THE MOMENT INDEX OF MINIMA (II)

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Abstract

The moment index $\kappa(X) = \sup\{k\colon \mathrm{E}(X^k) < \infty\}$ of a nonnegative random variable X has the property that $\kappa(\min(X, Y)) \geq \kappa(X) + \kappa(Y)$ for independent r.v.s X and Y. We characterize when equality holds for a given r.v. X and every independent nonnegative r.v. Y, and discuss extensions to related r.v.s and their distributions.

Key words and phrases: exponential index, moment index, regular variation.

1. Introduction

In Daley (2001) to which this note is a sequel, the *moment index* $\kappa(X)$ of a nonnegative random variable (r.v.) X is defined by

$$\kappa(X) = \sup\{k \geq 0 : \mathrm{E}(X^k) < \infty\}. \tag{1}$$

It was shown that for independent nonnegative r.v.s X and Y each with a finite moment index,

$$\kappa(\min(X, Y)) \geq \kappa(X) + \kappa(Y), \tag{2}$$

that equality holds when the tail of the d.f. of either X or Y is regularly varying, and an example in which X and Y have discrete supports that are 'increasingly sparse' and 'well interspersed' demonstrated that the inequality at (2) can be strict.

The main purpose of this paper is to prove the theorem below; it characterizes independent nonnegative r.v.s X and Y for which equality holds in (2). We precede its proof in Section 2 with further discussion. In Section 3 we note companion results for the exponential index of a r.v., and Section 4 looks at questions surrounding the finiteness or otherwise of $\mathrm{E}(X^{\kappa(X)})$.

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Definition 1. (a) $\mathcal{M}^\alpha$ *denotes the class of all nonnegative r.v.s* X *with moment index* $\kappa(X) = \alpha$ *for which, for every nonnegative r.v.* Y *independent of* X,

$$\kappa(\min(X, Y)) = \kappa(X) + \kappa(Y).$$

(b) *For* $\alpha < \infty$, $\mathcal{M}^{\alpha+} \subseteq \mathcal{M}^\alpha$ *consists of those r.v.s* $X \in \mathcal{M}^\alpha$ *for which* $\mathrm{E}(X^\alpha) < \infty$, *and* $\mathcal{M}^{\alpha-} = \mathcal{M}^\alpha \setminus \mathcal{M}^{\alpha+}$.

Note that $\mathcal{M}^{0+} = \mathcal{M}^0$. For intermediate values $0 < \alpha < \infty$ the inclusion (b) is proper.

Theorem 2. $X \in \mathcal{M}^\alpha$ *if and only if the tail* $\overline{F}$ *of its d.f.* F *satisfies*

$$\lim_{x\to\infty} [-\log \overline{F}(x)]/\log x = \alpha = \kappa(X). \tag{3}$$

We remark that the class $\mathcal{M}^\alpha$ of d.f.s for $\alpha < \infty$, which by Daley (2001) includes d.f.s with regularly varying tails of index α, is indeed larger than the latter family. This follows essentially as in Proposition 2.2.8 of Bingham, Goldie and Teugels (1989) (hereafter, [BGT]), where there is an example of a monotone function whose lower and upper orders coincide (see equations (5)–(6) below) but for which the representation theorem [BGT 2.2.7] is not of the form of the corresponding theorem [BGT Theorem 1.3.1] for regularly varying functions.

2. Discussion and proof of Theorem 2

The identification of α in (3) with $\kappa(X)$ is a matter of definition. Also, it is known (though perhaps not well known; see Baltrūnas, Daley and Klüppelberg, 2004) that

$$\liminf_{x\to\infty} \frac{-\log \overline{F}(x)}{\log x} = \kappa(X), \tag{4}$$

so we give it as Lemma BDK below and indicate its proof; note that for a positive function f, its *lower order* $\mu(f)$ is just

$$\mu(f) = \liminf_{x\to\infty} [\log f(x)]/\log x, \tag{5}$$

while the companion *upper order* $\nu(f)$ say is

$$\nu(f) = \limsup_{x\to\infty} [\log f(x)]/\log x \tag{6}$$

[BGT Section 2.2.2]. Consequently, what is new in Theorem 2 is the identification of $\mathcal{M}^\alpha$ with tails of d.f.s $\overline{F}$ for which $\mu(1/\overline{F}) = \nu(1/\overline{F})$.

Now the tail of the d.f. of $\min(X, Y)$ is just the function $\overline{F}\,\overline{G}$, where G is the d.f. of Y, and for any real-valued functions f and g with finite limits infima, $\liminf_{t\to\infty}[f(t) + g(t)] \geq \liminf_{t\to\infty} f(t) + \liminf_{t\to\infty} g(t)$, where equality holds for any given f and all g if and only if $\lim_{t\to\infty} f(t)$ exists. In exploiting this property to demonstrate that inequality at (2) may hold for given $\overline{F}$ for which $\nu(1/\overline{F}) > \mu(1/\overline{F}) = \alpha$, we need to ensure that the function $\overline{G}$ we construct with given lower moment order smaller than its upper order is indeed a distribution function.

Nevertheless, there are pairs of independent r.v.s X and Y for which the limits infima and suprema *are* different but for which equality holds at (2). One such pair is as in Lemma 3 (the proof is at the end of this section), which shows that for strict inequality to hold in (2), the regions where the ratios at (4) are close to their limits infima must not overlap but rather be well interspersed as in the example in Daley (2001). Indeed, Tu Anh Nguyen has given an example for which the limits infima are finite (and hence so too is the right-hand side of (2)) but the left-hand side of (2) is infinite because the limits suprema are infinite, the ratios being 'large' in different regions.

Lemma 3. *Let the nonnegative r.v. X have finite first moment, and let the nonnegative r.v. Y, independent of X, have probability density function proportional to the tail $\overline{F}(\cdot)$ of the d.f. of X. Then $\kappa(Y) = \kappa(X) - 1$ and $\kappa\big(\min(X, Y)\big) = 2\kappa(X) - 1$.*

Lemma BDK. *For a nonnegative r.v. X, the lower order of the reciprocal $1/\overline{F}(x)$ of the tail of its d.f. equals its moment index, i.e.*

$$\mu(1/\overline{F}) = \liminf_{x\to\infty} \frac{-\log \overline{F}(x)}{\log x} = \kappa(X) = \sup\{k \geq 0 :\ \mathrm{E}(X^k) < \infty\}. \tag{7}$$

PROOF. We first show that $\kappa(X) \geq \mu(1/\overline{F})$. If $\mu(1/\overline{F}) = 0$ there is nothing to prove. When $0 < \mu(1/\overline{F}) < \infty$ let $\lambda = \mu(1/\overline{F})$ and observe that for arbitrary $\varepsilon > 0$, $-\log \overline{F}(x) \geq (\lambda - \varepsilon) \log x = \log x^{\lambda-\varepsilon}$ for all $x \geq$ some $x_0 = x_0(\varepsilon)$. Then for such x, $\overline{F}(x) \leq 1/x^{\lambda-\varepsilon}$ and therefore

$$\int_{x_0}^{\infty} x^{\lambda-1-2\varepsilon}\overline{F}(x)\,\mathrm{d}x \leq \int_{x_0}^{\infty} \frac{\mathrm{d}x}{x^{1+\varepsilon}} < \infty. \tag{8}$$

Since

$$\mathrm{E}(X^k) = \int_0^{\infty} kx^{k-1}\overline{F}(x)\,\mathrm{d}x \quad \text{when } k > 0,$$

(8) implies that $\kappa(X) \geq \lambda - 2\varepsilon$, and as $\varepsilon > 0$ is otherwise arbitrary, we conclude $\kappa(X) \geq \mu(1/\overline{F})$. In the case $\mu(1/\overline{F}) = \infty$ the same argument works for all $\lambda > 0$, so we conclude $\kappa(X) = \infty$ as wanted.

For the converse assertion, that $\mu(1/\overline{F}) \geq \kappa(X)$, the argument is similarly structured. If $\kappa(X) = 0$ there is nothing to prove. If $0 < \kappa(X) < \infty$ we write $\lambda = \kappa(X)$ and note that for $0 < \varepsilon < \lambda$,

$$\infty > \int_0^{\infty} x^{\lambda-\varepsilon}\,\mathrm{d}F(x) = \mathrm{E}(X^{\lambda-\varepsilon}) \geq x^{\lambda-\varepsilon}\overline{F}(x),$$

so $(\lambda - \varepsilon)\log x + \log \overline{F}(x) \leq \log \mathrm{E}(X^{\lambda-\varepsilon})$ and

$$\frac{-\log \overline{F}(x)}{\log x} \geq \lambda - \varepsilon - \frac{\log \mathrm{E}(X^{\lambda-\varepsilon})}{\log x} \to \lambda - \varepsilon \qquad (x \to \infty)\,, \tag{9}$$

and hence $\mu(1/\overline{F}) \geq \lambda = \kappa(X)$. If $\kappa(X) = \infty$ this argument works for all $\lambda > 0$, hence $\mu(1/\overline{F}) = \infty$ as wanted. ∎

Proof of Theorem 2. The proof of (1.2) in Daley (2001) is probabilistic. Here we apply Lemma BDK to the r.v. $\min(X, Y)$, whose d.f. has the tail $\overline{F}\,\overline{G}$, in writing

$$\kappa(\min(X,Y)) = \liminf_{x\to\infty} \frac{-\log \overline{F}(x) - \log \overline{G}(x)}{\log x} \tag{10}$$

$$\geq \liminf_{x\to\infty} \frac{-\log \overline{F}(x)}{\log x} + \liminf_{x\to\infty} \frac{-\log \overline{G}(x)}{\log x} \;=\; \kappa(X) + \kappa(Y), \tag{11}$$

with equality holding when (3) holds.

For the converse, suppose X has moment index $\alpha < \infty$ but that $1/\overline{F}$ has upper order exceeding α, i.e.

$$\alpha = \liminf_{x\to\infty} \frac{-\log \overline{F}(x)}{\log x} < \limsup_{x\to\infty} \frac{-\log \overline{F}(x)}{\log x} \leq \infty.$$

Then there is a sequence $x_n \to \infty$ and constant $\varepsilon > 0$ such that

$$-\log \overline{F}(x_n) \geq (\alpha + 2\varepsilon) \log x_n \quad \text{for all } n, \tag{12}$$

and moreover we may choose the initial member x_0 of the sequence so large that

$$-\log \overline{F}(x) \geq (\alpha - \varepsilon) \log x \quad \text{for all } x \geq x_0. \tag{13}$$

Further, we can assume without loss of generality, just by taking a subsequence of x_n if need be, that

$$\log x_{n+1} > \log x_n'' \equiv \frac{\alpha + 2\varepsilon}{\alpha + \varepsilon} \log x_n\,. \tag{14}$$

The idea now is to choose $\beta > 2(\alpha + \varepsilon)$ and construct Y, or rather the tail $\overline{G}$ of its d.f., with $\kappa(Y) = \beta$ and the additional properties that

$$-\log \overline{G}(x) \geq (\beta + 2\varepsilon) \log x \quad \text{whenever} \quad -\log \overline{F}(x) \leq (\alpha + \varepsilon) \log x, \tag{15}$$

and that on some sequence $x_n' \to \infty$,

$$-\log \overline{G}(x_n') = \beta \log x_n' \quad \text{for all } n. \tag{16}$$

We will ensure that $-\log \overline{G}$ has lower order β, and hence $\kappa(Y) = \beta$, by insisting that

$$-\log \overline{G}(x) \geq \beta \log x \quad \text{for all } x \geq x_0. \tag{17}$$

If we can do all this then

$$-\log \overline{F}(x) - \log \overline{G}(x) \geq (\alpha + \beta + \varepsilon) \log x \quad \text{for all } x \geq x_0, \tag{18}$$

as this is so by (15) and (13) when $-\log \overline{F}(x) \leq (\alpha+\varepsilon)\log x$, and by (17) when $-\log \overline{F}(x) > (\alpha+\varepsilon)\log x$. By (18), (10) exceeds $\alpha+\beta = \kappa(X)+\kappa(Y)$, so $X \notin \mathcal{M}^\alpha$ as is to be proved.

There remains the construction of $\overline{G}$, i.e. of a non-decreasing function $-\log \overline{G}$ satisfying (15), (16) and (17). To the right of any x_n, because $-\log \overline{F}(x)$ is non-decreasing, we have

$$-\log \overline{F}(x) \geq -\log \overline{F}(x_n) \geq (\alpha+2\varepsilon)\log x_n > (\alpha+\varepsilon)\log x \quad \text{for all } x \in [x_n, x_n''),$$

where

$$\log x_n'' := \frac{\alpha+2\varepsilon}{\alpha+\varepsilon}\log x_n.$$

Within this interval, $\overline{F}$ does not satisfy the condition that activates (15). Let us define $\overline{G}$ to have the constant value

$$-\log \overline{G}(x) := (\beta+2\varepsilon)\log x_n \quad \text{for } x \in [x_n, x_n'], \tag{19}$$

where

$$\log x_n' := \frac{\beta+2\varepsilon}{\beta}\log x_n.$$

Our having fixed $\beta > 2(\alpha+\varepsilon)$ ensures that

$$\frac{\beta+2\varepsilon}{\beta} = 1 + \frac{2\varepsilon}{\beta} < \frac{\alpha+2\varepsilon}{\alpha+\varepsilon},$$

and so $x_n' < x_n''$ and thus (19) does not conflict with (15). Note that (19) implies (16).

For the rest of the definition of $\overline{G}$, just put

$$-\log \overline{G}(x) := (\beta+2\varepsilon)\log x \quad \text{for all } x \in (x_0,\infty) \setminus \Big(\bigcup_{n=1}^{\infty}(x_n, x_n']\Big). \tag{20}$$

Then $-\log \overline{G}$ is non-decreasing, and (20) and (19) together ensure (17). The construction is complete. ∎

Proof of Lemma 3. That $\kappa(Y) = \kappa(X) - 1$ follows from the definition (1) as noted in Daley (2001). For the rest, let $\alpha = \kappa(X)$, and let $\{x_n\}$ be an increasing sequence for which $x_n \to \infty$ as

$n \to \infty$ and

$$\lim_{n\to\infty} \frac{-\log \overline{F}(x_n)}{\log x_n} = \alpha. \tag{21}$$

For any fixed $c > 1$,

$$\frac{-\log \overline{F}(x_n/c)]}{\log(x_n/c)} \le \frac{-\log \overline{F}(x_n)}{\log x_n - \log c} \to \alpha \qquad (n \to \infty), \tag{22}$$

where we have used monotonicity of $\overline{F}$ and the limit from (21). From the limit infimum property of α, it then follows that the left-hand side of (22) must have α as its limit for $n \to \infty$.

Next consider

$$\frac{-\log\big(\int_{x_n/c}^{\infty} \overline{F}(u)\,\mathrm{d}u/\mathrm{E}(X)\big)}{\log(x_n/c)}\,, \tag{23}$$

whose limit infimum is bounded below by $\alpha - 1$ because this equals the moment index of Y, while the numerator with the factor $\mathrm{E}(X)$ omitted (without affecting the limit property) is bounded above by

$$-\log\left[\int_{x_n/c}^{x_n} \overline{F}(u)\,\mathrm{d}u\right] \;\le\; -\log\left[x_n(1-c^{-1})\overline{F}(x_n)\right].$$

It is readily checked that this quantity, when divided by $\log x_n$, has limit as $n \to \infty$ equal to $-1+\alpha$. Then (23) has a limit as $n \to \infty$, and it equals $\alpha - 1$, which result can be combined with the limit of (22) to give

$$\frac{-\log \overline{F}(x_n/c) - \log \int_{x_n/c}^{\infty} \overline{F}(u)\,\mathrm{d}u}{\log(x_n/c)} \;\to\; \alpha + (\alpha - 1) \;=\; 2\alpha - 1. \qquad \blacksquare$$

3. The exponential index of a nonnegative r.v.

We hope that the following discussion will facilitate greater use of moment generating functions as a handy technical device.

Definition 4. *The* exponential index $\varepsilon(X)$ *of a real-valued r.v.* X *for which* $\overline{F}(x) = \Pr\{X \ge x\}$, *is defined by*

$$\varepsilon(X) = \sup\big\{t\colon \mathrm{E}(\mathrm{e}^{tX}) < \infty\big\}. \tag{24}$$

We could now develop analogues of Lemma BDK and Theorem 2 for $\varepsilon(\cdot)$. However there is no point in reproducing the earlier construction and arguments because *the moment index of a positive r.v.* X *is related to the exponential index of the real r.v.* $Y = \log X$ *by* $\varepsilon(Y) = \kappa(X)$. This follows immediately, given a r.v. Y, from writing $Z = \mathrm{e}^Y$ in

$$\varepsilon(Y) = \sup\big\{t\colon \mathrm{E}(\mathrm{e}^{tY}) < \infty\big\} = \sup\big\{t\colon \mathrm{E}(Z^t) < \infty\big\} = \kappa(Z) \;=\; \kappa(\mathrm{e}^Y). \tag{25}$$

The analogues to which we have alluded can be stated as follows without need of further proof.

Lemma 5.

$$\varepsilon(X) = \liminf_{x\to\infty} \frac{-\log \overline{F}(x)}{x}\,. \tag{26}$$

Theorem 6. *For independent real-valued r.v.s* X *and* Y*,*

(i) $\varepsilon(X+Y) = \min(\varepsilon(X),\, \varepsilon(Y))$;

(ii) $\varepsilon(\max(X,Y)) = \min(\varepsilon(X),\, \varepsilon(Y))$; *and*

(iii) $\varepsilon(\min(X,Y)) \geq \varepsilon(X) + \varepsilon(Y)$, *where equality holds for all r.v.s* Y *if and only if*

$$\limsup_{x\to\infty} \frac{-\log \overline{F}(x)}{x} = \varepsilon(X).$$

In this theorem, parts (i) and (ii) are included for the sake of completeness (cf. Daley, 2001), while (iii) follows from Theorem 2.

4. Discussion

For independent r.v.s $X \in \mathcal{M}^{\alpha+}$ and $Y \in \mathcal{M}^{\beta+}$ it follows from $\mathrm{E}\big([\min(X,Y)]^{\alpha+\beta}\big) \leq \mathrm{E}(X^\alpha Y^\beta)$ $= \mathrm{E}(X^\alpha)\mathrm{E}(Y^\beta) < \infty$ that $\min(X,Y) \in \mathcal{M}^{(\alpha+\beta)+}$. Indeed, we have the following.

Lemma 7. *For independent r.v.s* $X_j \in \mathcal{M}^{\alpha_j}$ *and for nonnegative* α_j $(j = 1,\ldots,k)$, $Z \equiv \min(X_1,\ldots,X_k) \in \mathcal{M}^{\alpha}$ *where* $\alpha = \alpha_1 + \cdots + \alpha_k$.

If, for each j, α_j *is finite and* $X \in \mathcal{M}^{\alpha_j+}$, *then* $Z \in \mathcal{M}^{\alpha+}$.

On the other hand, there exist independent r.v.s X and $Y \in \mathcal{M}^{\alpha-}$ but $\min(X,Y) \in \mathcal{M}^{\alpha+}$. For example it is enough that they have d.f.s given by $\overline{F}(x) = \overline{G}(x) = 1/[x(1+\log x)^{3/5}]$ for $x \geq 1$.

Finiteness of the moment of the order of the moment index is also preserved for given X and Y under addition (without requiring independence: just use the c_r-inequality), and, for independent X and Y, when $\alpha = \kappa(X) < \kappa(Y)$ so that $\kappa(\max(X,Y)) = \min(\kappa(X), \kappa(Y)) = \alpha$, we have $X \in \mathcal{M}^{\alpha+}$ if and only if $\max(X,Y) \in \mathcal{M}^{\alpha+}$.

In work underlying Scheller-Wolf (2003), interest centres on independent nonnegative r.v.s X and Y with finite positive moment indexes α and β for which both

$$\mathrm{E}(X^\alpha) = \infty = \mathrm{E}(Y^\beta) \tag{27}$$

and

$$\mathrm{E}\big([\min(X,Y)]^{\alpha+\beta}\big) = \infty \tag{28}$$

hold. Clearly, $\mathrm{E}\big([\min(X,Y)]^{\alpha+\beta-\epsilon}\big) < \infty$ for arbitrary $0 < \epsilon < \alpha+\beta$, and elementary algebra yields the rest of Lemma 8.

Lemma 8. *Let independent nonnegative r.v.s X and Y satisfy $\kappa(X) = \alpha \in (0,\infty)$, $\kappa(Y) = \beta \in (0,\infty)$, and Condition* (27). *If also Condition* (28) *holds for given X and all Y as described, then $\kappa(\min(X,Y)) = \kappa(X) + \kappa(Y)$, and the d.f. F of X satisfies* (3).

A sufficient condition on X for (27) *to imply* (28) *is that*

$$\liminf_{x\to\infty} x^\alpha \overline{F}(x) > 0. \tag{29}$$

It remains to consider conditions under which (27) does *not* imply (28). Let $X \in \mathcal{M}^{\alpha-}$, and suppose that for some $\epsilon > 0$ and positive integer $k \geq 2$, its d.f. F satisfies

$$\limsup_{x\to\infty} \big(\operatorname{Log}_{k-1}(x)\big)^\epsilon x^\alpha \overline{F}(x) < \infty, \tag{30}$$

where the positive monotonic nondecreasing function $\mathrm{Log}_k(x)$ is defined for $x > 0$ by

$$\mathrm{Log}_k(x) = \begin{cases} \max(1, x) & (k = 0), \\ \max\big(1, \log \mathrm{Log}_{k-1}(x)\big) & (k = 1, 2, \ldots) \end{cases}$$

(this function Log_k is similar to but not the same as the functional iterate of log denoted $\log_k$ and defined in [BGT 1.3.3]). We assert that there exists a r.v. Y satisfying (27) such that $\mathrm{E}\big([\min(X, Y)]^{\alpha+\beta}\big) < \infty$.

To check this assertion, observe that the function $\mathrm{Log}_k(x)$ has derivative

$$\big(\mathrm{Log}_k(x)\big)' = \begin{cases} 0 & \text{if } \mathrm{Log}_k(x) = 1, \\ \dfrac{\big(\mathrm{Log}_{k-1}(x)\big)'}{\mathrm{Log}_{k-1}(x)} = \dfrac{1}{\mathrm{Log}_{k-1}(x) \cdots \mathrm{Log}_0(x)} & \text{otherwise.} \end{cases}$$

In the latter case, this function is monotone decreasing to 0 as $x \to \infty$. Its product with $\big(\max(1, x)\big)^{-(\beta-1)}$, where $\beta \geq 1$, can therefore be taken as the upper tail $\overline{G}$ say, of the d.f. of a nonnegative r.v. Y_L say for which $\mathrm{E}(Y_L^{\beta}) = \infty$, and $\mathrm{E}(Y_L^{\beta-\epsilon}) < \infty$ for every positive $\epsilon \leq \beta$. Thus, $Y_L \in \mathcal{M}^{\beta-}$.

From (30) it follows that for some finite positive A,

$$\overline{F}(x) \leq A x^{-\alpha} \big(\mathrm{Log}_{k-1}(x)\big)^{-\epsilon}.$$

Then $\mathrm{E}\big([\min(X, Y_L)]^{\alpha+\beta}\big) < \infty$ because the function $\overline{F}(x)\overline{G}(x)$, which equals the tail of the d.f. of the r.v. $\min(X, Y_L)$, has $x^{\alpha+\beta-1}\overline{F}(x)\overline{G}(x)$ bounded above for sufficiently large x, $x \geq x_0$ say, by

$$\frac{A}{\big(\mathrm{Log}_{k-1}(x)\big)^{1+\epsilon} \mathrm{Log}_{k-2}(x) \cdots \mathrm{Log}_0(x)},$$

and this function, being the derivative of $-A\big(\mathrm{Log}_{k-1}(x)\big)^{-\epsilon}/\epsilon$, is integrable on (x_0, ∞).

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