PERPETUITIES WITH THIN TAILS

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Abstract

We investigate the behaviour of $P(R \geq r)$ and $P(R \leq -r)$ as $r \to \infty$ for the random variable $R := \sum_{n=1}^{\infty} Q_n \prod_{k=1}^{n-1} M_k$, where $((Q_k, M_k))_{k \in \mathbb{N}}$ is an independent, identically distributed sequence with $P(-1 \leq M \leq 1) = 1$. Random variables of this type appear in insurance mathematics, as solutions of stochastic difference equations, in the analysis of probabilistic algorithms and elsewhere. Exponential and Poissonian tail behaviour can arise.

PERPETUITY; RANDOM AFFINE MAP; SELECTION ALGORITHM; STOCHASTIC DIFFERENCE EQUATION; STOCHASTIC DISCOUNTING; TAIL BEHAVIOUR

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1. Introduction

Let (Q, M), (Q_1, M_1), (Q_2, M_2), ... be independent and identically distributed (i.i.d.) $\mathbb{R}^2$-valued random elements on some probability space $(\Omega, \mathcal{A}, P)$, and put

$$R := \sum_{i=1}^{\infty} Q_i \prod_{j=1}^{i-1} M_j. \tag{1.1}$$

Random variables of this type occur in a variety of situations. R arises in insurance mathematics, where it is called a *perpetuity*. The interpretation is that it represents the present value of a commitment to pay a possibly fluctuating amount of some resource every year in the future, the future being subject to random discounting. Second, R solves a stochastic difference equation, that is, its distribution is the fixed point of a random affine map (see (1.5) below). Many references for the above are given in [Embrechts & Goldie 1994]: we here single out [Kesten 1973] and [Vervaat 1979] as the most fundamental papers on the topic. These three papers refer to a huge range of applications, from water density in Norwegian fjords to the ARCH processes of modern financial modelling.

We focus on cases where $P(-1 \leq M \leq 1) = 1$. These figure in many of the known applications but have had less theoretical development. In particular we treat the case when $Q = 1$ and $0 \leq M \leq 1$ almost surely (a.s.), when (1.1) becomes

$$R := 1 + \sum_{i=2}^{\infty} \prod_{j=1}^{i-1} M_j. \tag{1.2}$$

For this we have a new motivation: random variables (r.v.s) of this type arise in connection with the analysis of a probabilistic selection algorithm in the style of Quicksort, due to Hoare. If C_{nl} denotes the number of comparisons needed by the algorithm to find the l^{th} smallest element in a set of size n, then C_{nl} converges in distribution as $n \to \infty$ if l varies with n such that l/n tends to a limit $t \in [0,1]$. If $t = 0$ or $t = 1$ then the limit distribution is of the type of (1.2) with M uniformly distributed on the unit interval; for general t a stochastic upper bound can be found that is again of this type, now with M uniformly distributed on the interval $(\frac{1}{2}, 1)$ [Grübel & Rösler 1993/96]. The quantiles of these distributions are obviously of interest as they permit a quantification of the likelihood of excessively long searching times.

The case of (1.2) when $M = U^{1/\alpha}$, where U is uniformly distributed on the unit interval and α is a positive parameter, has been much studied and the behaviour of the tail $P(R \geq r)$ of R is known. For $R-1$ is then the r.v. $V := \sum_{n=1}^{\infty}(U_1 \cdots U_n)^{1/\alpha}$, which occurs in number theory, combinatorics, atomic cascades and elsewhere. Behaviour of the tail $P(V \geq v)$ for large v is treated in [de Bruijn 1951] and [Vervaat 1977, p. 94]. As V is infinitely divisible (i.d.), its tail behaviour to the accuracy we shall be aiming for here can be deduced from general results about i.d. laws: see §6.3 for details.

Søren Asmussen reminded us that perpetuities arise naturally in two distinct ways from shot-noise processes with exponentially decaying after-effect. For our purposes such a process is built from a stationary Poisson point process $\mathcal{P}$ on $\mathbb{R}$, though what follows can be extended with minor variations to the case when $\mathcal{P}$ is a stationary renewal process. $\mathcal{P}$ is thought of as occurring in time. Associated with each point $p \in \mathcal{P}$ is an impulse I_p. The I_p are real r.v.s which are conditionally i.i.d. given $\mathcal{P}$. The value of the shot-noise process at time t is $X_t := \sum_{p \in \mathcal{P} \cap (-\infty, t]} e^{-\lambda(t-p)} I_p$ where λ is a positive parameter. The sum converges absolutely a.s. provided that $E\big(\log(|I_p| \vee 1) \bigm| \mathcal{P}\big) < \infty$. Now it is immediate that X_t satisfies the variant form

$$R := \sum_{i=1}^{\infty} Q_i \prod_{j=1}^{i} M_j$$

of (1.1), with the Q_i being the impulses $I_{p_i^t}$ occurring at the sequence $\cdots < p_2^t < p_1^t < t$ of time-points of $\mathcal{P}$ prior to t, and the decay-factors M_i being respectively $\exp\big(-\lambda(p_i^t - p_{i-1}^t)\big)$ where $p_0^t := t$. On the other hand X_t is also given by (1.1) where the M_i are non-random, namely $M_i := e^{-\lambda}$ for all i, and $Q_i := \sum_{p \in \mathcal{P} \cap (t-i, t-i+1]} e^{-\lambda(t-i+1-p)} I_p$. These two formulations correspond respectively to sampling the shot-noise process at impulse times and at unit time-intervals.

Most of the existing literature deals with cases where R is heavy-tailed, meaning at least that not all its moments are finite. That can arise from Q being heavy-tailed [Grincevičius 1975], [Grey 1994], or, a deeper phenomenon, when neither Q nor M is heavy-tailed but M can take values outside the interval $[-1, 1]$ [Kesten 1973]. Again, see [Embrechts & Goldie 1994] for a fuller survey. Our interpretation is that in the first case, individual large values of the Q-sequence dominate R while in the second case clusters of large values of the sequence of products $(\prod_{k=1}^{n} M_k)_{n=1}^{\infty}$ do so. The third case, when $P(-1 \leq M \leq 1) = 1$, is qualitatively distinct from the other two, provided Q is sufficiently light-tailed; then R is necessarily light-tailed (has all moments finite). Interesting new phenomena arise as large values of the Q_n or of the products $\prod_1^n M_k$ no longer have a dominant contribution towards the tails of R. Also this case covers some of the most interesting applications, such as the one to Hoare's selection algorithm outlined above.

Our main results are given in the next three sections. §2 gives a generally-applicable exponential upper bound. Exponential tail decay of R arises mainly when M has an atom at ± 1. By contrast when M has a density in the neighbourhood of $+1$, and Q is a.s. bounded, Poisson-like tail decay ensues, as we show in §3. In §4 we show that the cases when $P(|M| > 1) > 0$ are fundamentally different from those when the contrary holds, in that they always yield a heavy-tailed R. Proofs are in §5, and the paper concludes with some complements in §6.

We conclude this introduction by summarizing our notation and those parts of the standard theory of perpetuities that we shall need. See [Vervaat 1979] or [Embrechts & Goldie 1994] for a full treatment.

Let $\mathcal{M}(\mathbb{R})$ (respectively $\mathcal{M}(\mathbb{R}^2)$) denote the set of all probability measures on the Borel subsets of $\mathbb{R}$ ($\mathbb{R}^2$). We write $\mathcal{L}(X)$ for the probability law or distribution of X, and will say $X =_{\mathcal{D}} Y$ if X and Y have the same distribution, i.e. $\mathcal{L}(X) = \mathcal{L}(Y)$. We consider the distribution $\nu := \mathcal{L}(Q, M)$ as a fixed element of $\mathcal{M}(\mathbb{R}^2)$, and define a function $T : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ by

$T\mu := \mathcal{L}(Q + MX)$, where X is independent of (Q, M) and $\mathcal{L}(X) = \mu$, $\mathcal{L}(Q, M) = \nu$.

Suppose

$$P(-1 \le M \le 1) = 1 \quad \text{and} \quad P(-1 < M < 1) > 0. \tag{1.3}$$

If, further,

$$P(M = 0) > 0 \quad \text{or} \quad E\log(1 \vee |Q|) < \infty, \tag{1.4}$$

then R is well defined by (1.1) in that the series converges with probability 1. Under (1.3), although condition (1.4) is not strictly necessary for existence of R, it covers all cases of interest. Indeed, if $P(M = 0) > 0$ then the series in (1.1) is actually a finite sum, whereas if on the other hand $E\log|M| > -\infty$ then the condition $E\log(1 \vee |Q|) < \infty$ is necessary for existence of R.

Under (1.3) and (1.4) the series in (1.1) is in fact a.s. absolutely convergent. Further R, or more precisely its probability law, is a fixed point of T:

$$R =_{\mathcal{D}} Q + MR, \text{ with } R \text{ independent of } (Q, M), \tag{1.5}$$

and is the unique probability law to solve (1.5).

The element δ_a of $\mathcal{M}(\mathbb{R})$ is unit mass at $a \in \mathbb{R}$. As usual T^n, for $n \in \mathbb{N}$, are the iterates of T. In $\mathcal{M}(\mathbb{R})$ the 'weak' convergence of probability laws is denoted $\to_{\mathrm{w}}$. Now it is not hard to check by induction that the n^{th} partial sum in (1.1) has distribution $T^n\delta_0$:

$$T^n\delta_0 = \mathcal{L}\Big(\sum_{i=1}^{n} Q_i \prod_{j=1}^{k-1} M_j\Big) \quad \text{for all } n \in \mathbb{N}. \tag{1.6}$$

Since the partial sums converge a.s. to R it follows that the sequence $(T^n\delta_0)_{n\in\mathbb{N}}$ converges weakly to μ_R, the distribution of R.

Turning to the special perpetuity (1.2), with $P(0 \le M \le 1) = 1$, we see that R is a.s. finite provided M is not degenerate at 1. The law of R is then the unique solution of

$$R =_{\mathcal{D}} 1 + MR, \text{ with } R \text{ independent of } M. \tag{1.7}$$

We shall touch upon cases when $P(-1 \le M \le 1) < 1$, violating (1.3), only in §4. Certain degenerate solutions of (1.5) are then possible but we shall rule them out by specifying that the law of R be non-degenerate and solve (1.5). According to the references cited above, such a probability law is then the unique solution to (1.5) and is the law of R in (1.1).

We use the notation

$$x^+ := \max(x, 0), \quad x^- := -\min(x, 0) \qquad (x \in \mathbb{R}).$$

2. Exponential upper bounds.

The result for this section is

THEOREM 2.1. *Let R satisfy (1.5), where $P(|M| \le 1) = 1$ and $P(|M| < 1) > 0$.*

(a) *If $Ee^{\varepsilon|Q|} < \infty$ for some $\varepsilon > 0$ then $P(|R| \ge r)$ decreases at least exponentially fast:*

$$\limsup_{r\to\infty} r^{-1} \log P(|R| \ge r) \le -\rho_0,$$

where $\rho_0 := \sup\{\rho : E(e^{\rho|Q|}|M|) < 1\} > 0$.

(b) *Assume $P(M \ge 0) = 1$, and that (1.4) holds. Then we gain separate results on each tail: if $Ee^{\varepsilon Q^+} < \infty$ for some $\varepsilon > 0$ then*

$$\limsup_{r\to\infty} r^{-1} \log P(R \ge r) \le -\rho_+, \tag{2.1}$$

where $\rho_+ := \sup\{\rho : E(e^{\rho Q^+} M) < 1\} > 0$, while if $Ee^{\varepsilon Q^-} < \infty$ for some $\varepsilon > 0$ then

$$\limsup_{r\to\infty} r^{-1} \log P(R \le -r) \le -\rho_-,$$

where $\rho_- := \sup\{\rho : E(e^{\rho Q^-} M) < 1\} > 0$.

When Q is degenerate the theorem becomes simpler to state. We record its form for the special perpetuity (1.2):

COROLLARY 2.2. *Let R satisfy (1.7), where $P(0 \le M \le 1) = 1$ and $P(M = 1) < 1$. Then*

$$\limsup_{r\to\infty} r^{-1} \log P(R \ge r) \le \log EM. \tag{2.2}$$

We now present some examples to show that the above bounds cannot be lowered unless we impose further restrictions on the distribution of M and/or Q. In the special setting of Corollary 2.2 the bound can be attained exactly:

Example 2.3. In (1.7) let $P(M=0) = 1 - P(M=1) = p$, for some p with $0 < p < 1$. Then it is easy to see that R has a geometric distribution with parameter p, hence $P(R \geq k) = (1-p)^{k-1}$ for all $k \in \mathbb{N}$ and consequently

$$\lim_{k\to\infty} k^{-1} \log P(R \geq k) = \log(1-p) = \log EM.$$

It is not surprising that the upper bound is achieved by a variable taking only the values 0 and 1, as such distributions maximize the atom at 1 among all distributions on the unit interval with prescribed mean. If M has an atom at 1 then a comparison argument, together with the calculation in the above example, can be used to obtain an exponentially decreasing lower bound for the tail of R in (1.2); see the remarks before the proof of Theorem 3.1 in §5 for a suitable ordering of distributions and the associated monotonicity of the function that maps the distribution of M to the distribution of R. Roughly, exponential tails are associated with atoms at 1.

The next two examples cover more cases than the above, but can approach the relevant bounds only as closely as desired rather than exactly. They respectively show that the bounds of Theorem 2.1(a) and (b) are sharp. They are of additional interest in being some of the rare instances where explicit calculations are practicable.

Example 2.4. Let $Q := 1$ and let M take values $1, -1, 0$ with respective probabilities p, q, r, all positive and with $p+q+r = 1$. Then it is easy to see that the perpetuity $R := \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} M_j$ has probability-generating function (p.g.f.)

$$Es^R = \frac{r(s-p+q)}{1 - p(s+s^{-1}) + p^2 - q^2},$$

convergent for $\kappa^{-1} < s < \kappa$, where κ, κ^{-1} are the zeroes of the denominator, namely

$$\kappa = \frac{1+p^2-q^2+\sqrt{(1+p^2-q^2)^2 - 4p^2}}{2p},$$
$$\kappa^{-1} = \frac{1+p^2-q^2-\sqrt{(1+p^2-q^2)^2 - 4p^2}}{2p}.$$

Expanding the p.g.f., one finds that for all $n \in \mathbb{N}$, $P(R = n) = a\kappa^{-n}$ and $P(R = -n) = b\kappa^{-n}$, for certain positive constants a and b. So both $n^{-1}\log P(R \geq n)$ and $n^{-1}\log P(R \leq -n)$ converge to $-\log\kappa$ as $n \to \infty$. The bound $-\rho_0$ of Theorem 2.1 simplifies here to $\log E|M| = \log(p+q)$. This exceeds the true value $-\log\kappa$, but if q is made small while the value of p is held fixed then both $-\log\kappa$ and the bound approach $\log p$, so the bound is sharp.

Example 2.5. Let $P(Q=1,\ M=1) = p$, $P(Q=-1,\ M=1) = q$, $P(Q=1,\ M=0) = r$, where p, q and r are positive and sum to 1. Then R is geometrically stopped random walk: for $n = 0, 1, 2, \ldots$ there is probability $(1-r)^n r$ that $M_1 = \cdots = M_n = 1$ while $M_{n+1} = 0$. On that event we have $R = Q_1 + \cdots + Q_{n+1}$, where $Q_{n+1} = 1$; further, $Q_1, \ldots, Q_n$ are conditionally independent, each 1 or -1 with respective probabilities $p/(p+q)$ and $q/(p+q)$. Hence R has p.g.f.

$$Es^R = \frac{(1-p-q)s}{1-ps-q/s},$$

convergent for $\kappa_- < s < \kappa_+$ where κ_- and κ_+ are the zeroes of the denominator:

$$\kappa_- = \frac{1-\sqrt{1-4pq}}{2p}, \quad \kappa_+ = \frac{1+\sqrt{1-4pq}}{2p}.$$

Expanding the p.g.f., we find this time that for all $n \in \mathbb{N}$, $P(R \geq n) = a\kappa_+^{-n}$ and $P(R \leq -n) = b\kappa_-^n$, for certain positive constants a and b. Therefore

$$\lim_{n\to\infty} n^{-1} \log P(R \geq n) = -\log \kappa_+, \qquad \lim_{n\to\infty} n^{-1} \log P(R \leq -n) = \log \kappa_-.$$

The upper bounds given by Theorem 2.1(b) are

$$-\rho_+ = -\log \frac{1-q}{p}, \quad -\rho_- = -\log \frac{1-p}{q}.$$

Again, these exceed the true values $-\log \kappa_+$ and $\log \kappa_-$. But as $q \downarrow 0$ with p fixed, both $-\rho_+$ and $-\log \kappa_+$ converge to $\log p$. Thus the first bound is sharp, and similarly for the second.

3. Poissonian tails.

We noted in §2 that exponential tails of R are associated with atoms of the multiplier at 1. Trivially, if $P(|Q| \leq a) = 1$ for some $a < \infty$ and $P(|M| \leq c) = 1$ for some $c < 1$, then $P(|R| \geq r) = 0$ for all $r > a/(1-c)$ ('there is no tail'). If Q is degenerate and M is uniformly distributed on $(\frac{1}{2}, 1)$ or $(0,1)$, as in one of the applications in §1, then it is possible to show that exponential decay of the tails implies faster-than-exponential decay (we will not give the details as the next result contains a stronger statement). What is the exact rate of tail decrease for such variables? This question leads us to consider distributions of M for which the ratio $P(1-\varepsilon \leq M \leq 1)/\varepsilon$ is bounded away from 0 and ∞ as ε tends to 0. That is the case considered in this section.

Known results on heavy-tailed perpetuities reveal a dichotomy between the case when M is a.s. non-negative, and the contrary case; cf. [Kesten 1973], [Goldie 1991]. The two tails can be of totally different magnitudes in the former situation, but not in the latter. A similar dichotomy is present here, and we give our results below for the two cases in separate theorems.

Theorem 3.1. *Let R satisfy (1.5), where Q is independent of M. Suppose that (1.4) holds, that $P(0 \leq M \leq 1) = 1$ and that for some $\varepsilon > 0$ there are constants $c > 0$ and $C < \infty$ such that*

$$c\delta \leq P(1-\delta \leq M \leq 1) \leq C\delta \quad \text{for all } \delta \in (0, \varepsilon]. \tag{3.1}$$

(a) *If $q_+ := \sup\{q : P(Q \geq q) > 0\}$ satisfies $0 < q_+ \leq \infty$ then*

$$\lim_{r\to\infty} \frac{\log P(R \geq r)}{r \log r} = -\frac{1}{q_+}.$$

(b) *If $q_- := \inf\{q : P(Q \leq q) > 0\}$ satisfies $-\infty \leq q_- < 0$ then*

$$\lim_{r\to\infty} \frac{\log P(R \leq -r)}{r \log r} = \frac{1}{q_-}.$$

Theorem 3.2. *Let R satisfy (1.5), where Q is independent of M. Suppose that (1.4) holds, that $P(-1 \le M \le 1) = 1$ and $P(M < 0) > 0$, and that for some $\varepsilon > 0$ there are constants $c > 0$ and $C < \infty$ such that*

$$P(1-\delta \le M \le 1) \ge c\delta \quad \text{and} \quad P(1-\delta \le |M| \le 1) \le C\delta$$

for all $\delta \in (0, \varepsilon]$. Suppose also that $P(Q \ne 0) > 0$. Then

$$\lim_{r\to\infty} \frac{\log P(R \ge r)}{r \log r} = -\frac{1}{\max(q_+, m_- q_-)} \tag{3.2}$$

and

$$\lim_{r\to\infty} \frac{\log P(R \le -r)}{r \log r} = \frac{1}{\min(q_-, m_- q_+)}, \tag{3.3}$$

where $q_- := \inf\{q : P(Q \le q) > 0\}$, $q_+ := \sup\{q : P(Q \ge q) > 0\}$, and $m_- := \inf\{m : P(M \le m) > 0\}$.

4. Heavy tails.

This section is devoted to the following result:

Theorem 4.1. *Suppose R is non-degenerate and satisfies (1.5). If $P(|M| > 1) > 0$ then $|R|$ has at least a power-law ('Pareto') tail:*

$$\liminf_{r\to\infty} \frac{\log P(|R| \ge r)}{\log r} > -\infty. \tag{4.1}$$

In combination with Theorem 2.1, this shows that when Q is well behaved, meaning that it has finite m.g.f. (moment-generating function) in some non-empty open strip, no tail decay for the perpetuity (1.1) in between power-law and exponential is possible. Which of power-law and exponential decay can occur is determined simply by whether M can take values outside the interval $[-1, 1]$.

Theorem 4.1 has the following immediate consequence:

Corollary 4.2. *Suppose R is non-degenerate, has all moments finite, and satisfies (1.5). Then $P(-1 \le M \le 1) = 1$.*

A proof is given of Corollary 4.2 in [Pötzelberger 1990, Theorem 1(b)], and another, independently, in [Karlsen 1990, Corollary 4.1]. However neither proof is in our view valid. First, in [Pötzelberger 1990, page 173, lines 21–28], translated into our notation, it is supposed that $P(|M| > 1) > 0$, and a pair of r.v.s $(\tilde{Q}, \tilde{M})$ is set up, having the conditional law of (Q, M) given $|M| > 1$. This is used to build a r.v. $Y_1 := \tilde{Q} + \tilde{M} Y_0$, where Y_0 has the probability law μ of R and is independent of $(\tilde{Q}, \tilde{M})$. It is claimed that Y_1 then has law μ. But that is not so: Y_1 has the conditional law of $Q + MR$ in our (1.5), *given* $|M| > 1$, and this differs from μ.

Turning to [Karlsen 1990] we have two difficulties. First is that the use of Burkholder's and Rosenthal's inequalities on page 133 does not seem to make sense.

Where, for instance, are the squared random variables that have such an intrinsic role in these inequalities? Second, the proof of the relevant result, Karlsen's Corollary 4.1, refers back to the proof of his Theorem 3.2, which theorem carries an extra independence assumption that is not part of the hypotheses for his Corollary 4.1.

Our proof for Theorem 4.1, at the end of §5, is different from those of the above two authors.

In §6.5 we note that there is a very rapid proof available of Corollary 4.2, for the special case when Q and M are non-negative.

5. Proofs.

The main task in proving Theorem 2.1 is to establish its special case when Q and M are non-negative. We set this out as a separate lemma:

LEMMA 5.1. *Let R satisfy (1.5), where $P(Q \geq 0,\ 0 \leq M \leq 1) = 1$ and $P(M < 1) > 0$. If $Ee^{\varepsilon Q} < \infty$ for some $\varepsilon > 0$ then*

$$\limsup_{r \to \infty} r^{-1} \log P(R \geq r) \leq -\rho_0,$$

where $\rho_0 := \sup\{\rho : E(e^{\rho Q} M) < 1\} > 0$.

PROOF: The assumptions on M make $EM < 1$. Since $E(e^{\rho Q} M)$ is continuous where finite, and is less than 1 when $\rho = 0$, it follows that ρ_0 is indeed positive.

Because Q and M are non-negative, so is R and we can work in the space $\mathcal{M}[0, \infty)$ of probability measures on the Borel sets of $[0, \infty)$ instead of in $\mathcal{M}(\mathbb{R})$. Clearly we can regard T in this case as a map from $\mathcal{M}[0, \infty)$ to $\mathcal{M}[0, \infty)$.

Let $\mathcal{M}_\rho$, for $\rho > 0$, be the set of all $\mu \in \mathcal{M}[0, \infty)$ with the property $\int_0^\infty e^{\rho x} \mu[x, \infty)\, dx < \infty$. For all $\mu,\ \nu \in \mathcal{M}_\rho$ let

$$d_\rho(\mu, \nu) := \int_0^\infty e^{\rho x} \big|\mu[x, \infty) - \nu[x, \infty)\big|\, dx.$$

This defines a metric on $\mathcal{M}_\rho$. If $(\mu_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{M}_\rho, d_\rho)$ then $(f_n)_{n \in \mathbb{N}}$ with $f_n : [0, \infty) \to \mathbb{R}$, $f_n(x) := e^{\rho x} \mu_n[x, \infty)$, is a Cauchy sequence in the $\mathcal{L}^1$-space of integrable functions on $[0, \infty)$. Hence $\int_0^\infty |f_n(x) - f_\infty(x)|\, dx \to 0$ for some integrable left-continuous $f_\infty : [0, \infty) \to \mathbb{R}$, and a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ converges pointwise to f_∞ outside some Lebesgue-null set $A \subset (0, \infty)$. Define $h : [0, \infty) \to \mathbb{R}$ by $h(x) := e^{-\rho x} f_\infty(x)$ for all x. Then h is left-continuous, and the pointwise limit on A^c of non-increasing functions $\mu_{n_k}[\cdot, \infty)$, so is itself non-increasing. Since h is monotone and integrable, $h(x) \downarrow 0$ as $x \to \infty$. So h is the tail function of some $\mu_\infty \in \mathcal{M}[0, \infty)$:

$$\mu_\infty[x, \infty) = h(x) = e^{-\rho x} f_\infty(x) \qquad \big(x \in [0, \infty)\big).$$

Integrability of f_∞ makes $\mu_\infty \in \mathcal{M}_\rho$, and $\mathcal{L}^1$-convergence of f_n to f_∞ says that $d_\rho(\mu_n, \mu_\infty) \to 0$ as $n \to \infty$. In sum: $(\mathcal{M}_\rho, d_\rho)$ is a complete metric space. As we also have $\lim_{n \to \infty} \mu_n[x, \infty) = \mu_\infty[x, \infty)$ for all $x \notin A$, we have also shown that d_ρ-convergence implies weak convergence.

$\mathcal{M}_\rho$ can be characterized as the set of probability laws μ of non-negative r.v.s X for which $Ee^{\rho X} < \infty$. For such a μ, $T\mu$ is the law of $Q + MX$, where (Q, M) is independent of X, and we have

$$Ee^{\rho(Q+MX)} \le E^{\rho(Q+X)} = Ee^{\rho Q}\, Ee^{\rho X}.$$

Thus if $Ee^{\rho Q} < \infty$, T maps $\mathcal{M}_\rho$ into $\mathcal{M}_\rho$.

Now fix $q \ge 0$ and $0 < m \le 1$, let X have law μ and Y have law ν, and consider

$$\begin{aligned} I(q,m) &:= \int_0^\infty e^{\rho t} \big| P(q + mX \ge t) - P(q + mY \ge t) \big|\, dt \\ &= \int_q^\infty e^{\rho t} \left| P\Big(X \ge \frac{t-q}{m}\Big) - P\Big(Y \ge \frac{t-q}{m}\Big) \right| dt \\ &= \int_0^\infty e^{\rho(q+mu)} m \big| P(X \ge u) - P(Y \ge u) \big|\, du. \end{aligned}$$

Observe that $e^{\rho(q+mu-u)}$ is non-increasing in $u \ge 0$, with maximum value $e^{\rho q}$. Thus $I(q, m) \le e^{\rho q} m d_\rho(\mu, \nu)$, and so

$$d_\rho(T\mu, T\nu) \le EI(Q, M) \le d_\rho(\mu, \nu) E(e^{\rho Q} M).$$

Thus if $E(e^{\rho Q}M) < 1$, T not only maps $\mathcal{M}_\rho$ into itself but is a contraction with respect to d_ρ. As the metric space $(\mathcal{M}_\rho, d_\rho)$ is complete, this contraction has a unique fixed point, μ_ρ say. Since $\delta_0 \in \mathcal{M}_\rho$, $(T^n\delta_0)_{n\in\mathbb{N}}$ converges in $(\mathcal{M}_\rho, d_\rho)$ to μ_ρ. It follows as above that $T^n\delta_0 \to_{\mathrm{w}} \mu_\rho$, so μ_ρ must coincide with the law of R, which law is thus in $\mathcal{M}_\rho$. We conclude that $\limsup_{r\to\infty} r^{-1} \log P(R \ge r) \le -\rho$, and the conclusion follows. $\square$

Proof of Theorem 2.1: *part (a).* (1.1) implies that

$$|R| \le R^* := \sum_{i=1}^{\infty} |Q_i| \prod_{j=1}^{i-1} |M_j|,$$

so that $P(|R| \ge r) \le P(R^* \ge r)$ for all r. The conditions of (a), applied to R^*, are the same as those on R, so the above lemma yields $\limsup_{r\to\infty} r^{-1} \log P(R^* \ge r) \le -\rho_0$, and (a) follows.

Part (b). Non-negativity of M allows us to employ a variant of the R^* trick in (a), this time using the $\cdot^+$ function which is sub-additive:

$$(x + y)^+ \le x^+ + y^+ \qquad (x, y \in \mathbb{R}).$$

For we have from (1.1) that in this case

$$R \le \sum_{i=1}^{\infty} \left(Q_i \prod_{j=1}^{i-1} M_j \right)^+ = \sum_{i=1}^{\infty} Q_i^+ \prod_{j=1}^{i-1} M_j =: R^*.$$

If $Ee^{\varepsilon Q^+} < \infty$ then R^* satisfies the conditions for R in (a), so by (a), $\limsup_{r\to\infty} r^{-1} \log P(R^* \geq r) \leq -\rho_+$, whence (2.1) since $P(R \geq r) \leq P(R^* \geq r)$ for all r.

For the other part of (b) note that $-R$ satisfies

$$-R =_{\mathcal{D}} -Q + M(-R); \tag{5.1}$$

that is, $-R$ is the perpetuity generated by the probability law of $(-Q, M)$ in place of that of (Q, M). Thus, on applying to $-R$ what we have just proved, we gain the claimed bound on the lower tail of R. $\square$

For the results of §3 we need elementary notions of stochastic order. For μ, $\nu \in \mathcal{M}(\mathbb{R})$ we write $\mu \preceq \nu$ if $\mu[x, \infty) \leq \nu[x, \infty)$ for all $x \in \mathbb{R}$; we then say that μ is less than or equal to ν, or that ν (stochastically) dominates μ. The main use we make of this is that if f is non-decreasing then $\mu \preceq \nu$ implies $\int f\,d\mu \leq \int f\,d\nu$ if the integrals exist. This can be established by using a quantile construction to set up r.v.s X, Y with laws μ, ν respectively, such that $X \leq Y$ a.s.

We remark that stochastic order can be applied in a more far-reaching way to the perpetuity models in this paper. Consider, for instance, the function $\Psi : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ that maps the distribution μ_M of the M-variables in (1.2) to the distribution μ_R of R. Now Ψ is non-decreasing: that $\mu_1 \preceq \mu_2$ implies $\Psi(\mu_1) \preceq \Psi(\mu_2)$ can easily be seen on representing the distributions involved by suitably chosen random variables M_1, M_2, ... with law μ_1, and M_1', M_2', ... with law μ_2, such that $M_k \leq M_k'$ for all $k \in \mathbb{N}$; these in turn can again be obtained on using the quantile transformation. Monotonicity of Ψ is very useful in comparison arguments, as mentioned in §2.

The proof of Theorem 3.1 needs the following lemma.

Lemma 5.2. *For every $\alpha \in (0,1)$ there exists a constant $1 \leq C_\alpha < \infty$ such that*

$$\frac{1}{\alpha}\int_{1-\alpha}^{1} \exp\bigl(C_\alpha e^{\theta u}\bigr)\,du \leq \exp\bigl(C_\alpha e^{\theta} - \theta\bigr) \quad \textit{for all } \theta \geq 0.$$

Proof: Let $x := e^\theta$, $\mathcal{L}(U) := \mathrm{unif}(1-\alpha, 1)$ and

$$f(x, C) := xEe^{Cx^U} - e^{Cx}.$$

Our aim is to find a constant $C \geq 1$ such that $f(x, C) \leq 0$ for all $x \geq 1$. We have

$$\begin{aligned}\frac{\partial}{\partial x} f(x, C) &= E\bigl((CUx^U + 1)e^{Cx^U}\bigr) - Ce^{Cx} \\ &\leq e^{Cx}\bigl(CE(Ux^U - 1) + 1\bigr) =: g(x, C).\end{aligned}$$

With $C_0 := 3/\alpha$ we get $g(1, C_0) < 0$; note that $C_0 > 1$. Since g is continuous, there exists $x_0 > 1$ such that $g(x, C_0) \leq 0$ for $1 \leq x \leq x_0$. For each such x we therefore have $C_0E(Ux^U - 1) + 1 \leq 0$, in particular $EUx^U < 1$. So, for all $C \geq C_0$,

$$\begin{aligned} g(x, C) &\leq e^{Cx}\bigl(C_0E(Ux^U - 1) + 1\bigr) \\ &\leq e^{C_0x}\bigl(C_0E(Ux^U - 1) + 1\bigr) \\ &= g(x, C_0) \leq 0.\end{aligned}$$

Combining this upper bound for the derivative with the fact that $f(1,C)=0$ we see that
$$f(x,C) \le 0 \quad \text{for } 1 \le x \le x_0,\ C \ge C_0. \tag{5.2}$$
On the other hand, using
$$Ex^{kU} = \frac{1}{\alpha}\int_{1-\alpha}^{1} e^{ku\log x}\,du \le \frac{1}{\alpha k \log x} x^k$$
for all $k \in \mathbb{N}$ and $x > 1$, we obtain
$$\begin{aligned} Ee^{Cx^U} &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} C^k Ex^{kU} \\ &\le 1 + \frac{1}{\alpha \log x} \sum_{k=1}^{\infty} \frac{1}{kk!}(Cx)^k \\ &\le 1 + \frac{2}{Cx\alpha\log x}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}(Cx)^{k+1} \\ &\le 1 + \frac{2}{Cx\alpha\log x} e^{Cx}. \end{aligned} \tag{5.3}$$
Now let x_0 and C_0 be as in (5.2). There exists $C \ge C_0$ such that
$$e^C\left(1 - \frac{2}{C\alpha\log x_0}\right) \ge 1.$$
As $x \mapsto e^{Cx}/x$ and $x \mapsto 1 - 2/(C\alpha\log x)$ are increasing on $(1,\infty)$, it follows with this C that
$$\frac{e^{Cx}}{x}\left(1 - \frac{2}{C\alpha\log x}\right) \ge 1 \quad \text{for all } x \ge x_0.$$
This implies that the bound (5.3) is at most e^{Cx}/x, and so $f(x,C) \le 0$ for all $x \ge x_0$. Combining this with (5.2) we obtain the desired inequality. $\square$

Proof of Theorem 3.1: (b) follows immediately from (a) since, as noted in the proof of Theorem 2.1(b), (1.5) is equivalent to (5.1).

To prove (a) we first show that
$$\limsup_{r\to\infty} \frac{\log P(R \ge r)}{r \log r} \le -\frac{1}{q_+}. \tag{5.4}$$
We prove (5.4) without assuming Q independent of M. That assumption is needed only for the lower bound. Now (5.4) has content only when $q_+ < \infty$, so we assume that. Since $M \ge 0$ and $Q \le q_+$, a.s., (1.1) gives $R \le \tilde{R} := \sum_{i=1}^{\infty} q_+ \prod_{j=1}^{i-1} M_j$. It suffices to prove (5.4) for $\tilde{R}$ in place of R. Since $\tilde{R}$ satisfies the random equation
$$\tilde{R} =_{\mathcal{D}} q_+ + M\tilde{R},$$

the import is that it suffices to prove (5.4) assuming Q is degenerate at some positive constant q, as we now will do.

Under the condition (3.1) we may find $\alpha \in (0,1)$ sufficiently small that $P(M \geq x) \leq \min(1,(1-x)/\alpha)$ for all $x \in (0,1)$; that is, so that M is stochastically dominated by a unif$(1-\alpha,1)$-distributed r.v. U.

Let $M_\mu : [0,\infty) \to [0,\infty]$, $M_\mu(\theta) := \int e^{\theta x}\mu(dx)$ be the m.g.f. associated with $\mu \in \mathcal{M}(\mathbb{R})$. Let C_α be as in Lemma 5.2 and assume that $\mu \in \mathcal{M}[0,\infty)$ is such that

$$M_\mu(\theta) \leq \exp\bigl(C_\alpha e^{q\theta}\bigr) \quad \text{for all } \theta \geq 0. \tag{5.5}$$

Let $X \geq 0$ be independent of M and have law μ. By using stochastic dominance, formula (5.5) and Lemma 5.2 we obtain

$$\begin{aligned} M_{T\mu}(\theta) = Ee^{\theta(q+MX)} &\leq e^{\theta q} Ee^{\theta UX} \\ &\leq e^{\theta q} E\exp\bigl(C_\alpha e^{qU\theta}\bigr) \\ &\leq \exp\bigl(C_\alpha e^{q\theta}\bigr). \end{aligned}$$

That is, if (5.5) holds for some $\mu \in \mathcal{M}[0,\infty)$, then it also holds for $T\mu$. Certainly (5.5) is satisfied for δ_0 and therefore, by (1.6), for an increasing sequence of r.v.s that converges pointwise to R. Hence, by monotone convergence, (5.5) holds for the m.g.f. associated with R, and with the usual appeal to Markov's inequality we complete the proof of the upper bound.

To complete the proof of (a) it suffices to prove that for any finite $q > 0$ such that $p := P(Q \geq q) > 0$,

$$\liminf_{r\to\infty} \frac{\log P(R \geq r)}{r \log r} \geq -\frac{1}{q}. \tag{5.6}$$

Fix such a q. We then have $q_0 := P(Q \geq 0) > 0$, and will prove in Lemma 5.3 that $r_0 := P(R \geq 0) > 0$. Take $\varepsilon > 0$ so small that $C\varepsilon < 1$. Fix $\delta \in (0,\varepsilon]$ and let $N_\delta := \inf\{n \in \mathbb{N} : M_n < 1-\delta \text{ or } Q_n < q\}$. Since N_δ is a stopping time, the perpetuity construction (1.1) assures us that

$$R = \sum_{k=1}^{N_\delta} Q_k \prod_{j=1}^{k-1} M_j + R' \prod_{j=1}^{N_\delta} M_j,$$

where R' has the same distribution as R and is independent of $\sigma\bigl((Q_k, M_k)_{k \leq N_\delta}\bigr)$. Pick $n \in \mathbb{N}$. It is easy to check that N_δ and Q_{N_δ} are independent. The events that $N_\delta > n$, $Q_{N_\delta} \geq 0$ and $R' \geq 0$ are thus independent, and if they all occur then

$$R \geq \sum_{k=1}^{n} q(1-\delta)^{k-1}.$$

For their probabilities we have $P(N_\delta > n) \geq (c\delta p)^n$, $P(R' \geq 0) = r_0$, and

$$\begin{aligned} q_1 := P(Q_{N_\delta} \geq 0) &= P(Q \geq 0 \mid M < 1-\delta \text{ or } Q < q) \\ &\geq P(Q \geq 0)P(M < 1-\delta) \geq q_0(1 - C\varepsilon). \end{aligned}$$

Thus

$$P\Big(\frac{R}{q} \geq \frac{1-(1-\delta)^n}{\delta}\Big) \geq (c\delta p)^n r_0 q_1. \tag{5.7}$$

That implies

$$P\Big(\frac{R}{q} \geq \frac{1-(1-\delta)^y}{\delta}\Big) \geq (c\delta p)^{y+1} r_0 q_1 \quad \text{for all } y \geq 0.$$

Fix β and r satisfying $0 < \beta \leq \varepsilon$ and $r > q$; then we may set $\delta := q\beta/r$ in the above. With the further choice

$$y := \frac{\log(1-\beta)}{\log(1-q\beta/r)}$$

we obtain $r/q = \big(1-(1-\delta)^y\big)/\delta$ and so

$$\log P(R \geq r) \geq \big(\log(cq\beta p) - \log r\big)\left(\frac{\log(1-\beta)}{\log(1-q\beta/r)} + 1\right) + \log(r_0 q_1),$$

which implies

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log P(R \geq r)}{r \log r} &\geq -\frac{\log(1-\beta)}{\lim_{r\to\infty} r\log(1-q\beta/r)} \\ &= \frac{\log(1-\beta)}{q\beta}.\end{aligned}$$

As $\beta \downarrow 0$, this tends to $-1/q$, which completes the proof of the asymptotic lower bound. □

Proof of Theorem 3.2: As in the previous proof we establish an upper bound and a lower bound, and begin with the upper bound, where again we shall not need to assume Q independent of M. We treat the cases $m_- = -1$ and $m_- > -1$ separately.

Upper bound when $m_- = -1$. We show that

$$\limsup_{r\to\infty} \frac{\log P(|R| \geq r)}{r \log r} \leq -\frac{1}{q_*} \tag{5.8}$$

where $q_* := \max(-q_-, q_+) = \sup\{q : P(|Q| \geq q) > 0\}$. (5.8) has content only when $q_* < \infty$, which we therefore assume. From (1.1),

$$|R| \leq \tilde{R} := \sum_{i=1}^{\infty} q_* \prod_{j=1}^{i-1} |M_j|,$$

and $\tilde{R}$ satisfies

$$\tilde{R} =_{\mathcal{D}} q_* + |M|\tilde{R}.$$

By Theorem 3.1, therefore, with $|M|$ replacing M, we obtain

$$\lim_{r\to\infty} \frac{\log P(\tilde{R} \geq r)}{r \log r} = -\frac{1}{q_*},$$

whence (5.8).

Upper bound when $-1 < m_- < 0$. We are to prove

$$\limsup_{r\to\infty} \frac{\log P(R \geq r)}{r \log r} \leq -\frac{1}{\max(q_+, m_- q_-)} \tag{5.9}$$

and

$$\limsup_{r\to\infty} \frac{\log P(R \leq -r)}{r \log r} \leq \frac{1}{\min(q_-, m_- q_+)}, \tag{5.10}$$

There is nothing to prove unless q_- and q_+ are finite, so we assume that. Pick a such that $-m_- < a < 1$. Then the conditional law of $-M/a$, given $M < 0$, is that of a r.v. with values in $(0, -m_-/a]$, so is dominated by the unif$(1-\alpha, 1)$ law if $\alpha > 0$ is at most $1 - (-m_-)/a$. By taking α small enough we can ensure that unif$(1-\alpha,1)$ also dominates the conditional law of M given $M > 0$. Let U have the unif$(1-\alpha,1)$ distribution. Thus the conditional law of $-M$, given $M < 0$, is dominated by the law of aU.

Set $b_+ := \max(q_+^+, aq_-^-) \geq 0$, $b_- := \max(q_-^-, aq_+^+) \geq 0$. Let X be a r.v. with law μ such that

$$Ee^{\theta X} \leq \begin{cases} \exp\bigl(C_\alpha e^{b_+\theta}\bigr), & \text{for all } \theta \geq 0, \\ \exp\bigl(C_\alpha e^{-b_-\theta}\bigr), & \text{for all } \theta \leq 0, \end{cases} \tag{5.11}$$

with C_α as in Lemma 5.2. We show that these bounds on the m.g.f. of μ are inherited by the m.g.f. of $T\mu$. Let (Q, M) be independent of X and set $m_0 := P(M \geq 0)$. Then for $\theta \geq 0$,

$$\begin{aligned} Ee^{(Q+MX)\theta} &= m_0 E(e^{(Q+MX)\theta} \mid M \geq 0) + (1-m_0) E(e^{(Q+MX)\theta} \mid M < 0) \\ &\leq m_0 E\bigl(\exp(q_+^+\theta + C_\alpha e^{b_+ M\theta}) \mid M \geq 0\bigr) \\ &\qquad + (1-m_0) E\bigl(\exp(q_+^+\theta + C_\alpha e^{b_-(-M)\theta}) \mid -M > 0\bigr) \\ &\leq m_0 E\exp(q_+^+\theta + C_\alpha e^{b_+U\theta}) + (1-m_0) E\exp(q_+^+\theta + C_\alpha e^{b_- aU\theta}) \\ &\leq E\exp(b_+\theta + C_\alpha e^{b_+U\theta}) \end{aligned}$$

(since $q_+^+ \leq b_+$ and $b_- a \leq b_+$)

$$\leq \exp(C_\alpha e^{b_+\theta})$$

by Lemma 5.2. And, again for $\theta \geq 0$,

$$\begin{aligned} Ee^{-(Q+MX)\theta} &= m_0 E(e^{-(Q+MX)\theta} \mid M \geq 0) + (1-m_0) E(e^{-(Q+MX)\theta} \mid M < 0) \\ &\leq m_0 E\bigl(\exp(q_-^-\theta + C_\alpha e^{b_- M\theta}) \mid M \geq 0\bigr) \\ &\qquad + (1-m_0) E\bigl(\exp(q_-^-\theta + C_\alpha e^{b_+(-M)\theta}) \mid -M > 0\bigr) \\ &\leq m_0 E\exp(q_-^-\theta + C_\alpha e^{b_-U\theta}) + (1-m_0) E\exp(q_-^-\theta + C_\alpha e^{b_+ aU\theta}) \\ &\leq E\exp(b_-\theta + C_\alpha e^{b_-U\theta}) \end{aligned}$$

(since $q_-^- \leq b_-$ and $b_+ a \leq b_-$)

$$\leq \exp(C_\alpha e^{b_-\theta})$$

by Lemma 5.2. It now follows as in the proof of (5.4) that the m.g.f. of R also has the bound (5.11).

Since at least one of q_+ and $-q_-$ is non-negative, $b_+ = \max(q_+, -aq_-)$. Similarly $-b_- = \min(q_-, -aq_+)$. Letting $a \downarrow -m_-$, we conclude that

$$Ee^{\theta R} \leq \begin{cases} \exp\bigl(C_\alpha e^{\max(q_+, m_- q_-)\theta}\bigr), & \text{for all } \theta \geq 0 \\ \exp\bigl(C_\alpha e^{\min(q_-, m_- q_+)\theta}\bigr), & \text{for all } \theta \leq 0. \end{cases}$$

Use of Markov's inequality now leads to (5.9) and (5.10).

Lower bounds. To complete the proof we must show that $(r \log r)^{-1} \log P(R \geq r)$ has $\liminf$ at least the right-hand side of (3.2), and that $(r \log r)^{-1} \log P(R \leq -r)$ has $\liminf$ at least the right-hand side of (3.3). We split these tasks into four, namely to show

$$\text{when } 0 < q_+ < \infty, \quad \liminf_{r\to\infty} (r \log r)^{-1} \log P(R \geq r) \geq -1/q_+; \tag{5.12}$$

$$\text{when } -\infty < q_- < 0, \quad \liminf_{r\to\infty} (r \log r)^{-1} \log P(R \geq r) \geq -1/(m_- q_-); \tag{5.13}$$

$$\text{when } -\infty < q_- < 0, \quad \liminf_{r\to\infty} (r \log r)^{-1} \log P(R \leq -r) \geq 1/q_-; \tag{5.14}$$

$$\text{when } 0 < q_+ < \infty, \quad \liminf_{r\to\infty} (r \log r)^{-1} \log P(R \leq -r) \geq 1/(m_- q_+). \tag{5.15}$$

But (5.14) follows from (5.12), and (5.15) from (5.13), on considering the triple $(-R, -Q, M)$ in place of (R, Q, M), and re-using the equivalence of (1.5) with (5.1).

Proof of (5.12). It is enough to show that for any finite $q > 0$ such that $p := P(Q \geq q) > 0$, (5.6) holds (under present assumptions).

We imitate the proof of (5.6). Fix such a q. We again have $q_0 := P(Q \geq 0) > 0$, and Lemma 5.3 gives $r_0 := P(R \geq 0) > 0$. Take $\varepsilon' > 0$ so small that $c\varepsilon' < 1$. Take $\varepsilon > 0$ so small that $C\varepsilon < c\varepsilon'$. Then

$$\begin{aligned} m_0 := P(1 - \varepsilon' \leq M < 1 - \varepsilon) &= P(M \geq 1 - \varepsilon') - P(M \geq 1 - \varepsilon) \\ &\geq c\varepsilon' - C\varepsilon > 0. \end{aligned}$$

Fix $\delta \in (0, \varepsilon]$ and set $N_\delta := \inf\{n \in \mathbb{N} : M_n < 1 - \delta \text{ or } Q_n < q\}$ as before. The calculations leading to (5.7) go through, provided we replace the event that $Q_{N_\delta} \geq 0$ by the event $\{Q_{N_\delta} \geq 0\} \cap \{M_{N_\delta} \geq 0\}$, which has probability

$$\begin{aligned} q_1 &:= P(Q_{N_\delta} \geq 0,\ M_{N_\delta} \geq 0) \\ &= P(Q \geq 0,\ M \geq 0 \mid M < 1 - \delta \text{ or } Q < q) \\ &\geq P(Q \geq 0) P(0 \leq M < 1 - \delta) \geq q_0 m_0 > 0. \end{aligned}$$

Hence (5.7) and then (5.6).

Proof of (5.13). Choose m satisfying $0 < m < -m_-$, then $p_m := P(M \le -m) > 0$. For $r > 0$,

$$P(R \ge r) = P(Q + MR \ge r) \ge p_m P\big(R \le -(r - q_-)/m\big),$$

so by (5.14) we obtain

$$\liminf_{r\to\infty} \frac{\log P(R \ge r)}{\left(\frac{r-q_-}{m}\log\frac{r-q_-}{m}\right)} \ge \frac{1}{q_-},$$

and hence

$$\liminf_{r\to\infty} \frac{\log P(R \ge r)}{r\log r} \ge \frac{1}{mq_-}.$$

Since $-m$ can be taken as close to m_- as we wish, we conclude (5.13). $\square$

Lemma 5.3. *Under the conditions of Theorem 3.1(a) or Theorem 3.2, $P(R \ge 0) > 0$.*

Proof: First suppose $P(Q > 0) > 0$, as must be so in Theorem 3.1(a), and may or may not be so in Theorem 3.2. Pick $r \in \mathbb{R}$. We shall show that

$$P(R \ge r) > 0, \tag{5.16}$$

implying the result claimed, for this case. To do so, fix $q > 0$ so that $P(Q \ge q) > 0$. Pick any $r' \le 0$ such that $P(R \ge r') \ge \frac{1}{2}$. Choose $\delta \in (0,1)$ so small that $q\delta^{-1} + r' > r$. Pick n so large that $\sum_{k=1}^{n} q(1-\delta)^{k-1} + r' \ge r$. Now

$$R = \sum_{k=1}^{n} Q_k \prod_{j=1}^{k-1} M_j + R' \prod_{j=1}^{n} M_j,$$

where all the component r.v.s on the right, namely R', $Q_1, \ldots, Q_n$, $M_1, \ldots, M_n$, are mutually independent, and $R' =_{\mathcal{D}} R$. There is positive probability that $Q_k \ge q$ and $M_k \ge 1-\delta$, for $k = 1, \ldots, n$, and that $R' \ge r'$, upon which we see that $R \ge r$, establishing (5.16).

In the contrary case in Theorem 3.2, when $P(Q > 0) = 0$, we must have $P(Q < 0) > 0$. By considering $(-R, -Q, M)$, satisfying (5.1), in place of (R, Q, M) satisfying (1.5), we see that (5.16) gives $P(R \le -r) > 0$ for all r. Choose $q' > 0$ and $m' > 0$ so that $P(Q \ge -q') > 0$ and $P(M \le -m') > 0$; then

$$\begin{aligned} P(R \ge 0) &= P(Q + MR \ge 0) \\ &\ge P(Q \ge -q')P(M \le -m')P(R \le -q'/m') > 0. \quad \square \end{aligned}$$

Proof of Theorem 4.1: First, if $P(M > 1) = 0$ then let (Q', M') be independent of (Q, M) and have the same probability law as (Q, M), and observe that

$$R =_{\mathcal{D}} Q + MQ' + MM'R,$$

and that $P(MM' > 1) > 0$. Thus without loss of generality we may assume $P(M > 1) > 0$. Then there exists $m > 1$ such that $P(M \ge m) > 0$.

By a *point of density* of (the law of) R we mean an r such that $P(R \in B) > 0$ for every open B containing r. Since R is non-degenerate it has points of density $r_1 < r_2$, with $r_2 - r_1 = 5\varepsilon$, say. We may find constants $a < b$, with $b - a \leq \varepsilon$, such that $\delta := P(M \geq m,\ a \leq -Q/(M-1) \leq b) > 0$. We then have either (a) $r_1 \leq a - 2\varepsilon$, so that $\eta_- := P(R \leq a - \varepsilon) > 0$, or (b) $r_2 \geq b + 2\varepsilon$, so that $\eta_+ := P(R \geq b + \varepsilon) > 0$, or both. In case (a) we can then prove by induction that $P(R \leq a - m^n\varepsilon) \geq \eta_-\delta^n$ for all $n = 0, 1, 2, \ldots$. For if $r \leq a - m^n\varepsilon$ then, with probability at least δ,

$$Q + Mr = Q + Ma + M(r - a) \leq a - m \cdot m^n\varepsilon,$$

so

$$\begin{aligned} P(R \leq a - m^{n+1}\varepsilon) &= P(Q + MR \leq a - m^{n+1}\varepsilon) \\ &\geq P(R \leq a - m^n\varepsilon)P(M \geq m,\ a \leq -Q/(M-1) \leq b) \\ &\geq \delta P(R \leq a - m^n\varepsilon), \end{aligned}$$

and the induction succeeds. In case (b) we may similarly prove $P(R \geq b + m^n\varepsilon) \geq \eta_+\delta^n$ for all $n \in \mathbb{N}_0$, for if $r \geq b + m^n\varepsilon$ then there is probability at least δ that

$$Q + Mr = Q + Mb + M(r - b) \geq b + m \cdot m^n\varepsilon.$$

Either conclusion leads to (4.1). For instance, in the second case pick $r > b + \varepsilon$ and let n be the least positive integer such that $b + m^n\varepsilon \geq r$; then

$$\begin{aligned} \log P(R \geq r) &\geq \log P(R \geq b + m^n\varepsilon) \\ &\geq \log \eta_+ + n \log \delta \\ &> \log \eta_+ + \left(1 + \frac{\log((r-b)/\varepsilon)}{\log m}\right) \log \delta, \end{aligned}$$

since $m^{n-1}\varepsilon < r - b$. So (4.1) follows. Similarly in the first case. □

6. Complements.

6.1. The proof of Lemma 5.1 can be adapted to yield existence and standard properties of the perpetuity (1.1) in the case of non-negative Q such that $Ee^{\varepsilon Q} < \infty$ for some $\varepsilon > 0$, and non-negative M with $P(M \leq 1) = 1$ and $P(M < 1) > 0$. For pick $\rho > 0$ so small that $E(e^{\rho Q}M) < 1$, then the proof shows that $T^n\delta_0 \to_{\mathrm{w}} \mu_\infty \in \mathcal{M}_\rho$ as $n \to \infty$. As T is continuous with respect to weak convergence (use [Billingsley 1968, Theorem 3.2]) we obtain $T\mu_\infty = \mu_\infty$

Recall (1.6), giving $T^n\delta_0$ as the probability law of the n^{th} partial sum in (1.1). As we are here assuming that the terms in (1.1) are non-negative, it follows that the partial sums converge a.s. to the possibly improper r.v. R. Since a.s. convergence implies weak convergence, and the distributions of the partial sums converge weakly to μ_∞, it follows that R has distribution μ_∞ and is thus proper.

We remark that the identification (1.6) shows that under the assumptions in force here the sequence of distributions $(T^n\delta_0)$ is stochastically non-decreasing: for each $x \in \mathbb{R}$, $\big(T^n\delta_0[x, \infty)\big)$ is non-decreasing in n.

For another recent treatment emphasizing monotone maps see [Bhattacharya & Waymire 1990, II.14].

6.2. For distributions with an atom at 1 a probabilistic heuristic for the exponential rate of decrease of the tails of R in (1.2) can be given in terms of the waiting time for the first value less than 1. The distributions considered in §3 generate tail behaviour that is similar to that of a Poisson distribution, and for that phenomenon we offer the following analytic heuristic. Consider the perpetuity (1.2) when $\mathcal{L}(M) = \mathrm{unif}(1-\alpha, 1)$ for some $\alpha \in (0, 1]$. The law of R is a fixed point of T where, if $\mathcal{L}(X) = \mu$, $T\mu := \mathcal{L}(1 + MX)$ with M independent of X. Now

$$T\mu[x,\infty) = \frac{1}{\alpha}\int_{1-\alpha}^{1} \mu\Big[\frac{x-1}{m},\infty\Big)\,dm.$$

If $\mu[x,\infty) \doteq e^{-ax\log x}$ for some constant $a > 0$ then the integrand here has maximum value $e^{-a(x-1)\log(x-1)}$, and is negligible in comparison with this value except where $(x-1)/m$ is comparable in magnitude to x; that is, roughly, when m is in the sub-range $[1 - 1/x, 1]$. Thus

$$\begin{aligned} T\mu[x,\infty) &\doteq \frac{1}{\alpha}\int_{1-1/x}^{1} e^{-a(x-1)\log(x-1)}\,dm \\ &= (\alpha x)^{-1} e^{-a(x-1)\log(x-1)}. \end{aligned}$$

For μ to be a fixed point this needs to be comparable to $\mu[x,\infty) \doteq e^{-ax\log x}$, and that implies $a = 1$.

6.3. As remarked in §1, the tail behaviour of R in (1.2) is known when $M = U^{1/\alpha}$ where $\mathcal{L}(U) = \mathrm{unif}(0, 1)$ and α is a positive parameter. We work with $V := R - 1 = \sum_{n=1}^{\infty}(U_1\cdots U_n)^{1/\alpha}$. As shown in [Vervaat 1979, §4], V can be embedded into a shot-noise process and thus is i.d. Its Lévy measure is that with density $x \mapsto \alpha x^{-1}\mathbf{1}_{0<x<1}$. As the supremum of the support of the Lévy measure is 1 it follows by a result of [Sato 1973] (cf. also [Kruglov 1970–71]) that $\lim_{v\to\infty}(v\log v)^{-1}\log P(V > v) = -1$, in agreement with our Theorem 3.1.

The connection with infinite divisibility can be extended to cover many cases of (1.1) when $M = U^{1/\alpha}$; see [Vervaat 1979, §4]. The result of Sato will gain the relevant conclusion of our Theorem 3.1 or 3.2 in each case.

6.4. Not all perpetuities, and in particular not all perpetuities with Poisson-like tail decay as in §3, have i.d. distributions. For consider the perpetuity that features in the selection algorithm mentioned in §1, namely that given by (1.2) with M distributed as $\mathrm{unif}(\frac{1}{2}, 1)$. If R is i.d. then its m.g.f. $M_R(t) = Ee^{tR}$ is expressible in terms of its Lévy measure ν by $M_R(t) = \exp\big(\int_{[0,\infty)}(e^{tx} - 1)x^{-1}\,d\nu(x)\big)$, so all its cumulants $(\log M_R)^{(k)}(0)$, for $k = 1, 2, \ldots$, are positive. Now the moments of R may be found iteratively as in [Vervaat 1979, §5], and hence the cumulants evaluated. Exact (rational arithmetic) calculations with Maple V reveal that the first 7 cumulants are positive but the 8^{th} is not. Thus R is not i.d.

6.5. There is a simple proof of the special case of Corollary 4.2 when Q and M are non-negative. This case includes (1.2). When Q and M are non-negative, R can be taken to be non-negative, and

$$E(R^n) = E\big((Q + MR)^n\big) \ge E(M^n)E(R^n),$$

so finiteness of moments of R forces $E(M^n) \le 1$ for all n, hence $P(0 \le M \le 1) = 1$.

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