

$$\mu(x) := \inf\{y > 0 : F(y,\infty) \le x^{-1}\} \tag{2.1}$$

on $x > 0$, and the non-decreasing sequence $\mu_n := \mu(n)$.

2.1. Differences bounded above

We seek conditions for $\limsup_{n\to\infty}(R^{(r)}_{n+s} - R^{(r)}_n) \le c$ a.s. or $\limsup_{n\to\infty}(M^{(r)}_n - M^{(r+s)}_n) \le c$ a.s., where r, $s \in \mathbb{N}$ and $c \ge 0$. For $s = 1$ these properties turn out to be equivalent to each other and to an integral criterion related to one of them which characterizes the a.s. stability of $M^{(r)}_n$.

THEOREM 2.1. *Fix $r \in \mathbb{N}$ and $c \ge 0$. The following are equivalent:*

$$\limsup_{n\to\infty}(R^{(r)}_{n+1} - R^{(r)}_n) \le c \text{ a.s.}; \tag{2.2}$$

$$-c \le \liminf_{n\to\infty}(M^{(r)}_n - \mu_n) \le 0 \le \limsup_{n\to\infty}(M^{(r)}_n - \mu_n) \le c \text{ a.s.}; \tag{2.3}$$

$$\limsup_{n\to\infty}(M^{(r)}_n - M^{(r+s)}_n) \le c \text{ a.s., for some (hence all) } s \in \mathbb{N}; \tag{2.4}$$

$$\limsup_{n\to\infty}(M_n^{(r)} - M_{n-s}^{(r)}) \le c \text{ a.s., } \textit{for some (hence all) } s \in \mathbb{N}; \tag{2.5}$$

$$I_r(c') := \int_{-\infty}^{\infty} \left(\frac{F[x,\infty)}{F[x-c',\infty)}\right)^r \frac{dF(x)}{F[x,\infty)} < \infty \textit{ for all } c' > c. \tag{2.6}$$

We remark that (2.6) is equivalent to

$$I_r^*(c,\varepsilon) := \sum_{n=0}^{\infty} \left(\frac{F(n\varepsilon,\infty)}{F(n\varepsilon - c - \varepsilon,\infty)}\right)^r = \sum_{n=0}^{\infty} e^{-r\eta(n\varepsilon - c - \varepsilon, n\varepsilon]} < \infty \tag{2.7}$$

for each $\varepsilon > 0$, as shown in the proof of Theorem 2.1, a condition which may be easier to check in practice.

We now give an 'unquantified' version of Theorem 2.1. (Similar versions can be formulated for some of the later results below, but we will not give them.)

COROLLARY 2.2. *Fix $r \in \mathbb{N}$. The following are equivalent:*

$$\limsup_{n\to\infty}(R_{n+1}^{(r)} - R_n^{(r)}) < \infty \text{ a.s.;} \tag{2.8}$$

for some non-random sequence $b_n \uparrow \infty$,

$$-\infty < \liminf_{n\to\infty}(M_n^{(r)} - b_n) \le \limsup_{n\to\infty}(M_n^{(r)} - b_n) < \infty \text{ a.s.;} \tag{2.9}$$

$$-\infty < \liminf_{n\to\infty}(M_n^{(r)} - \mu_n) \le 0 \le \limsup_{n\to\infty}(M_n^{(r)} - \mu_n) < \infty \text{ a.s.;} \tag{2.10}$$

$$\limsup_{n\to\infty}(M_n^{(r)} - M_n^{(r+s)}) < \infty \text{ a.s., } \textit{for some (hence all) } s \in \mathbb{N}; \tag{2.11}$$

$$\limsup_{n\to\infty}(M_n^{(r)} - M_{n-s}^{(r)}) < \infty \text{ a.s., } \textit{for some (hence all) } s \in \mathbb{N}; \tag{2.12}$$

$$I_r(c) < \infty \textit{ for some } c > 0. \tag{2.13}$$

If we specialize Theorem 2.1 to the case $c = 0$ we obtain equivalences for the existence of $b_n \uparrow \infty$ such that $M_n^{(r)} - b_n \to 0$ a.s., which is the a.s. *stability* of $M_n^{(r)}$. However, our conclusions then go much further, and perhaps the most surprising part of our next theorem is that the a.s. stability (also, the a.s. relative stability, see below) of $M_n^{(r)}$ is equivalent to that of $M_n^{(1)}$.

THEOREM 2.3. *The following are equivalent to each other, for all r, $s \in \mathbb{N}$:*

$$R_{n+s}^{(r)} - R_n^{(r)} \xrightarrow{a.s.} 0; \tag{2.14}$$

$$M_n^{(r)} - b_n \xrightarrow{a.s.} 0 \textit{ for a non-random sequence } b_n \uparrow \infty; \tag{2.15}$$

$$M_n^{(r)} - \mu_n \xrightarrow{a.s.} 0; \tag{2.16}$$

$$M_n^{(r)} - M_n^{(r+s)} \xrightarrow{a.s.} 0; \tag{2.17}$$

$$M_n^{(r)} - M_{n-s}^{(r)} \xrightarrow{a.s.} 0; \tag{2.18}$$

$$I_r(\varepsilon) < \infty \textit{ for all } \varepsilon > 0; \tag{2.19}$$

$$I_1(\varepsilon) < \infty \textit{ for all } \varepsilon > 0. \tag{2.20}$$

We emphasize that the theorem states that each of the statements for one particular r, s pair is equivalent to itself for any *other* pair r, $s \in \mathbb{N}$, as well as to the other statements. This is the case also for the results below, except where r or s is held fixed.

Theorem 2.3 generalizes and extends a number of previously proved results. Barndorff-Nielsen (1963) stated for $r \in \mathbb{N}$ and proved for $r = 1$ the equivalence of the a.s. stability condition (2.15) and the integral criterion (2.19). On the other hand Resnick and Tomkins (1973) proved the case $r = 1$ of, and Hall (1979a) the case $r \in \mathbb{N}$ of, the following equivalence for the a.s. *relative stability* of $M_n^{(r)}$: for a non-random sequence b_n,

$$M_n^{(r)}/b_n \to 1 \text{ a.s.} \tag{2.21}$$

if and only if

$$\tilde{I}_r(\lambda) := \int_0^\infty \left(\frac{F[x,\infty)}{F[\lambda x,\infty)}\right)^r \frac{dF(x)}{F[x,\infty)} < \infty \text{ for all } \lambda \text{ in } (0,1). \tag{2.22}$$

(See also Tomkins (1986).) Now note that, since $F(0) < 1$, for each fixed r we have $M_n^{(r)} > 0$ for large enough n, a.s., so we may without loss of generality assume the X_i are positive if convenient. Then we see that Theorem 2.3 also proves the equivalence of the a.s. relative stability condition (2.21) with (2.22), and likewise with $R_{n+s}^{(r)}/R_n^{(r)} \to 1$ for r, $s \in \mathbb{N}$, via the logarithmic transformation that maps X_i into $\log X_i$, with d.f. $F(e^x)$. For continuous F (a restriction we do not require) Maller and Resnick (1984) gave the multiplicative version of the equivalence of (2.14) and (2.16)–(2.17) when $r = s = 1$.

Note further that Theorem 2.3 shows that, for $r > 1$, $I_r(\varepsilon) < \infty$ for all $\varepsilon > 0$ if and only if $I_1(\varepsilon) < \infty$ for all $\varepsilon > 0$, and, via the logarithmic transform, that for $r > 1$, $\tilde{I}_r(\lambda) < \infty$ for all $0 < \lambda < 1$ if and only if $\tilde{I}_1(\lambda) < \infty$ for all $0 < \lambda < 1$. Thus we have the important corollary of Theorem 2.3: *$M_n^{(r)}$ is a.s. stable (respectively, a.s. relatively stable) for $r > 1$ if and only if the same is true of $M_n^{(1)}$.* This had been noticed by H. Kesten in 1990, in a personal communication to one of us, and it also follows from some results of Kinoshita and Resnick (1991), who show that, for a non-random sequence b_n, the random sets $\{X_i/b_n, 1 \le i \le n\}$ converge a.s. in the Hausdorff metric to the interval $[0,1]$ if and only if $M_n^{(1)}/b_n \to 1$ a.s. It is plausible, and in fact not difficult to show from their methods, that we must then have $M_n^{(r)}/b_n \to 1$ a.s. for $r = 2, \ldots$

(2.22) is not, however, implied by the convergence of $\tilde{I}_1(\lambda)$ for *some* $\lambda \in (0,1)$; similarly for (2.19) and (2.20). In its multiplicative version, Corollary 2.2 tells us that the convergence of $\tilde{I}_r(\lambda)$, defined in (2.22), for some λ in (0,1), is equivalent to

$$0 < \liminf_{n\to\infty} \frac{M_n^{(r)}}{b_n} \le \limsup_{n\to\infty} \frac{M_n^{(r)}}{b_n} < \infty \text{ a.s.}$$

for $r \in \mathbb{N}$ and some non-random $b_n \uparrow \infty$ (and then b_n may be replaced by μ_n).

If (2.22) holds, i.e., $\tilde{I}_r(\lambda) < \infty$ for *all* λ in $(0,1)$, then $F[\cdot,\infty)$ is rapidly varying, i.e.,

$$F[x,\infty)/F[\lambda x,\infty) \to 0 \qquad (x \to \infty) \tag{2.23}$$

for every $\lambda \in (0,1)$ (see (3.17) below). Thus the results in this section are characteristic of light-tailed distributions. Relative stability *in probability* of $M_n^{(r)}$ (i.e., $M_n^{(r)}/\mu_n \xrightarrow{P} 1$) for $r > 1$ was known to be equivalent to that of $M_n^{(1)}$; this is Hall ((1979b), Theorem 3). By the log transform, this is also true for stability in probability. Relative stability in probability is equivalent to rapid variation of $F[\cdot,\infty)$, i.e., (2.23), and this is evidently implied by convergence of $\tilde{I}_r(\lambda)$ for all λ in $(0,1)$, as follows also from our results. The device of moving between an additive and a multiplicative setting by means of the logarithmic transform is one that we shall often use. Geffroy (1958/59) developed this device in the context of extremes.

It is natural also to ask about the connection between the a.s. stability or relative stability of $M_n^{(r)}$ and that of $R_n^{(r)}$. From the point process point of view adopted herein, the a.s. stability of $R_n^{(1)} = R_n$, say, i.e., the condition that there exists a non–random sequence b_n for which $R_n - b_n \to 0$ a.s., is easily addressed by noting that, when F is continuous, the sequence $\eta(-\infty, R_n]$ is just the sequence of points of a unit Poisson process on $(0,\infty)$. It would be useful to convert this to an analytic equivalence (i.e., a condition on the tails of F) for a.s. stability, but currently no such condition is known. This issue is discussed by Resnick (1973) and de Haan and Resnick (1973). The former gives an example for which $M_n^{(1)}$ is a.s. relatively stable but R_n is not, and a similar example exists for a.s. stability, by the log transform. De Haan and Resnick (1973) give conditions of a generalized regular variation type for behaviour such as

$$0 < \liminf_{n\to\infty} \frac{R_n}{b_n} < \limsup_{n\to\infty} \frac{R_n}{b_n} < \infty \text{ a.s.},$$

where b_n is a non–random sequence, and compare them with the behaviour of $M_n^{(1)}$. Again the connection is not close. This adds interest to Theorems 2.1 and 2.3, which do establish equivalences between the a.s. behaviour of $M_n^{(r)}$ and $R_n^{(r)}$.

By means of an example (Example 3.7 in §3.2) we shall show that the extra equivalences given by Theorem 2.3 do not extend to the case of Theorem 2.1. We will however prove the following.

THEOREM 2.4. *Assume F is continuous. For fixed r, $s \in \mathbb{N}$ and $c \geq 0$,*

$$\limsup_{n\to\infty}(R_{n+s}^{(r)} - R_n^{(r)}) \leq c \text{ a.s.} \tag{2.24}$$

if and only if for all $\varepsilon > 0$,

$$\sum_{n=0}^{\infty}\left(1 + \left(\log\frac{F(n\varepsilon - c - \varepsilon, \infty)}{F(n\varepsilon, \infty)}\right)^{s-1}\right)\left(\frac{F(n\varepsilon,\infty)}{F(n\varepsilon - c - \varepsilon, \infty)}\right)^{r} < \infty. \tag{2.25}$$

Convergence of (2.25) for some s does not imply convergence for all s, as Example 3.7 will also demonstrate. We also give, in Proposition 3.6, a somewhat less explicit equivalence for (2.24) without the assumption of continuity of F.

2.2. Differences, and ratios, bounded below

In this sub-section we look for conditions that bound the differences of partial record values or order statistics away from 0, almost surely. This is a property of heavy-tailed distributions, by contrast with the light-tailed F in §2.1. We proceed by characterizing $\liminf_{n\to\infty}(R_{n+s}^{(r)} - R_n^{(r)}) > 0$ a.s.

THEOREM 2.5. *Suppose F is continuous. Fix $s \in \mathbb{N}$. The following are equivalent:*

$$R_{n+s}^{(r)} - R_n^{(r)} \xrightarrow{a.s.} \infty \text{ for some (hence all) } r \in \mathbb{N}; \tag{2.26}$$

$$\liminf_{n\to\infty}(R_{n+s}^{(r)} - R_n^{(r)}) > 0 \text{ a.s., for some (hence all) } r \in \mathbb{N}; \tag{2.27}$$

$$\int_{-\infty}^{\infty}\left(\frac{F[x-c,\infty)}{F[x,\infty)} - 1\right)^{s}\frac{dF(x)}{F[x,\infty)} < \infty \text{ for some (hence all) } c > 0. \tag{2.28}$$

Each of these assertions implies

$$\lim_{x\to\infty} \frac{F[x-c,\infty)}{F[x,\infty)} \to 1 \text{ for some (hence all) } c \in \mathbb{R}; \tag{2.29}$$

equivalently, $F(\log x, \infty)$ is slowly varying as $x \to \infty$.

The assumption of continuity of F in Theorem 2.5 is crucial, as we show in Propositions 3.13 and 3.15, where equivalences for (2.26) and (2.27) without that assumption are given. The case $r = s = 1$ of (2.26) appears (in its multiplicative form) in Maller and Resnick ((1984), Theorem 3.1). We remark also that the conditions of Theorem 2.5 are equivalent to

$$\sum_{n=0}^{\infty} \left(\frac{F(n-1,\infty)}{F(n,\infty)} - 1 \right)^{s+1} < \infty \tag{2.30}$$

(see the proof of Theorem 2.5), which may be convenient for checking. In Example 3.17 we show that the assertions of Theorem 2.5 are not equivalent for different s.

Now we come to equivalences for ratios of order statistics and trimmed sums. The *trimmed sum* of order r is defined by

$$^{(r)}S_n := S_n - M_n^{(1)} - \cdots - M_n^{(r)} \qquad (r = 1, 2, \ldots, n),$$

with $^{(0)}S_n := S_n = X_1 + X_2 + \cdots + X_n$, and is important in various areas of statistics. For these it is more natural to use a multiplicative setting and we restrict ourselves to non-negative X_i.

THEOREM 2.6. *Suppose $F(-\infty, 0) = 0$. Fix $s \in \mathbb{N}$. The following are equivalent, for all $r \in \mathbb{N}$:*

$$\liminf_{n\to\infty} \frac{M_n^{(r)}}{M_n^{(r+s)}} > 1 \text{ a.s.}; \tag{2.31}$$

$$\lim_{n\to\infty} \frac{M_n^{(r)}}{M_n^{(r+s)}} = \infty \text{ a.s.}; \tag{2.32}$$

$$\liminf_{n\to\infty} \frac{M_n^{(r)}}{^{(r+s-1)}S_n} > 1 \text{ a.s.}; \tag{2.33}$$

$$\lim_{n\to\infty} \frac{M_n^{(r)}}{^{(r+s-1)}S_n} = \infty \text{ a.s.}; \tag{2.34}$$

$$\text{for continuous } F,\ \liminf_{n\to\infty} \frac{R_{n+s}^{(r)}}{R_n^{(r)}} > 1 \text{ a.s.}; \tag{2.35}$$

$$\text{for continuous } F,\ \lim_{n\to\infty} \frac{R_{n+s}^{(r)}}{R_n^{(r)}} = \infty \text{ a.s.}; \tag{2.36}$$

for some (hence all) $\lambda \in (0,1)$,

$$J_s(\lambda) := \int_0^{\infty} \left(1 - \frac{F(x,\infty)}{F(\lambda x,\infty)}\right)^s \frac{dF(x)}{F(x,\infty)} < \infty. \tag{2.37}$$

Maller and Resnick ((1984), Theorem 3.1) gave a necessary condition and a separate sufficient condition for the case $r = s = 1$ of (2.33). Pruitt (1987) showed these conditions to be equivalent to (2.37) and gave other integral equivalences for (2.37). Our proof of (2.34) depends strongly upon one of these, but is otherwise different and more general in allowing arbitrary values of r. Note however that we consider only non-negative X_i, whereas one of the (difficult) equivalences in Pruitt's Theorem 1 concerns signed r.v.s in a crucial way; we have not generalized that result. The results in Maller and Resnick (1984) also allow signed X_i but, as Pruitt remarks, are essentially statements about $|X_i|$ for the most part. Note also the interesting feature that in Theorem 2.1 it is the index r, not s, that is significant, whereas in Theorems 2.5 and 2.6, s is significant rather than r. The multiplicative version of (2.28) is equivalent to (2.37), and each implies the slow variation of $F(x,\infty)$; see the proof of Theorem 2.6.

The following result generalizes Pruitt's (1987) new construction of the positive integers.

COROLLARY 2.7. *Suppose $F(-\infty,0)=0$. For each $r\geq 1$, the only possible values of $\limsup_{n\to\infty} {}^{(r-1)}S_n/M_n^{(r)}$ are the positive integers and ∞.*

2.3. Record differences bounded away from 0 and ∞

Recall that we write simply R_n for $R_n^{(1)}$. As we note in the proof of Corollary 2.2, the lim sup and lim inf of $R_{n+s}-R_n$, as $n\to\infty$, are degenerate r.v.s. We show that if both of these are finite and positive, and the first is less than $(s+1)/s$ times the second, then there is a non-random sequence $x_n\to\infty$ with which the sequence R_n coincides, sufficiently far along. In particular, this is the only way $R_{n+s}-R_n$ can converge a.s. to a finite positive limit, and so that behaviour is entirely restricted to non-continuous distributions.

THEOREM 2.8. *Let s be a fixed positive integer and let $0<c<d<\infty$ be constants with $d<(s+1)c/s$. Then*

$$c\leq \liminf_{n\to\infty}(R_{n+s}-R_n)\leq \limsup_{n\to\infty}(R_{n+s}-R_n)\leq d \text{ a.s.} \tag{2.38}$$

if and only if there is a strictly increasing non-random sequence (x_k), with

$$c\leq \liminf_{k\to\infty}(x_{k+s}-x_k)\leq \limsup_{k\to\infty}(x_{k+s}-x_k)\leq d, \tag{2.39}$$

such that

$$\sum_{k=1}^{\infty}\frac{F(x_k,x_{k+1})+F(x_{k+1},\infty)}{F\{x_{k+1}\}}<\infty. \tag{2.40}$$

Furthermore, there then exist r.v.s N, T, such that $R_n=x_{T+n}$ for all $n\geq N$, almost surely.

Condition (2.40) is most easily satisfied by a discrete F, concentrated on the x_k and with probability masses that decay sufficiently fast; take, for instance $F\{x_k\}\propto e^{-k^2}$. The condition $d<(s+1)c/s$ is intrinsic, in that without it the characterization of (2.38) by (2.39)–(2.40) is not valid and a much more complicated one is needed. We discuss this point in §3.6, after the proof.

The following is an immediate consequence of Theorem 2.8.

COROLLARY 2.9. *Fix $s\in\mathbb{N}$. Then $R_{n+s}-R_n\to c$ a.s., where $0<c<\infty$, if and only if there is a non-random sequence (x_n) with $x_{n+s}-x_n\to c$, such that (2.40) holds.*

In Remark 3.23 in §3.5 we mention some analogues of the above for partial maxima. For r, $s\in\mathbb{N}$, we always have $M_n^{(r)}-M_{n-s}^{(r)}\xrightarrow{P}0$ and $M_n^{(r)}/M_{n-s}^{(r)}\xrightarrow{P}1$, and

$M_n^{(r+s)}/M_n^{(r)}$ can converge in probability (hence, a.s.) to a constant only if that constant is 0 or 1, in which case $F(x,\infty)$ is either slowly or rapidly varying.

These ideas suggest some useful applications in Statistics that are specifically concerned with stability concepts. Green (1976), developing ideas of Neyman and Scott (1971), declared a distribution to be *Absolutely (resp., Relatively) Outlier Resistant* (AOR or ROR) if $M_n^{(1)} - M_n^{(2)} \xrightarrow{P} 0$ (resp., $M_n^{(2)}/M_n^{(1)} \xrightarrow{P} 1$). For such distributions, we will not expect to see 'outlying' observations manifested by large values of the maximum $M_n^{(1)}$ remote from $M_n^{(2)}$. Green showed that F is AOR (resp., ROR) if and only if, in our terminology, $M_n^{(1)}$ is stable (resp., relatively stable) in probability, mentioning also the relevant analytic conditions. From Hall (1979a) we know that $M_n^{(1)}$ is stable if and only if the same is true of $M_n^{(r)}$, and then $M_n^{(r)} - M_n^{(r+s)} \xrightarrow{P} 0$ for all r, $s \in \mathbb{N}$. Thus in fact we do not expect to see large values of the spacings $M_n^{(r)} - M_n^{(r+s)}$ for any lag s, if F is AOR. Our results in Theorem 2.3 give the analogues of these for almost-sure convergence, so we can define outlier resistance in a strong sense. As a bonus we obtain from Theorem 2.3 that again $M_n^{(r)} - M_n^{(r+s)} \to 0$ for all r, $s \in \mathbb{N}$, this time a.s., if and only if $M_n^{(1)} - M_n^{(2)} \to 0$ a.s.

Theorems 2.3–2.6 allow statisticians to make the following general classification: distributions may be very light-tailed, thus strongly outlier-resistant, ($M_n^{(1)} - M_n^{(2)} \to 0$ a.s.), or very heavy-tailed and thus strongly outlier-prone ($M_n^{(1)} - M_n^{(2)} \to \infty$ a.s.). There is no intermediate case, in the sense that $M_n^{(1)} - M_n^{(2)} \to c$ a.s. for some $c \in (0,\infty)$ cannot occur, though we could have convergence in distribution to a non-degenerate limit distribution.

Extensions of Green's results are given by Mathar (1984), who merely requires, for example, that $M_n^{(1)} - M_n^{(2)}$ be bounded in probability. The generalization of this to the a.s. case is available from our Theorem 2.1. For a quite different application of a.s. relative stability, see Nayak and Wali (1992), who study the moments of $N(\delta)$, the number of times $M_n^{(1)}$ exceeds $(1+\delta)b_n$, when $M_n^{(1)}/b_n \to 1$ a.s. These kinds of results have applications to boundary-crossing and optimal-stopping problems. In another statistical context of current interest, Sepanski (1993) connects the heavy-tailed results of Maller and Resnick (1984) and Pruitt (1987), related to Theorem 2.6 above, to the large-sample behaviour of the randomly normed and centred bootstrap mean.

Among extensions of the basic model treated here it is natural to consider (i) multivariate versions and (ii) weakenings of the independence assumption. Considering first (i), the results in the present paper can be applied to each univariate margin of a stream of i.i.d. multivariate r.v.s. However, as concerns genuine multivariate extensions, we remark that while the theory of multivariate extremes is quite well developed (cf. Resnick (1987), Chapter 5), the theory of multivariate records is much less so. There are competing possibilities even for how to define multivariate record values (Goldie and Resnick (1989)). In many cases of interest the number of simultaneous record values of a stream of bivariate r.v.s is a.s. finite (Gnedin (1993, 1994)), so there is no asymptotic behaviour to be considered.

Concerning (ii), we mention Haiman (1987), where it is proved that for certain stationary m–dependent sequences the bivariate sequence of record times and values is representable as the corresponding sequence from a stream of i.i.d. r.v.s, subject to a certain lag. This permits a number of our results to be extended to suitable m–dependent sequences of input r.v.s.

3. Proofs

This section contains proofs of all results together with some complements and examples. For the results up to Theorem 2.3, the proofs naturally divide themselves into those about partial records, and those about order statistics. We deal with the former in §3.2 and with the latter in §3.3, assembling our interim results in separately stated propositions. Thus Propositions 3.3 and 3.10 together will establish Theorem 2.1. In §3.3, after concluding the proof of Proposition 3.10, we will prove Corollary 2.2, and then a few lines of proof will finish Theorem 2.3. Theorem 2.4, concerning records, is proved in §3.2. Theorems 2.5 and 2.6 are proved in §3.4 and §3.5, while Theorem 2.8 is proved in the final §3.6.

3.1. Preliminaries

We need some further development of the point process set-up of §1. The tail measure η defined in (1.3) is to be distinguished from the *hazard measure* H defined by

$$dH(x) := \frac{dF(x)}{F[x,\infty)} \qquad (x \in \mathbb{R}).$$

We have $H(B) \le \eta(B)$ for all $B \in \mathcal{B}$, and H and η are the same measure when, and only when, F is continuous. Shanbhag ((1979), Theorem 2) proves that $H(-\infty, x] < \infty$, and $H(x, \infty] = \infty$, for all $x \in \mathbb{R}$. This dichotomy also holds trivially for η, from its definition.

Denote the set of atoms of a sigma-finite measure ν by D_ν and decompose ν into its continuous and discrete components: $\nu = \nu^c + \nu^d$, where $\nu^d(B) := \sum_{x \in D_\nu \cap B} \nu\{x\}$ and $\nu^c(B) := \nu(B \setminus D_\nu)$, the set D_ν being countable. With this notation we obtain a completely explicit representation of the point process ρ_i (or of any Shorrock process):

$$\rho_i = \sum_{x \in D_F} I_x \delta_x + \rho_i', \tag{3.1}$$

where all the I_x and ρ_i' are mutually independent, the I_x are Bernoulli r.v.s:

$$P(I_x = 0) = e^{-\eta\{x\}} = 1 - H\{x\} = 1 - P(I_x = 1),$$

and ρ_i' is a Poisson process of intensity measure $H^c = \eta^c$. In particular, we have $E(\rho_i(B)) = H(B)$ for Borel B and $i = 1, \ldots, r$.

From (1.2) and the independence of the ρ_i we obtain the following useful formula for the avoidance function of ζ_r:

$$P(\zeta_r(B) = 0) = e^{-r\eta(B)} \qquad (B \in \mathcal{B}). \tag{3.2}$$

Similarly we get $E(\zeta_r(B)) = rH(B)$ for all $B \in \mathcal{B}$, so in particular H is the mean measure of ζ_1. When $r = 1$, ζ_r is itself a Shorrock process, while if F is continuous it is a Poisson process of intensity $rH(\cdot)$, and in each of those cases it is a simple point process.

The point process ρ_i is always simple but not in general Poisson. Rather, it is the *simplification* (Engelen *et al.* (1988)) of a Poisson process of intensity measure H: its distribution is that of the simple point process obtained from such a Poisson process by replacing every multiple point by a singleton.

We need some preliminary results before the main proofs; first, a generalization of the formula $r\int_y^\infty F^{r-1}(x,\infty)\,dF(x) = F^r(y,\infty)$, which holds when F is continuous. (Recall from §2 that $F^r(y,\infty) := (F(y,\infty))^r$.) In Kesten and Maller (1992) it is shown that the general version is

$$F^r(y,\infty) = \int_{(y,\infty)} \sum_{j=0}^{r-1} F^j(x,\infty) F^{r-1-j}[x,\infty)\,dF(x). \tag{3.3}$$

So we obtain

$$\int_{(y,\infty)} F^{r-1}[x,\infty)\,dF(x) \le F^r(y,\infty) \le r\int_{(y,\infty)} F^{r-1}[x,\infty)\,dF(x); \tag{3.4}$$

and similarly

$$\int_{[y,\infty)} F^{r-1}[x,\infty)\,dF(x) \le F^r[y,\infty) \le r\int_{[y,\infty)} F^{r-1}[x,\infty)\,dF(x). \tag{3.5}$$

We also need two lemmas about sequences.

LEMMA 3.1. *For a real sequence $r_n \uparrow \infty$ and constant $c \ge 0$, it is the case that* $\limsup_{n\to\infty}(r_{n+1}-r_n) > c$ *if and only if there exists $\varepsilon > 0$ such that infinitely many of the intervals $(n\varepsilon - c - \varepsilon, n\varepsilon]$, for $n = 0, 1, 2, \ldots$, contain no element of the sequence.*

Proof. 'If' is immediate, as the latter assertion guarantees infinitely many gaps of length $c+\varepsilon$, with left end-points tending to infinity.

Suppose $\limsup_{n\to\infty}(r_{n+1}-r_n) > c$, then there exists $\varepsilon > 0$ such that $r_{n'+1} - r_{n'} > c + 2\varepsilon$ along some subsequence $r_{n'}$. If m is the least integer such that $r_{n'} \le m\varepsilon - c - \varepsilon$ then

$$m\varepsilon = (m\varepsilon - c - \varepsilon - r_{n'}) + r_{n'} + c + \varepsilon < \varepsilon + r_{n'} + c + \varepsilon < r_{n'+1},$$

so the interval $(m\varepsilon - c - \varepsilon, m\varepsilon]$ contains none of the r_n. □

LEMMA 3.2. *If $(s_n)_{n\ge 0}$ is a sequence such that $s_n \downarrow 0$ and $\sum_{n=m}^\infty (s_n - s_{n+1})/s_{n-m} < \infty$ for some positive integer m, then $\sum_{n=m}^\infty s_n/s_{n-m} < \infty$.*

Proof. First, for $n \ge m$,

$$s_n = \sum_{k=n}^\infty s_{k-m}\frac{s_k - s_{k+1}}{s_{k-m}} \le s_{n-m}\sum_{k=n}^\infty \frac{s_k - s_{k+1}}{s_{k-m}} = \mathrm{o}(s_{n-m})$$

as $n \to \infty$. So there exists $n_0 \ge m$ such that $s_n \le 2^{-m}s_{n-m}$ for all $n \ge n_0$. Iterating this gives $s_{n+km} \le 2^{-km}s_n$ for $k = 0, 1, 2, \ldots$ and $n \ge n_0 - m$. Hence there is a constant $a < \infty$ such that $s_{n+k} \le a2^{-k}s_n$ for $k = 0, 1, 2, \ldots$ and $n \ge n_0 - m$. Then $as_{n-m} \ge 2^{k-n}s_{k-m}$ for $n_0 \le n \le k$, so

$$\begin{aligned}\sum_{n=n_0}^\infty \frac{s_n}{s_{n-m}} &= \sum_{n=n_0}^\infty \frac{1}{s_{n-m}}\sum_{k=n}^\infty (s_k - s_{k+1}) \\ &= \sum_{k=n_0}^\infty (s_k - s_{k+1})\sum_{n=n_0}^k \frac{1}{s_{n-m}} \\ &\le \sum_{k=n_0}^\infty (s_k - s_{k+1})\sum_{n=n_0}^k \frac{a2^{-(k-n)}}{s_{k-m}} < 2a\sum_{k=n_0}^\infty \frac{s_k - s_{k+1}}{s_{k-m}},\end{aligned}$$

and the result follows. □

3.2. Proofs: upper bounds on record differences

PROPOSITION 3.3. *Fix $r \in \mathbb{N}$ and $c \geq 0$. Then (2.2), (2.6), and (2.7) are equivalent.*

Proof. Since $R_n^{(r)} \uparrow \infty$ a.s. as $n \to \infty$, Lemma 3.1 gives

$$\begin{aligned} P\big(\limsup_{n\to\infty}(R_{n+1}^{(r)} - R_n^{(r)}) > c\big) &= P\left(\bigcup_{\varepsilon>0}\{\zeta_r(n\varepsilon - c - \varepsilon, n\varepsilon] = 0 \text{ i.o.}\}\right) \\ &= P\left(\bigcup_{k=1}^{\infty}\{\zeta_r(n2^{-k} - c - 2^{-k}, n2^{-k}] = 0 \text{ i.o.}\}\right) \end{aligned}$$

(where 'i.o.' stands for 'infinitely often in n'). So (2.2) is equivalent to

$$P\big(\zeta_r(n\varepsilon - c - \varepsilon, n\varepsilon] = 0 \text{ i.o.}\big) = 0 \text{ for all } \varepsilon > 0. \tag{3.6}$$

By the complete randomness property of ζ_r (see §1), the events that $\zeta_r(n\varepsilon-c-\varepsilon, n\varepsilon] = 0$ are independent when n varies so that the intervals $(n\varepsilon - c - \varepsilon, n\varepsilon]$ are disjoint, i.e. for any set of values of n separated by at least $(c+\varepsilon)/\varepsilon$. Pick an integer $m \geq (c+\varepsilon)/\varepsilon$ and let n run through each of the residue classes $[0]_m, [1]_m, \ldots, [m-1]_m$. Applying the Borel-Cantelli Lemmas to each, we see that (3.6) is equivalent to

$$\sum_{n=0}^{\infty} P(\zeta_r(n\varepsilon - c - \varepsilon, n\varepsilon] = 0) < \infty \text{ for all } \varepsilon > 0. \tag{3.7}$$

But by the avoidance formula (3.2), and (1.3), this is (2.7).

Assume next that (2.7), i.e., (3.7), holds with $r = 1$. By writing $F(n\varepsilon, \infty)$ as $\sum_{k=n}^{\infty} F(k\varepsilon, (k+1)\varepsilon]$ and interchanging the sums, we get from (2.7) that

$$I_1^*(c, \varepsilon) = \sum_{k=0}^{\infty} F(k\varepsilon, (k+1)\varepsilon] \sum_{n=0}^{k} \frac{1}{F(n\varepsilon - c - \varepsilon, \infty)} < \infty$$

for all $\varepsilon > 0$. Taking the term for $n = k$, we see that the left-hand side is at least

$$\begin{aligned} &\sum_{k=0}^{\infty} \int_{(k\varepsilon,(k+1)\varepsilon]} \frac{dF(x)}{F(k\varepsilon - c - \varepsilon, \infty)} \\ &\geq \sum_{k=0}^{\infty} \int_{(k\varepsilon,(k+1)\varepsilon]} \frac{dF(x)}{F(x - c - 2\varepsilon, \infty)} = \int_{(0,\infty)} \frac{dF(x)}{F(x - c - 2\varepsilon, \infty)}. \end{aligned}$$

Now $\int_{(-\infty,0]} dF(x)/F(x - c - 2\varepsilon, \infty)$ is finite in any case, since it is bounded above by $\int_{(-\infty,0]} dF(x)/F(0,\infty) < \infty$. So we conclude $I_1(c + 2\varepsilon) < \infty$, whence (2.6) (for $r = 1$).

Conversely, assume (2.6) for $r = 1$. Choose $\delta > 0$ and let m be the integer such that $(m-1)\delta \leq c < m\delta$. Take $\varepsilon := m\delta - c$. Then from $I_1(c + \varepsilon) < \infty$ we conclude

$$\sum_{n=0}^{\infty} \frac{F(n\delta, \infty) - F((n+1)\delta, \infty)}{F((n-m)\delta, \infty)} \leq \sum_{n=0}^{\infty} \int_{(n\delta,(n+1)\delta]} \frac{dF(x)}{F(x - m\delta, \infty)} < \infty.$$

We now have the hypothesis of Lemma 3.2, with $s_n := F(n\delta,\infty)$, and its conclusion is $\sum_{n=0}^{\infty} F(n\delta,\infty)/F(n\delta-c-\varepsilon,\infty)<\infty$, which implies $I_1^*(c,\delta)<\infty$ since $\varepsilon\le\delta$. Thus we have (2.7) (for $r=1$).

Now we extend the equivalence of (2.6) and (2.7) to general r. Define a d.f. G by $G(x,\infty):=F^r(x,\infty)=(F(x,\infty))^r$, which is the tail of a proper probability law. Applying what we have just proved to G, we find (2.7) equivalent to

$$\int_{-\infty}^{\infty}\frac{dG(x)}{G(x-c-\varepsilon,\infty)}<\infty \text{ for all } \varepsilon>0.$$

Now by (3.3), $dG(x)=dF(x)\sum_{j=0}^{r-1}F^j(x,\infty)F^{r-1-j}[x,\infty)$, and the sum is bounded below by $F^{r-1}[x,\infty)$ and above by $rF^{r-1}[x,\infty)$. So finally (2.7) is equivalent to

$$\int_{-\infty}^{\infty}\left(\frac{F[x,\infty)}{F(x-c-\varepsilon,\infty)}\right)^r\frac{dF(x)}{F[x,\infty)}<\infty \text{ for all } \varepsilon>0.$$

We can change $F(x-c-\varepsilon,\infty)$ to $F[x-c-\varepsilon,\infty)$ by sandwiching, so this condition is in fact (2.6). □

Proposition 3.4. *(2.14), (2.19) and (2.20) are equivalent, for all r, $s\in\mathbb{N}$.*

Proof. For a real sequence r_n it is the case that $r_{n+1}-r_n\to 0$ implies $r_{n+s}-r_n=\sum_{t=1}^{s}(r_{n+t}-r_{n+t-1})\to 0$, and vice versa. So (2.14) with $s=1$ is equivalent to (2.14) itself. We can thus restrict ourselves to $s=1$ in what follows.

We know (2.20) and $I_r^*(0,\varepsilon)<\infty$ for all $\varepsilon>0$ are equivalent, by Proposition 3.3, and we know by the same result that for each fixed r the a.s. convergence of $R_{n+1}^{(r)}-R_n^{(r)}$ to 0 is equivalent to $I_r^*(0,\varepsilon)<\infty$ for all $\varepsilon>0$. Thus it suffices to show that finiteness of $I_r^*(0,\varepsilon)$ for all $\varepsilon>0$ is equivalent to finiteness of $I_1^*(0,\varepsilon)$ for all $\varepsilon>0$. But since $I_r^*(0,\varepsilon)$ is non-increasing in r, it is in fact enough to show that $I_{2r}^*(0,\varepsilon)<\infty$ for all $\varepsilon>0$ implies $I_r^*(0,\varepsilon)<\infty$ for all $\varepsilon>0$. In fact the $r=1$ case suffices, since we may then replace $F(x,\infty)$ by $(F(x,\infty))^r$. So assume simply that $I_2^*(0,\varepsilon)<\infty$. Then by (2.7), $\sum_{n=0}^{\infty}e^{-2\eta((n-2)\varepsilon,(n-1)\varepsilon]}$ is also finite. Doubling n and taking the average, we find

$$\infty>\frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-2\eta((2n-2)\varepsilon,(2n-1)\varepsilon]}+e^{-2\eta((2n-1)\varepsilon,2n\varepsilon]}\right)\ge\sum_{n=0}^{\infty}e^{-\eta((2n-2)\varepsilon,2n\varepsilon]},$$

the latter by the Arithmetic Mean-Geometric Mean inequality. But the last sum is $I_1^*(0,2\varepsilon)$, as was wanted. □

Remark 3.5. By a similar use of the A.M.-G.M. inequality we may show that $\limsup_{n\to\infty}(R_{n+1}^{(r)}-R_n^{(r)})\le c$ a.s. implies $\limsup_{n\to\infty}(R_{n+1}-R_n)\le rc$ a.s. However the converse is false: consider, for instance, η measure consisting of a mass n at each integer $n\ge 1$. Then, with probability 1, for all large n it will be the case that ζ_1 places unit mass at each n while ζ_2 places mass 2 there; thus $\limsup_{n\to\infty}(R_{n+1}^{(2)}-R_n^{(2)})=1$, while $R_{n+1}-R_n=1$ for all n. Further, if the atoms of η are replaced by the same masses spread over narrow intervals, so as to make F continuous, essentially the same behaviour results, in that $\limsup_{n\to\infty}(R_{n+1}^{(r)}-R_n^{(r)})=1$ for $r=1$ and $r=2$.

PROPOSITION 3.6. *Fix r, $s \in \mathbb{N}$ and $c \geq 0$. For general F, (2.24) is equivalent to*

$$\sum_{n=0}^{\infty} P(\zeta_r(n\varepsilon - c - \varepsilon, n\varepsilon] < s) < \infty \text{ for all } \varepsilon > 0. \tag{3.8}$$

Proof. This is a variation on some of the previous proofs. First, in Lemma 3.1 replace $r_{n+1} - r_n$ by $r_{n+s} - r_n$, and alter 'no element' to 'fewer than s elements'. The proof of the lemma is virtually the same. Thus

$$\begin{aligned} P\big(\limsup_{n\to\infty}(R_{n+s}^{(r)} - R_n^{(r)}) > c\big) &= P\left(\bigcup_{\varepsilon>0}\{\zeta_r(n\varepsilon - c - \varepsilon, n\varepsilon] < s \text{ i.o.}\}\right) \\ &= P\left(\bigcup_{k=1}^{\infty}\{\zeta_r(n2^{-k} - c - 2^{-k}, n2^{-k}] < s \text{ i.o.}\}\right) \end{aligned}$$

Now copy the rest of the proof of Proposition 3.3, with $(\zeta_r(n\varepsilon - c - \varepsilon, n\varepsilon] = 0)$ replaced by $(\zeta_r(n\varepsilon - c - \varepsilon, n\varepsilon] < s)$. □

We see from the above proposition that to obtain an explicit criterion for (2.24) we need to evaluate $P(\zeta_r(\cdot) < s)$. However, unlike the avoidance function $P(\zeta_r(\cdot) = 0) = e^{-r\eta(\cdot)}$ that figured in earlier proofs, for $s > 1$ and general F this expression has no simple evaluation in terms of η or anything else. (Recall that for any Borel set B the r.v. $\zeta_r(B)$ is the sum of a $\mathrm{Pois}(H^c(B))$ r.v. and independent $\mathrm{Bin}(r, H\{x\})$ r.v.s at each jump-point x of F in B.) So for Theorem 2.4 we have assumed F to be continuous, which makes ζ_r a Poisson process with mean measure $r\eta(\cdot) = rH(\cdot)$, as discussed in §3.1. Thus when $H(B) \to \infty$,

$$P(\zeta_r(B) < s) = \sum_{t=0}^{s-1} e^{-rH(B)}(rH(B))^t/t! \sim e^{-rH(B)}(rH(B))^{s-1}/(s-1)! \tag{3.9}$$

Proof of Theorem 2.4. Evaluating the Poisson probabilities in (3.8) as in (3.9), we see that for the terms of the series to tend to 0 we need

$$\lim_{n\to\infty} H(n\varepsilon - c - \varepsilon, n\varepsilon] = \infty \text{ for all } \varepsilon > 0. \tag{3.10}$$

Assuming that is so, (3.9) shows that convergence of the series (3.8) is equivalent to

$$\sum_{n=0}^{\infty} H^{s-1}(n\varepsilon - c - \varepsilon, n\varepsilon]e^{-rH(n\varepsilon - c - \varepsilon, n\varepsilon]} < \infty \text{ for all } \varepsilon > 0. \tag{3.11}$$

We have thus shown that (2.24) holds if and only if both (3.10) and (3.11) hold. The first of these says that $F(n\varepsilon, \infty)/F(n\varepsilon - c - \varepsilon, \infty) \to 0$ as $n \to \infty$, while the second is

$$\sum_{n=0}^{\infty}\left(\log\frac{F(n\varepsilon - c - \varepsilon, \infty)}{F(n\varepsilon, \infty)}\right)^{s-1}\left(\frac{F(n\varepsilon, \infty)}{F(n\varepsilon - c - \varepsilon, \infty)}\right)^{r} < \infty.$$

These two conditions together are equivalent to (2.25). □

EXAMPLE 3.7. This will demonstrate that for continuous F it is possible for $P\big(\limsup_{n\to\infty}(R_{n+s} - R_n) \leq 1\big)$ to be 1 for $s = 1$ and 0 for $s = 2$. For each $n \geq 3$, the measure H is to have a mass $\log n + 2\log\log n$, uniformly spread over the interval $(n, n + n^{-1}]$. We call these the 'mass blocks' of H. Elsewhere it is to have no mass.

Choose $k \in \mathbb{N}$; then for $n \geq 2^{2k+1}$, each interval $(n2^{-k}-1-2^{-k}, n2^{-k}]$ either contains the whole of one mass block, when $n \not\equiv 1 \pmod{2^k}$, or two mass blocks, when $n \equiv 1 \pmod{2^k}$. So

$$\sum_{n=2^{2k+1}}^{\infty} e^{-H(n2^{-k}-1-2^{-k}, n2^{-k}]} < \sum_{n=2^{2k+1}}^{\infty} \frac{1}{n\log^2 n} < \infty,$$

and (3.11) follows for $r = s = c = 1$. Thus $\limsup_{n\to\infty}(R_{n+1} - R_n) \leq 1$. However with $s = 2$ and $\varepsilon = 2^{-k}$ we find that for high $n \not\equiv 1 \pmod{2^k}$ the terms of the series in (3.11) are $(\log n + 2\log\log n)/(n\log^2 n) > 1/(n\log n)$ and so (3.11) fails; thus $\limsup_{n\to\infty}(R_{n+2} - R_n) > 1$ a.s.

3.3. Proofs: upper bounds on order statistics

LEMMA 3.8. *Fix $c > 0$ and $r \in \mathbb{N}$. Then $I_r(c) < \infty$ is equivalent to $\sum_{n\geq 1} n^{r-1}F^r[\mu_n + c, \infty) < \infty$ and also to $\sum_{n\geq 1} n^{r-1}F^r(\mu_n + c, \infty) < \infty$.*

Proof. This is the additive formulation of a special case of Theorem 1 of Li and Tomkins ((1991), equivalence of (2.6) and (2.7)). Their proof implicitly assumes continuity of F when $r > 1$. In general we proceed as follows. Suppose $I_r(c) < \infty$. By (3.5) we have

$$\begin{aligned}\sum_{n\geq 2} n^{r-1}F^r[\mu_n + c, \infty) &\leq r\sum_{n\geq 2} n^{r-1}\int_{[\mu_n+c,\infty)} F^{r-1}[x,\infty)\,dF(x)\\ &= r\sum_{n\geq 2} n^{r-1}\sum_{j\geq n}\int_{[\mu_j,\mu_{j+1})} F^{r-1}[x+c,\infty)\,dF(x+c)\\ &\leq r\sum_{j\geq 2} j^r\int_{[\mu_j,\mu_{j+1})} F^{r-1}[x+c,\infty)\,dF(x+c).\end{aligned}$$

Let us remove the lower end-point μ_j of each of the intervals of integration and include it in the next interval below (or in a new interval when $j = 2$). The above is thus at most

$$\begin{aligned}&r\sum_{j\geq 1}(j+1)^r\int_{(\mu_j,\mu_{j+1}]} F^{r-1}[x+c,\infty)\,dF(x+c)\\ &\qquad\leq r2^{r+1}\sum_{j\geq 1} j^r\int_{(\mu_j,\mu_{j+1}]} F^{r-1}[x+c,\infty)\,dF(x+c).\end{aligned}$$

Note from (2.1) that $F[\mu_n,\infty) \geq n^{-1} \geq F(\mu_n,\infty)$. When $x > \mu_j$ we have $F(\mu_j,\infty) \geq F[x,\infty)$, so $j \leq 1/F(\mu_j,\infty) \leq 1/F[x,\infty)$. Thus

$$\begin{aligned}\sum_{n\geq 2} n^{r-1}F^r[\mu_n + c,\infty) &\leq r2^{r+1}\sum_{j\geq 1}\int_{(\mu_j,\mu_{j+1}]}\frac{F^{r-1}[x+c,\infty)}{F^r[x,\infty)}\,dF(x+c)\\ &= r2^{r+1}\int_{(\mu_1-c,\infty)}\left(\frac{F[x,\infty)}{F[x-c,\infty)}\right)^r\frac{dF(x)}{F[x,\infty)}\\ &\leq r2^{r+1}I_r(c).\end{aligned}$$

Conversely, suppose $\sum_{n\geq 1} n^{r-1}F^r(\mu_n+c,\infty)$ converges. By (3.4), and interchanging summations as before, we obtain

$$\sum_{j\geq 2} j^r \int_{(\mu_{j-1},\mu_j]} F^{r-1}[x+c,\infty)\,dF(x+c)<\infty.$$

Again we can remove the lower end-point μ_j of each of the intervals of integration and include it in the next interval below (or in a new interval when $j=2$), obtaining

$$\sum_{j\geq 2} j^r \int_{[\mu_{j-1},\mu_j]} F^{r-1}[x+c,\infty)\,dF(x+c)<\infty.$$

Now when $x\leq\mu_j$ we have $j\geq 1/F[\mu_j,\infty)\geq 1/F[x,\infty)$, so we get

$$\int_{[\mu_1,\infty)} \frac{F^{r-1}[x+c,\infty)}{F^r[x,\infty)}\,dF(x+c)<\infty,$$

and hence $I_r(c)<\infty$. □

LEMMA 3.9. *Let $b_n\uparrow\infty$ be a non-random sequence. Fix $r\in\mathbb{N}$. We have*
(i) $P(M_n^{(r)}>b_n$ i.o.$)=0$ if and only if $\sum_{n=1}^{\infty} n^{r-1}F^r(b_n,\infty)<\infty$.
(ii) Also, if $P(M_n^{(r)}\geq b_n$ i.o.$)=1$ then $P(M_n^{(r)}<b_n$ i.o.$)=0$ provided that $\sum_{n=1}^{\infty} n^{r-1}F^r[b_{n+1},\infty)e^{-nF[b_n,\infty)}<\infty$.

Proof. (i) is Lemma 3 of Mori (1976), while for (ii) we use a lemma of Barndorff-Nielsen (1961, 1963): if $P(A_n^c$ i.o.$)=1$ and $\sum_{n=1}^{\infty}P(A_nA_{n+1}^c)$ converges then $P(A_n$ i.o.$)=0$. Apply this with $A_n:=\{M_n^{(r)}<b_n\}$. We have, when $r=2,3,\ldots$, that

$$\begin{aligned}A_nA_{n+1}^c&=\{M_n^{(r)}<b_n\leq b_{n+1}\leq M_{n+1}^{(r)}\}\\&=\{M_n^{(r)}<b_n\leq b_{n+1}\leq M_n^{(r-1)},\ X_{n+1}\geq b_{n+1}\}.\end{aligned}$$

Thus

$$\begin{aligned}P(A_nA_{n+1}^c)&=P(\text{exactly } r-1 \text{ of } X_1,\ldots,X_n \text{ exceed or equal } b_{n+1}\\&\qquad\text{and the remainder are strictly less than } b_n)P(X_{n+1}\geq b_{n+1})\\&=\binom{n}{r-1}F^{r-1}[b_{n+1},\infty)F^{n-r+1}(-\infty,b_n)F[b_{n+1},\infty)\\&<\binom{n}{r-1}F^{r}[b_{n+1},\infty)F^{-r+1}(-\infty,b_n)e^{-nF[b_n,\infty)}\\&\sim n^{r-1}F^r[b_{n+1},\infty)e^{-nF[b_n,\infty)}/(r-1)!,\end{aligned}$$

from which (ii) follows. The proof is similar when $r=1$. □

PROPOSITION 3.10. *Fix $r\in\mathbb{N}$ and $c\geq 0$. Then (2.3)–(2.6) are equivalent.*

Proof. Suppose $I_r(c)<\infty$ for some $c>0$. Then by Lemma 3.8, $\sum_{n=1}^{\infty}n^{r-1}F^r(\mu_n+c,\infty)$ converges, so by Lemma 3.9(i), $P(M_n^{(r)}-\mu_n>c$ i.o.$)=0$. Thus $\limsup_{n\to\infty}(M_n^{(r)}-\mu_n)\leq c$ a.s. On replacing c by $c+\varepsilon$, and letting $\varepsilon\downarrow 0$ through some sequence, we conclude that (2.6) implies the right-hand inequality of (2.3).

Note that, for $r=1,2,\ldots$,

$$P(M_n^{(r)}\leq\mu_n)\geq P(M_n^{(1)}\leq\mu_n)=F^n(-\infty,\mu_n]\sim e^{-nF(\mu_n,\infty)}.$$

Since $nF(\mu_n,\infty)\le 1$, this shows that

$$\liminf_{n\to\infty} P(M_n^{(r)}\le \mu_n)>0. \tag{3.12}$$

Next, suppose that $nF[\mu_n,\infty)\to a$ through a subsequence. Then, as $n\to\infty$ through this sequence,

$$P(M_n^{(r)}<\mu_n)=\sum_{j=0}^{r-1}\binom{n}{j}F^j[\mu_n,\infty)F^{n-j}(-\infty,\mu_n)\to e^{-a}\sum_{j=0}^{r-1}\frac{a^j}{j!}.$$

(If $a=\infty$, interpret the limit as 0). Since a is positive, indeed $a\ge \liminf_{n\to\infty} nF[\mu_n,\infty)\ge 1$, the limit is less than 1. Consequently $\limsup_{n\to\infty} P(M_n^{(r)}<\mu_n)<1$, or equivalently,

$$\liminf_{n\to\infty} P(M_n^{(r)}\ge \mu_n)>0. \tag{3.13}$$

Since $\mu_n\to\infty$, the events $\{M_n^{(r)}\le\mu_n \text{ i.o.}\}$ and $\{M_n^{(r)}\ge\mu_n \text{ i.o.}\}$ are tail events for the sequence (X_n), and so (3.12) and (3.13) together with Kolmogorov's Zero-One Law imply that

$$P(M_n^{(r)}\le\mu_n \text{ i.o.})=1=P(M_n^{(r)}\ge\mu_n \text{ i.o.}). \tag{3.14}$$

Thus we have the middle inequalities of (2.3).

For the left-hand inequality of (2.3) we fix $c>0$ and use Lemma 3.9(ii) to show that $I_r(c)<\infty$ implies

$$P(M_n^{(r)}-\mu_n<-c \text{ i.o.})=0 \tag{3.15}$$

by checking that $P(M_n^{(r)}-\mu_n\ge -c \text{ i.o.})=1$, which follows from (3.14), and

$$\sum_n n^{r-1}F^r[\mu_{n+1}-c,\infty)e^{-nF[\mu_n-c,\infty)}<\infty. \tag{3.16}$$

The implication from (2.6) to the left inequality of (2.3) then follows as before on replacing c by $c+\varepsilon$. So it remains to check (3.16). The convergence of $I_r(c)$ implies for arbitrary $\delta>0$ and for y large enough that

$$\begin{aligned}\delta &\ge \int_{[y,\infty)}\left(\frac{F[x,\infty)}{F[x-c,\infty)}\right)^r\frac{dF(x)}{F[x,\infty)}\ge\int_{[y,\infty)}F^{r-1}[x,\infty)\frac{dF(x)}{F^r[y-c,\infty)}\\ &\ge\frac{1}{r}\left(\frac{F[y,\infty)}{F[y-c,\infty)}\right)^r\end{aligned}$$

by (3.5). Thus

$$\frac{F[y,\infty)}{F[y-c,\infty)}\to 0 \qquad (y\to\infty), \tag{3.17}$$

and so since $F[\mu_n,\infty)\ge n^{-1}$ we have

$$r_n:=nF[\mu_n-c,\infty)\ge\frac{F[\mu_n-c,\infty)}{F[\mu_n,\infty)}\to\infty.$$

Since $\mu_{n+1}\ge\mu_n$, the terms in the series in (3.16) satisfy

$$\begin{aligned}n^{r-1}F^r[\mu_{n+1}-c,\infty)e^{-nF[\mu_n-c,\infty)}&\le n^{r-1}\frac{F^{2r}[\mu_n-c,\infty)}{F^r[\mu_{n+1}-c,\infty)}e^{-r_n}\\ =n^{-r-1}\frac{r_n^{2r}e^{-r_n}}{F^r[\mu_{n+1}-c,\infty)}&=\mathrm{o}\left(\frac{1}{n^2}\frac{F^{r-1}[\mu_{n+2},\infty)}{F^r[\mu_{n+1}-c,\infty)}\right).\end{aligned}$$

Here we used $n^{-r+1} \sim (n+2)^{-r+1} \le F^{r-1}[\mu_{n+2},\infty)$, and the fact that $x^{2r}e^{-x} \to 0$ when $x \to \infty$. Note also that

$$F[\mu_{n+1},\mu_{n+2}] = F[\mu_{n+1},\infty) - F(\mu_{n+2},\infty) \ge \frac{1}{n+1} - \frac{1}{n+2} \ge \frac{1}{6n^2}.$$

Thus

$$\begin{aligned}\frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{F^{r-1}[\mu_{n+2},\infty)}{F^r[\mu_{n+1}-c,\infty)} &\le \sum_{n=1}^{\infty}\int_{[\mu_{n+1},\mu_{n+2}]}\frac{F^{r-1}[x,\infty)}{F^r[x-c,\infty)}\,dF(x)\\ &\le 2\int_{[\mu_2,\infty)}\left(\frac{F[x,\infty)}{F[x-c,\infty)}\right)^r\frac{dF(x)}{F[x,\infty)}.\end{aligned}$$

Consequently, we have the convergence in (3.16), and (3.15) holds. This shows that (2.6) implies (2.3). Note also that (3.17) is an additive version of (2.23).

Conversely, the right-hand inequality alone of (2.3) implies that $\sum_{n=1}^{\infty} n^{r-1}F^r(\mu_n + c+\varepsilon,\infty) < \infty$ for all $\varepsilon > 0$, by Lemma 3.9(i). This in turn implies (2.6), by Lemma 3.8.

Next in proving Proposition 3.10 we show that $I_r(c) < \infty$ for a fixed $c > 0$ implies (2.4). Then (2.6) implies (2.4) by the device we used above of replacing c by $c+\varepsilon$ and letting $\varepsilon \downarrow 0$ sequentially. Fix $s \in \mathbb{N}$. The sequence $\delta_n := M_n^{(r)} - M_n^{(r+s)}$ can have $\delta_n > \delta_{n-1}$ only when $M_n^{(r)} > M_{n-1}^{(r)}$ (and not always then), and that needs $X_n > M_{n-1}^{(r)}$. So if δ_n is to exceed c (> 0) infinitely often then it must exceed c infinitely often at times at which $X_n > M_{n-1}^{(r)}$. So for (2.4) it suffices to show that

$$P\big(\{M_n^{(r)} - M_n^{(r+s)} > c,\ X_n > M_{n-1}^{(r)}\}\ \text{i.o.}\big) = 0,$$

and for that, by the first Borel-Cantelli lemma, it suffices to have convergence of

$$\sum_{k=r+s}^{\infty} P(M_k^{(r)} - M_k^{(r+s)} > c,\ X_k > M_{k-1}^{(r)}). \tag{3.18}$$

For the present we keep to the cases $r = 2, 3, \ldots$, and we separate the events that $X_k \le M_{k-1}^{(r-1)}$ or $X_k > M_{k-1}^{(r-1)}$. When $M_{k-1}^{(r)} < X_k \le M_{k-1}^{(r-1)}$ we have $X_k = M_k^{(r)}$ and $M_k^{(j)} = M_{k-1}^{(j-1)}$ for $j = r+1,\ r+2, \ldots$. When $X_k > M_{k-1}^{(r-1)}$ we have $M_k^{(j)} = M_{k-1}^{(j-1)}$ for $j = r,\ r+1, \ldots$. Thus we can split (3.18) into two components:

$$\begin{aligned}&\sum_{k=r+s}^{\infty}\Big\{P(M_{k-1}^{(r+s-1)} < X_k - c,\ M_{k-1}^{(r)} < X_k \le M_{k-1}^{(r-1)})\\ &\qquad + P(M_{k-1}^{(r+s-1)} < M_{k-1}^{(r-1)} - c,\ M_{k-1}^{(r-1)} < X_k)\Big\}\\ &\le \sum_{k=r+s}^{\infty}\Big\{P(M_{k-1}^{(r+s-1)} < X_k - c,\ X_k \le M_{k-1}^{(r-1)})\\ &\qquad + P(M_{k-1}^{(r+s-1)} < M_{k-1}^{(r-1)} - c,\ M_{k-1}^{(r-1)} < X_k)\Big\}\\ &= \sum_{k=r+s-1}^{\infty}\int_{-\infty}^{\infty}\Big\{P(M_k^{(r+s-1)} < y - c,\ y \le M_k^{(r-1)})\\ &\qquad + P(M_k^{(r+s-1)} < M_k^{(r-1)} - c,\ M_k^{(r-1)} < y)\Big\}\,dF(y)\end{aligned}$$

$$= I + II, \text{ say.} \tag{3.19}$$

To calculate I note that the event $\{M_k^{(r+s-1)} < y - c,\ y \le M_k^{(r-1)}\}$ may be expressed as that at least $r-1$ of $X_1, \ldots, X_k$ are greater than or equal to y and yet that at most $r+s-2$ of $X_1, \ldots, X_k$ are greater than or equal to $y-c$. Thus by the multinomial theorem,

$$\begin{aligned}
&\sum_{k=r+s-1}^{\infty} P(M_k^{(r+s-1)} < y-c,\ y \le M_k^{(r-1)}) \\
&\quad = \sum_{k=r+s-1}^{\infty} \sum_{i=r-1}^{r+s-2} \sum_{j=i}^{r+s-2} \frac{k!}{i!(j-i)!(k-j)!} F^i[y,\infty) F^{j-i}[y-c,y) F^{k-j}(-\infty, y-c) \\
&\quad = \sum_{i=r-1}^{r+s-2} \sum_{j=i}^{r+s-2} \binom{j}{i} F^i[y,\infty) F^{j-i}[y-c,y) \sum_{k=r+s-1}^{\infty} \binom{k}{j} F^{k-j}(-\infty, y-c).
\end{aligned}$$

The following formula is easy to establish when $j = 0, 1, \ldots,$ and $0 < F < 1$:

$$\sum_{k=j}^{\infty} \binom{k}{j} F^{k-j} = \frac{1}{(1-F)^{j+1}}. \tag{3.20}$$

Thus by extending the innermost sum in the above we obtain

$$\begin{aligned}
&\sum_{k=r+s-1}^{\infty} P(M_k^{(r+s-1)} < y-c,\ y \le M_k^{(r-1)}) \\
&\qquad \le \sum_{i=r-1}^{r+s-2} \sum_{j=i}^{r+s-2} \binom{j}{i} \frac{F^{j-i}[y-c,y) F^i[y,\infty)}{F^{j+1}[y-c,\infty)}.
\end{aligned}$$

Each term $F[y-c,y)$ or $F[y,\infty)$ in the numerator is majorized by a corresponding term $F[y-c,\infty)$ in the denominator, so the above is at most

$$\sum_{i=r-1}^{r+s-2} \sum_{j=i}^{r+s-2} \binom{j}{i} \frac{F^{r-1}[y,\infty)}{F^r[y-c,\infty)} \le 2^{r+s-1} \frac{F^{r-1}[y,\infty)}{F^r[y-c,\infty)}.$$

Thus if we integrate with respect to $dF(y)$ our upper bound on I in (3.19) is $2^{r+s-1} I_r(c)$.

Now for II in (3.19). We need the formula

$$\begin{aligned}
&P(M_k^{(r+s-1)} < M_k^{(r-1)} - c,\ M_k^{(r-1)} < y) \\
&= \int_{(-\infty,y)} \sum{}' F^i(z,\infty) F^{j-1}\{z\} F^l[z-c,z) F^{k-i-j-l}(-\infty, z-c)\, dF(z),
\end{aligned} \tag{3.21}$$

where $\sum'$ denotes

$$\sum_{i=0}^{r-2} \sum_{j=r-1-i}^{r+s-2-i} \sum_{l=0}^{r+s-2-j-i} \frac{k!}{i!j!l!(k-i-j-l)!}.$$

Actually it will suffice to know the right-hand side is an upper bound on the left, so we shall prove only that.

Pick $t > 0$ and let z be in the support of F. For $a < z \le b$, with $b - a < t$,

$$\begin{aligned} &P(M_k^{(r+s-1)} < M_k^{(r-1)} - t,\ a < M_k^{(r-1)} \le b) \\ &\quad \le P(M_k^{(r+s-1)} < b - t,\ a < M_k^{(r-1)} \le b) \\ &\quad = \sum\nolimits' F^i(b,\infty)F^j(a,b]F^l[b-t,a)F^{k-i-j-l}(-\infty,b-t). \end{aligned}$$

On dividing the left by $F(a,b]$ and letting $a \uparrow z$ and $b \downarrow z$, the resulting limit (which must exist) is an evaluation of the Radon-Nikodym derivative $h(z) := dP(M_k^{(r+s-1)} < M_k^{(r-1)} - t,\ M_k^{(r-1)} \le z)/dF(z)$. (See Zaanen (1958), Section 32, Theorem 3.) Thus

$$h(z) \le \sum\nolimits' F^i(z,\infty)F^{j-1}\{z\}F^l(z-t,z)F^{k-i-j-l}(-\infty,z-t].$$

Integrate this with respect to $dF(z)$ over the interval $(-\infty, y)$. The resulting integral of h is $P(M_k^{(r+s-1)} < M_k^{(r-1)} - t,\ M_k^{(r-1)} < y)$. Let $t \downarrow c$, and (3.21) follows (with $\le$ in place of $=$).

Next, integrate (3.21):

$$\begin{aligned} &\int_{\mathbb{R}} P(M_k^{(r+s-1)} < M_k^{(r-1)} - c,\ M_k^{(r-1)} < y)\,dF(y) \\ &\quad \le \int_{\mathbb{R}} \sum\nolimits' F^{i+1}(z,\infty)F^{j-1}\{z\}F^l[z-c,z)F^{k-i-j-l}(-\infty,z-c)\,dF(z) \end{aligned}$$

(where, actually, equality holds). Now sum over all $k \ge r + s - 1$, taking the k-sum innermost on the right, and extending it to run over all $k \ge i + j + l$. By (3.20) it follows that

$$\begin{aligned} II \le &\sum_{i=0}^{r-2} \sum_{j=r-1-i}^{r+s-2-i} \sum_{l=0}^{r+s-2-j-i} \frac{(i+j+l)!}{i!j!l!} \times \\ &\times \int_{\mathbb{R}} \frac{F^{i+1}(z,\infty)F^{j-1}\{z\}F^l[z-c,z)}{F^{i+j+l+1}[z-c,\infty)} dF(z). \end{aligned}$$

On bounding the integrand as we did with a similar integral in evaluating I, we find that the integrand is at most $F^{i+j}[z,\infty)/F^{i+j+1}[z-c,\infty)$. Indeed, since $i+j \ge r-1$ this is itself bounded above by $F^{r-1}[z,\infty)/F^r[z-c,\infty)$, and thus the integral is bounded above by $I_r(c)$. We conclude that II is at most $3^{r+s-2}I_r(c)$. That completes the proof that (2.6) implies (2.4).

Next we show that (2.4) implies (2.5). This is immediate from

$$M_n^{(r+s)} \le M_{n-s}^{(r)}. \tag{3.22}$$

To see (3.22), we always have $M_{n+1}^{(r+1)} \le M_n^{(r)}$, indeed with equality if $X_{n+1} \ge M_n^{(r)}$. Iterating this inequality gives us $M_{n+s}^{(r+s)} \le M_n^{(r)}$, from which (3.22) follows on changing n to $n - s$.

The final implication we need links (2.5) with (2.6) via what we know for partial record values from Proposition 3.3. We need prove only that for fixed r, $s \in \mathbb{N}$, (2.5) implies (2.2). For when (2.5) holds for some $s \in \mathbb{N}$, it holds with $s = 1$. It thus implies $\limsup_{n\to\infty}\big(M_{L_n^{(r)}}^{(r)} - M_{L_n^{(r)}-1}^{(r)}\big) \le c$ almost surely. But $M_{L_n^{(r)}-1}^{(r)} = M_{L_{n-1}^{(r)}}^{(r)}$, so this, by (1.1), is (2.2). □

REMARK 3.11. The converse implication in (ii) of Lemma 3.9 is true, as shown by Klass (1986) for $r = 1$. For $r \in \mathbb{N}$, the result is also true and due to Kesten (personal communication, 1991). See also Wang and Tomkins (1992).

Proof of Corollary 2.2. First note that the lim sup in (2.8) is measurable with respect to the tail σ-algebra of the point processes $\zeta_r(\cdot \cap (n, n+1])$ which are independent by the complete randomness of ζ_r, so by the Kolmogorov Zero-One Law it is a degenerate r.v. So (2.8) holds if and only if (2.2) holds for a particular non-random c, and thus (2.8) is equivalent to (2.13) by Proposition 3.3.

The equivalence of (2.10)–(2.12) with (2.13) may be dealt with similarly.

Clearly (2.10) implies (2.9), so the only task left is to prove that (2.9) implies (2.10). Let (2.9) hold. Then there is a $c_1 > 0$ such that $nF(b_n - c_1, \infty) \to \infty$. To see this, suppose not. Then for every $A > 0$ there would be a sequence $n_i = n_i(A) \to \infty$ such that $n_i F(b_{n_i} - A, \infty) \to a < \infty$. But then

$$P(M_{n_i}^{(1)} \leq b_{n_i} - A) \sim e^{-n_i F(b_{n_i} - A, \infty)} \to e^{-a} > 0,$$

from which it follows that $P(M_{n_i}^{(1)} \leq b_{n_i} - A \text{ i.o.}) = 1$. This implies $P(M_n^{(1)} \leq b_n - A \text{ i.o.}) = 1$ and thus $P(M_n^{(r)} \leq b_n - A \text{ i.o.}) = 1$, contradicting $\liminf_{n\to\infty}(M_n^{(r)} - b_n) > -\infty$ a.s. By the definition of μ_n (see the line following (2.1)), we conclude that $\mu_n \geq b_n - c_1$ for all large n. The right-hand inequality of (2.9) thus implies the right-hand inequality of (2.10). In proving Proposition 3.10 we established that the right-hand inequality of (2.3) implies (2.6). Thus (2.9) implies (2.6) for some $c \geq 0$, whence (2.3) holds for that c, by Proposition 3.10, and we conclude (2.10). □

Proof of Theorem 2.3. All the assertions except (2.15) appear in Proposition 3.4 and the $c = 0$ case of Theorem 2.1 and are thus proved equivalent. As for (2.15), it implies (2.18) and is implied by (2.16). □

REMARK 3.12. Li and Tomkins ((1991), Corollary 1) show that finiteness of $\sum_{n=r}^{\infty} n^{-1} P(M_n^{(r)} \geq \mu_n + \varepsilon)$ for all $\varepsilon > 0$ is equivalent to the a.s. stability of $M_n^{(r)}$, and hence, by Theorem 2.3, to the convergence of $I_1(\varepsilon)$ for every $\varepsilon > 0$.

3.4. Proof of Theorem 2.5

PROPOSITION 3.13. *For any fixed r, $s \in \mathbb{N}$ and $c > 0$, and for general F,*

$$\liminf_{n\to\infty}(R_{n+s}^{(r)} - R_n^{(r)}) \geq c \text{ a.s.} \tag{3.23}$$

if and only if

$$\sum_{n=0}^{\infty} P(\zeta_r(n\varepsilon - c + \varepsilon, n\varepsilon] > s) < \infty \text{ for all } \varepsilon > 0. \tag{3.24}$$

Proof. The argument of proof of Proposition 3.3 can be used. Thus by considering deterministic sequences we find (3.23) equivalent to

$$P(\zeta_r(n\varepsilon - c + \varepsilon, n\varepsilon] > s \text{ i.o.}) = 0 \text{ for all } \varepsilon > 0. \tag{3.25}$$

Again, for fixed ε the events $\{\zeta_r(n\varepsilon - c + \varepsilon, n\varepsilon] > s\}$ are independent for any set of values of n separated by at least $(c - \varepsilon)/\varepsilon$. The Borel-Cantelli Lemmas thus give the equivalence of (3.25) with (3.24). □

LEMMA 3.14. *For continuous F, (2.27) implies (2.26).*

Proof. Let (2.27) hold for some $r \geq 1$. Then it holds for $r = 1$. As in the proof of Corollary 2.2, $\liminf_{n\to\infty}(R_{n+s} - R_n)$ is a degenerate r.v. so (2.27) asserts that (3.23) holds for some $c > 0$ and $r = 1$. We thus have (3.24) for $r = 1$. Fix $\varepsilon \in (0, c)$ and write $B_n := (n\varepsilon - c + \varepsilon, n\varepsilon]$. The terms in the series in (3.24) tend to 0, and since by the continuity of F, $\zeta_r(B_n)$ is Pois$(rH(B_n))$-distributed, this implies $H(B_n) \to 0$ and

$$P(\zeta_r(B_n) > s) = e^{-rH(B_n)} \sum_{j=s+1}^{\infty} (rH(B_n))^j/j! \sim r^{s+1}H^{s+1}(B_n)/(s+1)!$$

as $n \to \infty$. So (3.24), for any chosen r, is equivalent to

$$\sum_{n=0}^{\infty} H^{s+1}(n\varepsilon - c + \varepsilon, n\varepsilon] < \infty \text{ for all } \varepsilon > 0. \tag{3.26}$$

We can now lengthen the interval B_n, as follows. For any n, let n' be the least integer such that $n\varepsilon - c + \varepsilon \leq n'\varepsilon$. Then $n'\varepsilon < n\varepsilon - c + 2\varepsilon$, so $(n\varepsilon - 2c + 3\varepsilon, n\varepsilon] \subset (n'\varepsilon - c + \varepsilon, n'\varepsilon] \cup B_n$, whence

$$\begin{aligned}\sum_{n=0}^{\infty} H^{s+1}(n\varepsilon - 2c + 3\varepsilon, n\varepsilon] &\leq \sum_{n=0}^{\infty} (H(n'\varepsilon - c + \varepsilon, n'\varepsilon] + H(B_n))^{s+1} \\ &\leq 2^s \sum_{n=0}^{\infty} (H^{s+1}(n'\varepsilon - c + \varepsilon, n'\varepsilon] + H^{s+1}(B_n)) < \infty,\end{aligned}$$

where we used the elementary inequality

$$|a+b|^{s+1} \leq 2^s(|a|^{s+1} + |b|^{s+1}). \tag{3.27}$$

The sum on the left reduces to $\sum_{n=0}^{\infty} H^{s+1}(n\varepsilon' - 2c + \varepsilon', n\varepsilon']$, with $\varepsilon' := 3\varepsilon$, if we restrict n to run through multiples of 3. So we conclude by the above equivalence of (3.24) with (3.26) that $\liminf_{n\to\infty}(R_{n+s}^{(r)} - R_n^{(r)}) \geq 2c$ a.s. Continue doubling c in this way, and (2.26) follows. ☐

We now characterize when $R_{n+s}^{(r)} - R_n^{(r)} \xrightarrow{a.s.} \infty$. As with Propositions 3.6 and 3.13, for general F we have only a semi-explicit result, as follows, but we will be able to use it to take the proof of Theorem 2.5 further.

PROPOSITION 3.15. *For general F and for any fixed r, $s \in \mathbb{N}$, we have $R_{n+s}^{(r)} - R_n^{(r)} \xrightarrow{a.s.} \infty$ if and only if*

$$\sum_{n=0}^{\infty} P(\zeta_r(n-k, n] > s) < \infty \qquad (k = 1, 2, \ldots). \tag{3.28}$$

Proof. A non-random sequence (r_n) has $r_{n+s} - r_n \to \infty$ if and only if for each $k \in \mathbb{N}$, only finitely many of the intervals $(1-k, 1]$, $(2-k, 2]$, $(3-k, 3]$, ... contain more than s elements. So $R_{n+s}^{(r)} - R_n^{(r)} \xrightarrow{a.s.} \infty$ if and only if

$$P(\zeta_r(n-k, n] > s \text{ i.o.}) = 0 \qquad (k = 1, 2, \ldots). \tag{3.29}$$

The events $\{\zeta_r(n-k, n] > s\}$ are independent for any set of values of n at least k apart from each other. So a use of the Borel-Cantelli Lemmas similar to that in the proof of Proposition 3.3 gives the equivalence of (3.29) with (3.28). ☐

Proof of Theorem 2.5: conclusion. We first prove that (2.26) is equivalent to

$$\sum_{n=0}^{\infty} H^{s+1}(n-1,n] < \infty. \tag{3.30}$$

If (2.26) holds, for any one r we may deduce the $k=1$ case of (3.28), for that r, and, as calculated in the proof of Lemma 3.14, the n^{th} term of the series is asymptotic to $r^{s+1}H^{s+1}(n-1,n]/(s+1)!$ That proves (3.30). Conversely, (3.30) implies (3.28), for all r, hence (2.26). These are also equivalent to (2.30): recall from §3.1 that the measures H and η coincide, as F is continuous. Now (2.30) says that $\sum_n (e^{H(n-1,n]}-1)^{s+1} < \infty$, and since this implies $H(n-1,n] \to 0$ as does (3.30), (3.30) and (2.30) are equivalent via the fact that $e^x - 1 \sim x$ as $x \to 0$.

(3.30) implies $\sum H^{s+1}(n-2,n] < \infty$ (by utilizing (3.27)), and we may perform this doubling repeatedly, so $\sum H^{s+1}(n-a,n] < \infty$ for each $a > 0$. From $\sum H^{s+1}(n-c-1,n] < \infty$ we deduce $\sum H^s(n-c,n+1)\int_n^{n+1} dH(x) < \infty$, whence

$$\sum_{n=0}^{\infty} \int_n^{n+1} H^s(x-c,x]\,dH(x) < \infty. \tag{3.31}$$

Note that (3.30) also implies $H(x-c,x] \to 0$ as $x \to \infty$, i.e., (2.29). This with (3.31) yields $\int_0^\infty (e^{H(x-c,x]}-1)^s\,dH(x) < \infty$, which is (2.28). Thus (3.30) implies (2.28) for all $c > 0$.

Conversely, assume one case of (2.28), say $\int_0^\infty (e^{H(x-c,x]}-1)^s\,dH(x) < \infty$, for a single value of c. Since $e^x - 1 \ge x$ for $x \ge 0$ this implies the corresponding case of (3.31), namely $\int_0^\infty H^s(x-c,x]\,dH(x) < \infty$. That becomes a weaker statement if we reduce c, so we assume now that $c \le 1$. We deduce

$$\sum_{n=0}^{\infty} \int_{(n-1)c}^{nc} H^s((n-1)c,x]\,dH(x) < \infty,$$

which is $\sum_{n=0}^{\infty} H^{s+1}((n-1)c,nc] < \infty$. If $c=1$ this is (3.30), while if $c<1$ then as argued above we can successively double the length of the interval to get $\sum_{n=0}^{\infty} H^{s+1}(nc,(n+2^j)c] < \infty$ for any j we choose. This clearly implies (3.30). □

Remark 3.16. We noted above that (2.29) follows from (3.30) and thus from (2.28). Suppose

$$\int_{-\infty}^{\infty} \left(1 - \frac{F[x,\infty)}{F[x-c,\infty)}\right)^s \frac{dF(x)}{F[x,\infty)} < \infty; \tag{3.32}$$

note that this is the additive version of (2.37). Then if $\varepsilon > 0$ is given, for y large enough we have

$$\begin{aligned}\varepsilon \ge \int_y^{y+c} \left(1 - \frac{F[x,\infty)}{F[x-c,\infty)}\right)^s \frac{dF(x)}{F[x,\infty)} &\ge \frac{1}{F[y,\infty)} \int_y^{y+c} \left(1 - \frac{F[x,\infty)}{F[y,\infty)}\right)^s dF(x) \\ &= \frac{1}{s+1}\left(1 - \frac{F[y+c,\infty)}{F[y,\infty)}\right)^{s+1}.\end{aligned}$$

So $F[y+c,\infty)/F[y,\infty) \to 1$ as $y \to \infty$, for each $c > 0$, and again (2.29) holds. Then, as $x \to \infty$,

$$\frac{F[x-c,\infty)}{F[x,\infty)} - 1 \sim 1 - \frac{F[x,\infty)}{F[x-c,\infty)},$$

so (2.28) and (3.32) are equivalent.

EXAMPLE 3.17. The assertions of Theorem 2.5 are not equivalent for different s. To see that, take $H(-\infty, x] := \sqrt{x}$ for $x \geq 0$. Then for $n > c/\varepsilon$,

$$H(n\varepsilon - c + \varepsilon, n\varepsilon] = \frac{c - \varepsilon}{\sqrt{n\varepsilon} + \sqrt{n\varepsilon - c + \varepsilon}},$$

so when $s = 1$ (3.26) holds for no c while when $s = 2$ (3.26) holds for all c. Thus, recalling the degeneracy of the lim inf, we conclude that $\liminf_{n\to\infty}(R_{n+s}^{(r)} - R_n^{(r)})$ is 0 when $s = 1$, and ∞ when $s = 2$, for all values of r, almost surely.

3.5. Proofs of Theorem 2.6 and Corollary 2.7

Equivalence of (2.35), (2.36), and (2.37) for continuous F is just a multiplicative reformulation of part of Theorem 2.5, so is already proved.

For the main proof we work at first under the assumption that F is continuous and remove that restriction at the end. The first major task is to prove that finiteness of $J_s(\lambda)$ (i.e., (2.37)) for some λ in (0,1) implies (2.31). We prove that in Lemma 3.20. A preparatory step is to derive, in Lemma 3.19, the intermediate conclusion

$$P\left(\frac{M_n^{(r)}}{M_n^{(r+s)}} < \lambda^{-1} \text{ i.o.}\right) = 0, \tag{3.33}$$

but before even that we need to establish the following formula.

LEMMA 3.18. *For continuous F, any λ in $(0,1)$, and any r, $s \in \mathbb{N}$,*

$$\sum_{j=1}^{\infty} P\big(M_{L_j^{(r)}-1}^{(r+s)} > \lambda M_{L_j^{(r)}-1}^{(r)}\big)$$
$$= r\int_0^{\infty} \sum_{i=s}^{\infty} \binom{r+i}{i} \left(\frac{F[x,\infty)}{F[\lambda x,\infty)}\right)^{r+1} \left(1 - \frac{F[x,\infty)}{F[\lambda x,\infty)}\right)^{i} \frac{dF(x)}{F[x,\infty)}. \tag{3.34}$$

Proof. We calculate the left-hand side by conditioning on $L_j^{(r)}$ and then using the fact that an r-record occurs at time k if and only if $X_k > M_{k-1}^{(r)}$. Thus the left-hand side is

$$\sum_{j=1}^{\infty} \sum_{k=\max(j,r+s+1)}^{\infty} P(M_{k-1}^{(r+s)} > \lambda M_{k-1}^{(r)},\ L_j^{(r)} = k)$$
$$= \sum_{k=r+s+1}^{\infty} \sum_{j=1}^{k} P(M_{k-1}^{(r+s)} > \lambda M_{k-1}^{(r)},\ L_j^{(r)} = k)$$
$$= \sum_{k=r+s+1}^{\infty} P(M_{k-1}^{(r+s)} > \lambda M_{k-1}^{(r)}, \text{ an } r\text{-record occurs at } k)$$
$$= \sum_{k=r+s+1}^{\infty} P(M_{k-1}^{(r+s)} > \lambda M_{k-1}^{(r)},\ X_k > M_{k-1}^{(r)})$$
$$= \sum_{k=r+s}^{\infty} E\big(P(X_{k+1} > M_k^{(r)} \mid X_1, \dots, X_k)\mathbf{1}\{M_k^{(r+s)} > \lambda M_k^{(r)}\}\big)$$

$$= \sum_{k=r+s}^{\infty} E\big(F(M_k^{(r)},\infty)\mathbf{1}\{M_k^{(r+s)} > \lambda M_k^{(r)}\}\big).$$

The last expression can be evaluated since the joint distribution of $M_k^{(r+s)}$ and $M_k^{(r)}$ can be written down under the assumption of continuity of F. We obtain

$$\sum_{k=r+s}^{\infty} \int_0^{\infty} \int_{\lambda x}^{x} \frac{k!F[x,\infty)}{(r-1)!(s-1)!(k-r-s)!} F^{r-1}(x,\infty) \times$$
$$\times F^{s-1}(y,x]F^{k-r-s}(y)\, dF(y)dF(x)$$
$$= \int_0^{\infty} \int_{\lambda x}^{x} \frac{(r+s)!}{(r-1)!(s-1)!} F^r(x,\infty) \frac{F^{s-1}(y,x]}{F^{r+s+1}(y,\infty)}\, dF(y)dF(x),$$

where we used (3.20). Now, again using the continuity of F, we may integrate by parts to get

$$\int_{\lambda x}^{x} \frac{(r+s)!}{(s-1)!} \frac{F^{s-1}(y,x]}{F^{r+s+1}(y,\infty)}\, dF(y) = \sum_{i=s}^{\infty} \frac{(r+i)!}{i!} \frac{F^i(\lambda x, x]}{F^{r+i+1}(\lambda x,\infty)}.$$

Insert this into the above, and the Lemma follows. □

Lemma 3.19. *Suppose F is continuous and $J_s(\lambda) < \infty$ for some $\lambda \in (0,1)$. Then (3.33) holds.*

Proof. Note that

$$\max_{n \ge r+s} \frac{M_n^{(r+s)}}{M_n^{(r)}} \le \max_{j>s} \max_{L_{j-1}^{(r)} \le n < L_j^{(r)}} \frac{M_n^{(r+s)}}{M_n^{(r)}}$$
$$= \max_{j>s} \max_{L_{j-1}^{(r)} \le n < L_j^{(r)}} \frac{M_n^{(r+s)}}{M_{L_j^{(r)}-1}^{(r)}} = \max_{j>s} \frac{M_{L_j^{(r)}-1}^{(r+s)}}{M_{L_j^{(r)}-1}^{(r)}}.$$

So by the first Borel-Cantelli Lemma it suffices for (3.33) that $\sum_j P\big(M_{L_j^{(r)}-1}^{(r+s)} > \lambda M_{L_j^{(r)}-1}^{(r)}\big) < \infty$. By Lemma 3.18 we thus need to show that the right-hand side of (3.34) converges. In fact it exceeds $rJ_s(\lambda)$ but is at most

$$\frac{(r+s)(r+s-1)\cdots r}{s!} J_s(\lambda).$$

To see this, write the right hand of (3.34) as

$$r\int_0^{\infty} \sum_{i=0}^{\infty} c_i \binom{r+i}{i} \left(\frac{F[x,\infty)}{F[\lambda x,\infty)}\right)^{r+1} \left(1 - \frac{F[x,\infty)}{F[\lambda x,\infty)}\right)^{s+i} \frac{dF(x)}{F[x,\infty)},$$

where

$$c_i := \frac{(r+s+i)\cdots(r+1+i)}{(s+i)\cdots(1+i)} = \left(1 + \frac{r}{s+i}\right)\cdots\left(1+\frac{r}{1+i}\right).$$

As the product of factors that are decreasing in i, c_i decreases in i, so $c_0 \ge c_i \ge 1$. Thus the right-hand side of (3.34) lies between r and rc_0 times

$$\int_0^{\infty} \sum_{i=0}^{\infty} \binom{r+i}{i} \left(\frac{F[x,\infty)}{F[\lambda x,\infty)}\right)^{r+1} \left(1 - \frac{F[x,\infty)}{F[\lambda x,\infty)}\right)^{s+i} \frac{dF(x)}{F[x,\infty)} = J_s(\lambda). \qquad \square$$

LEMMA 3.20. *For continuous F, if $J_s(\lambda) < \infty$ for some $\lambda \in (0,1)$ then (2.31) and (2.32) hold.*

Proof. (2.31) is immediate from Lemma 3.19. Then (2.32) follows from the fact that $J_s(\lambda) < \infty$ for some λ in (0,1) implies $J_s(\lambda) < \infty$ for all λ in (0,1), as shown (in its additive form) in Theorem 2.5 (and see also Pruitt (1987)). $\square$

Next we prove the converse to what we have established so far.

LEMMA 3.21. *For continuous F, (2.31) implies $J_s(\lambda) < \infty$ for all $\lambda \in (0,1)$.*

Proof. We first show that for $s = 1, 2, \ldots,$

$$M^{(r+s)}_{L^{(r)}_n} \geq R^{(r)}_{n-s}. \tag{3.35}$$

This follows because, by definition, r values of X_i for i in $[1, L^{(r)}_{n-s}]$ equal or exceed $M^{(r)}_{L^{(r)}_{n-s}}$. Also at least s values of X_i for i in $(L^{(r)}_{n-s}, L^{(r)}_n]$ exceed $M^{(r)}_{L^{(r)}_{n-s}}$; in particular, such values occur at times $L^{(r)}_{n-s+1}, \ldots, L^{(r)}_n$. Thus at least $r+s$ values of X_i in $[1, L^{(r)}_n]$ equal or exceed $M^{(r)}_{L^{(r)}_{n-s}}$, and so $M^{(r+s)}_{L^{(r)}_n} \geq M^{(r)}_{L^{(r)}_{n-s}}$. Thus, by (1.1), (3.35) is proved.

Now our assumption (2.31) implies $\liminf_{n\to\infty} M^{(r)}_{L^{(r)}_n}/M^{(r+s)}_{L^{(r)}_n} > 1$ a.s., hence $\liminf_{n\to\infty} R^{(r)}_n/R^{(r)}_{n-s} > 1$ a.s. by (1.1) and (3.35). This is the multiplicative form of (2.27) and since we assumed F continuous, (3.32) in its multiplicative form proves $J_s(\lambda) < \infty$ for some (hence all) λ in $(0,1)$, as desired. $\square$

The last task for continuous F is to prove the following.

LEMMA 3.22. *When F is continuous (2.37), (2.33) and (2.34) are equivalent.*

Proof. Suppose again that $J_s(\lambda) < \infty$ for some λ in $(0,1)$. We will show that $P\big({}^{(r+s-1)}S_n/M^{(r)}_n > \lambda \text{ i.o.}\big) = 0$, which will prove (2.33). We have

$$\max_{n>r+s} \frac{{}^{(r+s-1)}S_n}{M^{(r)}_n} \leq \max_{j>s} \max_{L^{(r)}_{j-1} \leq n < L^{(r)}_j} \frac{{}^{(r+s-1)}S_n}{M^{(r)}_{L^{(r)}_j - 1}} \leq \max_{j>s} \frac{{}^{(r+s-1)}S_{L^{(r)}_j - 1}}{M^{(r)}_{L^{(r)}_j - 1}},$$

because when $n < L^{(r)}_j$,

$${}^{(r+s-1)}S_n = \sum_{i=r+s}^{n} M^{(i)}_n \leq \sum_{i=r+s}^{L^{(r)}_j - 1} M^{(i)}_{L^{(r)}_j - 1} = {}^{(r+s-1)}S_{L^{(r)}_j - 1}.$$

So by the first Borel-Cantelli Lemma it will suffice to have the convergence of

$$\sum_j P\big({}^{(r+s+1)}S_{L^{(r)}_j - 1} > \lambda M^{(r)}_{L^{(r)}_j - 1}\big) = \sum_k P(A_k), \text{ say},$$

where $A_k := \big\{\sum_{i=r+s}^{k-1} M^{(i)}_{k-1} > \lambda M^{(r)}_{k-1}, X_k > M^{(r)}_{k-1}\big\}$. The above equality is proved by the same argument used in Lemma 3.18. To establish this convergence we need to split the event A_k as follows. Choose δ in $(0, \lambda)$; then

$$\begin{aligned} P(A_k) &= P(A_k \cap \{M^{(s+r)}_{k-1} > \delta M^{(r)}_{k-1}\}) + P(A_k \cap \{M^{(s+r)}_{k-1} \leq \delta M^{(r)}_{k-1}\}) \\ &\leq P(X_k > M^{(r)}_{k-1},\ M^{(s+r)}_{k-1} > \delta M^{(r)}_{k-1}) + P(A_k \cap \{M^{(s+r)}_{k-1} \leq \delta M^{(r)}_{k-1}\}). \end{aligned}$$

The first term here is summable over k if $J_s(\delta) < \infty$, as we proved in Lemmas 3.18–3.19. However we know from the first part of the proof that $J_s(\lambda) < \infty$ implies $J_s(\delta) < \infty$ for all $\delta < \lambda$. So we need show only that the second term is summable over k, i.e., writing k for $k-1$, we need convergence of

$$\begin{aligned}
&\sum_k P\left(\sum_{i=r+s}^{k} M_k^{(i)} > \lambda M_k^{(r)},\ X_{k+1} > M_k^{(r)},\ M_k^{(s+r)} \le \delta M_k^{(r)}\right) \\
&= \sum_k \int_0^\infty P\left(\sum_{i=r+s}^{k} M_k^{(i)} > \lambda M_k^{(r)},\ M_k^{(s+r)} \le \delta M_k^{(r)} < \delta y\right) dF(y) \\
&\le \sum_k \int_0^\infty P\left(\sum_{i=r+s}^{k} M_k^{(i)} > \frac{\lambda}{\delta} M_k^{(s+r)},\ M_k^{(s+r)} < y\right) dF(y) \\
&\le \sum_k \int_0^\infty P\left(\sum_{i=r+s+1}^{k} M_k^{(i)} > \left(\frac{\lambda}{\delta} - 1\right) M_k^{(s+r)},\ M_k^{(s+r)} < y\right) dF(y).
\end{aligned}$$

We evaluate this by conditioning on $M_k^{(s+r)}$ and using Lemma 1.1 of Maller and Resnick (1984), as

$$\sum_k \int_0^\infty \int_0^y P\left(\sum_{i=r+s+1}^{k} M_{k-s-r}^{(i-s-r)}(z) > \left(\frac{\lambda}{\delta} - 1\right) z\right) P(M_k^{(s+r)} \in dz)\, dF(y),$$

where $M_{k-s-r}^{(1)}(z) \ge \cdots \ge M_{k-s-r}^{(k-s-r)}(z)$ are the order statistics of a random sample $X_1(z), \ldots, X_{k-s-r}(z)$ drawn from the distribution of the X_i conditional on $X_i \le z$. Thus the above is

$$\begin{aligned}
\sum_k \int_0^{F[z,\infty)} P\left(\sum_{i=1}^{k-s-r} M_{k-s-r}^{(i)}(z) > \left(\frac{\lambda}{\delta} - 1\right) z\right) \times \\
\times (s+r)\binom{k}{s+r} F^{s+r-1}(z,\infty) F^{k-s-r}(z)\, dF(z). \qquad (3.36)
\end{aligned}$$

We need an upper bound for the probability here. This can be obtained by applying Lemma 3.2 of Maller and Resnick (1984) with $x = 1$ (noting that $X_i(z)/z \le 1$ a.s.) and with δ chosen so small that $\lambda/\delta - 1 > a_s$, where a_s is the constant in that lemma. Thus we get for some constant b_s that

$$\begin{aligned}
&P\left(\sum_{i=1}^{k-s-r} M_{k-s-r}^{(i)}(z) > \left(\frac{\lambda}{\delta} - 1\right) z\right) = P\left(\sum_{i=1}^{k-s-r} X_i(z) > \left(\frac{\lambda}{\delta} - 1\right) z\right) \\
&\le P\left(\sum_{i=1}^{k-s-r} \frac{X_i(z)}{z} > a_s\right) \le b_s\left((k-s-r) E\left(\frac{X_i(z)}{z}\right)\right)^s \le b_s k^s \frac{\nu^s(z)}{z^s F^s(z)},
\end{aligned}$$

where

$$\frac{\nu(z)}{F(z)} = E(X \mid X \le z) = \frac{\int_0^z x\, dF(x)}{F(z)}.$$

So we have an upper bound for (3.36), of order of magnitude

$$\sum_k \int_0^\infty k^{2s+r} \frac{\nu^s(z)}{z^s} F^{s+r}(z,\infty) F^{k-2s-r}(z)\, dF(z).$$

By using (3.20) we deduce that $\sum_k k^{2s+r} F^{k-2s-r}(z) = \mathrm{O}\big(F^{-2s-r-1}(z,\infty)\big)$, so we get a multiple of

$$\int_0^\infty \frac{\nu^s(z)}{z^s F^s(z,\infty)} \frac{dF(z)}{F[z,\infty)}$$

as an upper bound for (3.36). But Pruitt (1987) shows that convergence of this is equivalent to $J_s(\lambda) < \infty$ for some and hence all $0 < \lambda < 1$. Thus we have shown that $J_s(\lambda) < \infty$ for some and hence all $0 < \lambda < 1$ implies (2.33) and (2.34).

Conversely, ${}^{(r+s-1)}S_n \geq M_n^{(r+s)}$ (when $X_i \geq 0$ a.s.), so (2.33) implies (2.31), which, as we saw, implies (2.37). □

Proof of Theorem 2.6: conclusion. It remains to remove the assumption of continuity of F. We use a device of Pruitt (1987). Let $X_i^\# := X_i + Z_i$, where Z_i are i.i.d. uniform r.v.s on $(0,1)$, independent of X_i. The $X_i^\#$ are independent r.v.s with continuous d.f. $F^\#$, say, the tail of which satisfies $F(x,\infty) \leq F^\#(x,\infty) \leq F(x-1,\infty)$ for all x. Also the order statistics $M_n^{\#(j)}$ of a sample $X_1^\#, \ldots, X_n^\#$ satisfy $|M_n^{\#(j)} - M_n^{(j)}| \leq 1$ for $j = 1, \ldots, n$, so $M_n^{\#(j)} \sim M_n^{(j)}$ a.s., as $n \to \infty$, for each fixed j. Let

$$J_s^\#(\lambda) := \int_0^\infty \left(1 - \frac{F^\#(x,\infty)}{F^\#(\lambda x,\infty)}\right)^s \frac{dF^\#(x)}{F^\#(x,\infty)} \qquad (0 < \lambda < 1).$$

We know that (2.31)–(2.34) and (2.37) are equivalent for the # functions and now prove they are equivalent as stated. We proceed via some of Pruitt's relations. So suppose $J_s(\lambda) < \infty$ for some λ in $(0,1)$. Pruitt shows this is equivalent to the convergence of the series $\sum u_k^{s+1}$, where $u_k := 1 - F(2^{k+1},\infty)/F(2^k,\infty)$, and that this in turn is equivalent to convergence of $\sum (u_k^\#)^{s+1}$, where the $u_k^\#$ are defined in terms of $F^\#$ in the obvious way. Thus $J_s^\#(\lambda) < \infty$ (for the given λ and hence) for all λ in $(0,1)$. So (2.31)–(2.34) hold for the # functions. Since $M_n^{\#(j)} \sim M_n^{(j)}$ a.s., (2.31) and (2.32) hold as stated. Also, let ${}^{(r)}S_n^\# := \sum_{i=r+1}^n M_n^{\#(i)}$; then

$$\left|{}^{(r+s-1)}S_n^\# - {}^{(r+s-1)}S_n\right| = \left|\sum_{j=r+s}^n (M_n^{\#(j)} - M_n^{(j)})\right| \leq n.$$

As a consequence of the slow variation of $F(x,\infty)$ (which follows easily from $u_k \to 0$), $M_n^{(r+s)}/n \to \infty$ a.s. Thus we have (2.33) and (2.34) as stated.

For the converse, (2.33) implies (2.31) since ${}^{(r+s-1)}S_n \geq M_n^{(r+s)}$, while (2.31) implies $\liminf_{n\to\infty} M_n^{\#(r)}/M_n^{\#(r+s)} > 1$ a.s., since $M_n^{\#(j)} \sim M_n^{(j)}$ a.s. for $1 \leq j \leq r$. So by the continuous part of the proof, $J_s^\#(\lambda) < \infty$ for λ in (0,1), and thus $\sum (u_k^\#)^{s+1} < \infty$ by Pruitt's proof. Hence $\sum u_k^{s+1} < \infty$, and $J_s(\lambda) < \infty$ for all $\lambda \in (0,1)$, again by Pruitt (1987). This completes Theorem 2.6. □

REMARK 3.23. Note that Theorem 2.6 says nothing about the ratio $M_n^{(r)}/M_{n-s}^{(r)}$ diverging to ∞, as might at first be expected by analogy with Corollary 2.2, because such cannot occur. For $r,\, s \in \mathbb{N}$ we have

$$\begin{aligned} P(M^{(r)}_{n-s} < M^{(r)}_n) &= P\big(X_i > M^{(r)}_{n-s} \text{ for some } i \in (n-s, n]\big) \\ &\le s \int_{-\infty}^{\infty} P(M^{(r)}_{n-s} < x)\, dF(x) \to 0 \qquad (n \to \infty), \end{aligned}$$

by dominated convergence, since $M^{(r)}_n \xrightarrow{P} \infty$. (Recall $F(x) < 1$ for all x.) Thus $M^{(r)}_{n-s} = M^{(r)}_n$ with probability approaching 1, so $M^{(r)}_{n-s} - M^{(r)}_n \xrightarrow{P} 0$, $M^{(r)}_{n-s}/M^{(r)}_n \xrightarrow{P} 1$ and $\liminf_{n\to\infty} M^{(r)}_n/M^{(r)}_{n-s} = 1$ a.s., always.

Next note that for $r,\, s \in \mathbb{N}$, $M^{(r+s)}_n/M^{(r)}_n$ can converge in probability to a constant only if that constant is 0 or 1; because if $P(M^{(r+s)}_n \le \varepsilon M^{(r)}_n) \to 1$ for some $\varepsilon \in (0,1)$ then the argument of Maller and Resnick ((1984), Theorem 2.1) shows that $F(\varepsilon x, \infty)/F(x,\infty) \to 1$ as $x \to \infty$ for that ε, hence for all $\varepsilon \in (0,1)$ by Bingham *et al.* (1989), Proposition 1.10.1. Consequently $F(x,\infty)$ is slowly varying and so $M^{(r+s)}_n/M^{(r)}_n \xrightarrow{P} 0$ by Theorem 2.1 of Maller and Resnick (1984).

REMARK 3.24. Notice also that the proof of Theorem 2.1 gave somewhat more than was stated in the Theorem, in that convergence of $I_r(c)$ for a particular $c > 0$ is equivalent to (2.3). This is probably true for some of the other conditions too, but we have not proved this.

Proof of Corollary 2.7. Let $r \ge 1$ and suppose $c := \limsup_{n\to\infty} {}^{(r-1)}S_n/M^{(r)}_n$ is finite. Let $s \ge r$ be an integer such that $\limsup_{n\to\infty} M^{(s)}_n/M^{(r)}_n > 0$ a.s.; for example, $s = r$ is such an integer. Then by Theorem 2.6 we have in fact $\limsup_{n\to\infty} M^{(s)}_n/M^{(r)}_n = 1$ a.s. Now

$$\frac{{}^{(r-1)}S_n}{M^{(r)}_n} = \frac{M^{(r)}_n + M^{(r-1)}_n + \cdots + M^{(s)}_n}{M^{(r)}_n} + \frac{{}^{(s)}S_n}{M^{(r)}_n} \ge (s-r+1)\frac{M^{(s)}_n}{M^{(r)}_n},$$

so $c \ge s - r + 1$. Suppose s is maximal, i.e., $s - r + 1 \le c < s - r + 2$. Then $M^{(s+1)}_n/M^{(r)}_n \to 0$ a.s., and thus ${}^{(s)}S_n/M^{(r)}_n \to 0$ a.s. by Theorem 2.6. It follows that

$$\frac{{}^{(r-1)}S_n}{M^{(r)}_n} = \frac{M^{(r)}_n + M^{(r-1)}_n + \cdots + M^{(s)}_n}{M^{(r)}_n} + \frac{{}^{(s)}S_n}{M^{(r)}_n} \le (s-r+1) + \mathrm{o}(1) \text{ a.s.}$$

Thus $c \le s - r + 1$, so $c = s - r + 1$. □

3.6. Proof of Theorem 2.8

LEMMA 3.25. *Let (B_n) be a sequence of Borel sets such that for some integers $s' \ge s \ge 1$, $P(\zeta_1(B_n) \ge s) \to 1$ but $P(\zeta_1(B_n) > s') \to 0$. Then for all large n there are in each B_n distinct points $x^{(1)}_n, \ldots, x^{(s)}_n$ with $\min(\eta\{x^{(1)}_n\}, \ldots, \eta\{x^{(s)}_n\}) \ge a$, for some constant $a > 0$.*

Proof. By enlarging the probability space if necessary we can assume the existence on it of ϕ_1, a Poisson process of which $\zeta_1 = \rho_1$ is the simplification (see §3.1). So ϕ_1 is Poisson with mean measure (or intensity, or rate) H, and $\phi_1(B) \ge \zeta_1(B)$ for every B, almost surely. Thus $P(\phi_1(B_n) \ge s) \ge P(\zeta_1(B_n) \ge s) \to 1$. This implies $H(B_n) \to \infty$.

Now B_n is non-empty for all large n; let $x^{(1)}_n$ be a point of B_n of maximal η-mass. Suppose there is a sequence of integers $n' \to \infty$ such that $\eta\{x^{(1)}_{n'}\} \to 0$. Then

$\eta\{x\} \to 0$ uniformly in $x \in B_{n'}$, as $n' \to \infty$, so $H\{x\} = 1 - e^{-\eta\{x\}}$ also tends to 0 uniformly in $x \in B_{n'}$. For fixed $t \in (0,1)$ we have from (3.1) that

$$\begin{aligned} Et^{\zeta_1(B_{n'})} &= e^{-H^c(B_{n'})(1-t)} \prod_{x \in D(H)\cap B_{n'}} (1 - H\{x\}(1-t)) \\ &= \exp\Big(- H^c(B_{n'})(1-t) + \sum_{x \in D(H)\cap B_{n'}} \log(1 - H\{x\}(1-t))\Big) \\ &\le \exp\Big(-H^c(B_{n'})(1-t) - (1-t) \sum_{x \in D(H)\cap B_{n'}} H\{x\}\Big) \end{aligned}$$

since $H\{x\} < 1$ for all $x \in B_{n'}$ if n' is large enough, and $\log(1-x) \le -x$ for $0 < x < 1$. Thus

$$Et^{\zeta_1(B_{n'})} \le e^{-H(B_{n'})(1-t)} \to 0 \qquad (n' \to \infty).$$

This behaviour of its probability generating function implies that $\zeta_1(B_{n'}) \xrightarrow{P} \infty$, which contradicts the assumption $P(\zeta_1(B_n) > s') \to 0$. So we conclude that $\eta\{x_n^{(1)}\}$ is bounded away from 0 for all large n.

If $s = 1$ we are done. Otherwise, observe that

$$P\big(\zeta_1(B_n \setminus \{x_n^{(1)}\}) \ge s-1\big) \ge P\big(\zeta_1(B_n) \ge s\big) \to 1,$$

and $P\big(\zeta_1(B_n \setminus \{x_n^{(1)}\}) > s'\big) \to 0$, so we may re-apply the above to $B_n \setminus \{x_n^{(1)}\}$, whence the existence of $x_n^{(2)} \ne x_n^{(1)}$ such that $\eta\{x_n^{(2)}\}$ is also eventually bounded away from 0. We can do this in all s times, and that gives the result. □

LEMMA 3.26. *Let (x_n) be a strictly increasing sequence such that*

$$c \le x_{n+s} - x_n \le d \qquad (n = 1, 2, \ldots), \tag{3.37}$$

where $0 < c < d < c(s+1)/s < \infty$.

(i) Suppose from the sequence (x_n) we remove an element x_n, where $n > s$, or we insert $t > x_s$ as a new element, and that in either case we re-index the modified sequence in increasing order. The resulting sequence then violates (3.37).

(ii) Every $t \ge x_s$ is contained in an interval of length less than c that contains s or more elements of the sequence (x_n).

Proof. (i) Try inserting t, where $x_{n_0} < t < x_{n_0+1}$ for some $n_0 \ge s$. The re-indexed sequence is (y_n), where $y_n = x_n$ for $n \le n_0$, $y_{n_0+1} = t$, and $y_n = x_{n-1}$ for $n > n_0 + 1$. We have

$$\sum_{n=n_0+1}^{n_0+s+1} (y_n - y_{n-s}) = \sum_{n=n_0+1}^{n_0+s} (x_n - x_{n-s}) \le sd,$$

so at least one of the summands on the left has value at most $sd/(s+1) < c$. Thus the modified sequence violates (3.37).

Suppose we remove x_n, where $n > s$, from the original sequence, and that on re-indexing we get a sequence conforming to (3.37). Then reinstatement of the removed x_n constitutes a legal insertion into the latter sequence, contradicting what we have just proved. So removals also violate (3.37).

(ii) In the first part of the above proof we have shown that every $t > x$, that is not an element of (x_n), is contained in an interval of length $sd/(s+1)$ that contains

s elements of (x_n). Taking limits, we find that the same holds when $t = x_n$ for any $n \geq s$. Since $sd/(s+1) < c$, we have (ii). $\square$

LEMMA 3.27. *(2.40) holds if and only if both of the following hold:*

$$\sum_{k=1}^{\infty} e^{-\eta\{x_k\}} < \infty; \tag{3.38}$$

$$\eta(\mathbb{R} \setminus \{x_1, x_2, \ldots\}) < \infty. \tag{3.39}$$

Proof. By (1.3), (3.38) may be re-expressed as $\sum_{k=1}^{\infty} F(x_k, \infty)/F[x_k, \infty) < \infty$. The summand is $F(x_k, \infty)/(F\{x_k\} + F(x_k, \infty))$, so this condition is equivalent to

$$\sum_{k=1}^{\infty} \frac{F(x_k, \infty)}{F\{x_k\}} < \infty. \tag{3.40}$$

Again by (1.3), (3.39) is equivalent to $\sum_{k=1}^{\infty} \log\bigl(F(x_k, \infty)/F[x_{k+1}, \infty)\bigr) < \infty$. Using $\log(1+x) \sim x$ as $x \to 0$ we may re-express this as $\sum_{k=1}^{\infty} F(x_k, x_{k+1})/F[x_{k+1}, \infty) < \infty$. In the summand the denominator is asymptotic to $F\{x_{k+1}\}$, since $F(x_{k+1}, \infty) = \mathrm{o}(F\{x_{k+1}\})$ by (3.40), so this last condition becomes $\sum_{k=1}^{\infty} F(x_k, x_{k+1})/F\{x_{k+1}\} < \infty$. In combination with (3.40), this is (2.40). $\square$

Proof of Theorem 2.8. Suppose that (2.38) holds. We may choose $\varepsilon > 0$ so small that

$$d + 2\varepsilon < \frac{s+1}{s}(c - 2\varepsilon). \tag{3.41}$$

By Proposition 3.6 we know that $\sum_{n=0}^{\infty} P(\zeta_1(J_n) < s) < \infty$, where $J_n := (n\varepsilon - d - \varepsilon, n\varepsilon]$. So $P(\zeta_1(J_n) \geq s) \to 1$ as $n \to \infty$. On the other hand, since $\liminf_{n\to\infty}(R_{n+2s} - R_n) \geq 2c > d$ a.s., by Proposition 3.13 we have $P(\zeta_1(J_n) > 2s) \to 0$. Lemma 3.25 now gives us the existence within each interval J_n, for $n \geq n_0$, of points $y_n^{(1)}, \ldots, y_n^{(s)}$ such that $\eta\{y_n^{(1)}\} \geq \cdots \geq \eta\{y_n^{(s)}\} > a > 0$. These are distinct within each interval but can coincide as we move between n-values, so let $x_1 < x_2 < \ldots$ be the set of all $y_n^{(i)}$ with repetitions removed and in increasing order. The sequence (x_n) then has the properties that $\eta\{x_n\} \geq a > 0$ and that every interval J_n contains at least s of its elements, so $x_{n+s} - x_n \leq d + 2\varepsilon$ for all $n \geq n_0$. Although (x_n) might at this stage depend on choice of ε, we shall soon see that it does not in fact do so.

Suppose $\liminf_{n\to\infty}(x_{n+s} - x_n) < c$. Then we can find $\delta > 0$ and a sequence of integers $n' \uparrow \infty$ such that every interval $(n'\delta - c + \delta, n'\delta]$ contains more than s of the x_k; say it contains $x_{k(n')}, x_{k(n')+1}, \ldots, x_{k(n')+s}$ at least. Then

$$P(\zeta_1(n'\delta - c + \delta, n'\delta] > s) \geq \prod_{j=0}^{s} \bigl(1 - e^{-\eta\{x_{k(n')+j}\}}\bigr) \geq (1 - e^{-a})^{s+1}.$$

By virtue of Proposition 3.13, this contradicts (2.38). So $\liminf_{n\to\infty}(x_{n+s} - x_n) \geq c$.

Denote $(n\varepsilon - c + \varepsilon, n\varepsilon]$ by I_n. By what we have just proved, each I_n for all large n contains at most s terms of the sequence (x_k). Let $I_{n'}$ be the sequence of those I_n that contain at least s terms. Then

$$\begin{aligned} P\bigl(\zeta_1(I_{n'}) > s\bigr) &\geq P\bigl(\zeta_1(I_{n'} \setminus \{x_1, x_2, \ldots\}) > 0\bigr) \prod_{\{k : x_k \in I_{n'}\}} \bigl(1 - e^{-\eta\{x_k\}}\bigr) \\ &\geq \bigl(1 - e^{-\eta(I_{n'} \setminus \{x_1, x_2, \ldots\})}\bigr)(1 - e^{-a})^s \end{aligned}$$

for all large n'. By Proposition 3.13 the sum of the left-hand sides is finite, hence $1 - e^{-\eta(I_{n'}\setminus\{x_1,x_2,\ldots\})} \sim \eta(I_{n'} \setminus \{x_1, x_2, \ldots\})$ and the sum of these converges. Now we know $c - 2\varepsilon \le x_{n+s} - x_n \le d + 2\varepsilon$ for all $n \ge n_1$, say, and, because of (3.41), Lemma 3.26 applies to $(x_n)_{n\ge1}$. So every $t \ge x_{n_1+s}$ is contained in some interval of length below $c - 2\varepsilon$ that also contains s elements of the sequence (x_n). Therefore $t \in I_{n'}$ for some n'. So the union of the $I_{n'}$ contains the interval $[x, \infty)$, where $x = x_{n_1+s}$. So $\eta\big([x,\infty) \setminus \{x_1, x_2, \ldots\}\big) < \infty$. Since $\eta(-\infty, x] < \infty$ we conclude (3.39).

In the above calculation we have also shown that $P\big(\zeta_1(I_{n'} \setminus \{x_1, x_2, \ldots\}) > 0\big)$ is summable, so with probability 1 there are only finitely many points of ζ_1 in the set $\bigcup_{n'}(I_{n'} \setminus \{x_1, x_2, \ldots\})$, and hence only finitely many of the R_n are not members of the sequence (x_k). But then also all but finitely many of the x_k must be points of ζ_1, for otherwise by Lemma 3.26(i) and (3.41) there would be infinitely many $R_{n+s} - R_n$ exceeding $d + \varepsilon$. We thus have the last statement of the theorem, and further the sequence (x_k) must satisfy (2.39). Also the probability that infinitely many of the events $\{\zeta_1\{x_k\} = 0\}$ occur must be 0, and since those events are independent with respective probabilities $e^{-\eta\{x_k\}}$, that gives (3.38). Hence (2.40), by Lemma 3.27.

The converse is easy. Suppose we have (x_k) with (2.39) and (2.40), hence (3.38) and (3.39). The converse of our deduction of (3.38) above holds, so $\zeta_1\{x_k\} = 1$ for all large k, a.s. Also the events $\zeta_1((n-1, n] \setminus \{x_1, x_2, \ldots\}) > 0$ are independent, with respective probabilities $1 - e^{-\eta((n-1,n]\setminus\{x_1,x_2,\ldots\})}$ that by (3.39) are asymptotic to $\eta((n-1, n] \setminus \{x_1, x_2, \ldots\})$ and so have finite sum. So only finitely many of those events occur. Thus with probability 1 there is an x such that ζ_1 on the interval $[x, \infty)$ coincides with the restriction of (x_n) to that set. (2.38) follows. □

REMARK 3.28. When $d \ge c(s+1)/s$ we can still show the existence of (x_n), satisfying (2.39), such that all x_n for large n are record values; but now some more complicated things can happen in between the x_n values. Take the $s = 1$ case. We can establish the existence of the atoms x_k of η as before. By considering the intervals $I_{n'}$ as before, we can establish that no records in the intervals $(x_n - c + \varepsilon, x_n + c - \varepsilon)$, other than at x_n, eventually occur, and that records at the x_n must eventually occur. All this is as in Theorem 2.8 but we now must also consider the intervals $[x_n + c, x_n - c]$ of which there may be now infinitely many. In them, records can occur, separated by distance $c + \mathrm{o}(1)$, but need not, and η-characterizations of this behaviour could be given, but would be complicated.

When s is a general positive integer, and $d \ge c(s+1)/s$, although sequences exist satisfying both (3.37) and the conclusions of Lemma 3.26(i), there always exists some sequence satisfying (3.37) but failing these conclusions. Consider, for instance, (x_n) for which for some n_0, $x_{n_0+1} - x_{n_0} = 2c/s$ but $x_{n+1} - x_n = c/s$ for $n = n_0 - s + 1$, $\ldots$, $n_0 - 1$ and for $n = n_0 + 1$, $\ldots$, $n_0 + s - 1$. With suitably defined other x_n, this satisfies (3.37); yet $t := \frac{1}{2}(x_{n_0} + x_{n_0+1})$ is insertible without any consequent violations of (3.37).

The characterization of (2.38) for such s, c and d cannot therefore demand of the R_n as Theorem 2.8 does that they eventually be restricted to stepping through a single non-random sequence.

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