ORDERED INDEPENDENT SCATTERING

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ABSTRACT

Suppose $(\mathbf{X}_n)_{n=1,2,\dots}$ is a sequence of independent, identically distributed random vectors in $\mathbb{R}^2$. Say $\mathbf{X}_n$ is a record if there is a record simultaneously in both coordinates at index n. For $A \subset \mathbb{R}^2$, let N_A be the number of records falling in region A. If $A < B$ are two regions of $\mathbb{R}^2$ such that every point of A is south-west of every point of B, then N_A and N_B are independent. We call this property ordered independent scattering and discuss which other notions of record give rise to it.

Keywords: Multivariate record values, independent scattering, order statistics, point process, Poisson process, set-indexed process

1. Introduction.

Let $<$ be a strict partial order on $\mathbb{R}^2$, that is, $\{(\mathbf{x},\mathbf{y}) : \mathbf{x} < \mathbf{y}\}$ is a subset of $\mathbb{R}^2 \times \mathbb{R}^2$ such that the properties of

$$\text{antisymmetry: } \mathbf{x} < \mathbf{x} \text{ for no } \mathbf{x} \in \mathbb{R}^2,$$
$$\text{transitivity: if } \mathbf{x} < \mathbf{y} \text{ and } \mathbf{y} < \mathbf{z}, \text{ then } \mathbf{x} < \mathbf{z}$$

hold in $\mathbb{R}^2$. It is standard (Birkhoff (1967), p. 1) that $<$ determines a weak partial order $\le$ by

$$\mathbf{x} \le \mathbf{y} \quad \text{iff} \quad \mathbf{x} < \mathbf{y} \text{ or } \mathbf{x} = \mathbf{y}.$$

Thus, it is indifferent whether we start with $<$ or $\le$.

Denote the Borel σ-algebra on $\mathbb{R}^2$ by $\mathcal{B}$. Corresponding to the partial ordering $<$ on $\mathbb{R}^2$ we may define an ordering of subsets A, B of $\mathbb{R}^2$ as follows: $A < B$ iff $\mathbf{a} \in A$ and $\mathbf{b} \in B$ implies $\mathbf{a} < \mathbf{b}$.

A set-indexed process $(M(A))_{A\in\mathcal{B}}$ has the *ordered independent scattering property* if whenever $(A_i)_{i\in I}$ is a disjoint, totally ordered collection of Borel sets in $\mathbb{R}^2$, the random variables $(M(A_i))_{i\in I}$ are mutually independent. The purpose of this paper is to discuss some contexts from multivariate extreme value theory which give rise to this notion.

Let $\mathbf{X}_n := (X_n^{(1)}, X_n^{(2)})$, for $n = 1, 2, \ldots$, be independent identically distributed (iid) $\mathbb{R}^2$-valued random vectors with common distribution F. Relative to the partial order "$<$", we say that

$$\mathbf{X}_n \text{ is a } \textit{record} \text{ if } \mathbf{X}_k < \mathbf{X}_n \text{ for } k = 1, \ldots, n-1. \tag{1.1}$$

Thus necessarily $\mathbf{X}_1$ is a record.

In Goldie & Resnick (1989) we considered the following wider list of plausible notions of 'record' in $\mathbb{R}^2$:

(i) $\mathbf{X}_n$ is a 'record' iff $X_n^{(1)} > \bigvee_{k=1}^{n-1} X_k^{(1)}$ and $X_n^{(2)} > \bigvee_{k=1}^{n-1} X_k^{(2)}$;

(ii) $\mathbf{X}_n$ is a 'record' iff $X_n^{(1)} \geq \bigvee_{k=1}^{n-1} X_k^{(1)}$ and $X_n^{(2)} \geq \bigvee_{k=1}^{n-1} X_k^{(2)}$ and at least one of these inequalities is strict;

(iii) $\mathbf{X}_n$ is a 'record' iff $X_n^{(1)} > \bigvee_{k=1}^{n-1} X_k^{(1)}$ or $X_n^{(2)} > \bigvee_{k=1}^{n-1} X_k^{(2)}$;

(iv) $\mathbf{X}_n$ is a 'record' iff $\mathbf{X}_n$ falls outside the convex hull of $\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$;

(v) $\mathbf{X}_n$ is a 'record' iff $|\mathbf{X}_n| > \bigvee_{k=1}^{n-1} |\mathbf{X}_k|$,

where $|\cdot|$ in (v) denotes Euclidean (or supremum, or some other) norm. We also considered a further notion, suggested by George O'Brien in a personal communication:

(vi) X_n is a 'record' iff $\mathbf{X}_k \not\geq \mathbf{X}_n$ for $k = 1, \ldots, n-1$.

We noted that (iii) and (iv) are not consistent with (1.1) for any partial order $<$, and nor is (vi). The others are consistent with specific definitions of $<$, and so the many results of Goldie & Resnick (1989) for a general partial order apply to them. Some of these results we were able to extend to (vi) in an *ad hoc* way, but it was rather unsatisfactory that this could be done and yet (vi) did not fit into a common framework.

For any specific definition of record, let $M(B)$ be the number of records that fall in $B \in \mathcal{B}$. The counting function $\{N_B, B \in \mathcal{B}\}$ mentioned in the abstract is a particular case. In Theorem 3.2, Goldie & Resnick (1989), we proved that every definition of record reconcilable with a partial order satisfying a measurability condition has the property of ordered independent scattering. Thus, in particular, notions (i), (ii) and (v) of record have the ordered independent scattering property.

We would like to bring (vi) into the picture. In the next section we give a framework for set-indexed processes arising naturally in extreme-value theory which guarantees that the property holds. The framework includes (i) and (vi) among other particular cases.

We have sought a general characterization of the ordered independent scattering property but so far such a characterization has eluded us.

The notion of multivariate record most frequently employed is (i), corresponding to the partial order

$$\mathbf{x} < \mathbf{y} \quad \text{iff} \quad x^{(1)} < y^{(1)} \text{ and } x^{(2)} < y^{(2)}.$$

See Deuschel & Zeitouni (1995), Gnedin (1994a, b, c) and Goldie & Resnick (1995) for recent developments with this definition. Also, the idea that will be developed in the next section, of defining records in terms of cones, is similar to the idea of conical extremes and kth layers discussed recently in Gnedin (1994c).

2. Ordered independent scattering.

Recall that a set-indexed process $(M(A))_{A\in\mathcal{B}}$ has the *ordered independent scattering property* if whenever $(A_i)_{i\in I}$ is a disjoint, totally ordered collection of Borel sets in $\mathbb{R}^2$, the random variables $(M(A_i))_{i\in I}$ are mutually independent. Notice that if we define the *restriction* of a set-indexed process M to a set $B \in \mathcal{B}$ to be the process M^B given by

$$M^B(A) := M(A \cap B) \qquad (A \in \mathcal{B}),$$

then it is immediate that ordered independent scattering implies that the restrictions $M^{A_1}, \ldots, M^{A_k}$ are independent processes whenever $A_1 < \cdots < A_k$. The ordered independent scattering property can thus alternatively be presented at process level, in terms of independent restrictions.

Suppose

$$N = \sum_k \varepsilon_{(t_k, \mathbf{j}_k)}$$

is the counting function of a point process on $(0,\infty) \times \mathbb{R}^2$ with points $(t_k, \mathbf{j}_k)$. We make the following assumptions.

(A) The realizations of N have only finitely many points in $[s,t] \times (\mathbb{R}^2 \setminus (-\infty, \mathbf{x}))$, for every compact interval $[s,t] \subset (0,\infty)$ and every $\mathbf{x} \in \mathbb{R}^2$.

(B) If $I_1, \ldots, I_k$ is any finite collection of disjoint closed intervals in $(0,\infty)$ then the restrictions of N to $I_1 \times \mathbb{R}^2, \ldots, I_k \times \mathbb{R}^2$ are mutually independent.

We have two cases in mind. First, in discrete time, N is of the form

$$N = \sum_{k=1}^{\infty} \varepsilon_{(k, \mathbf{X}_k)}$$

where $(\mathbf{X}_k)$ is a sequence of iid random vectors with common distribution F. Second, in continuous time, N is a Poisson process with mean measure $dt \times d\nu$ where ν is a finite measure. Both of these examples satisfy (A) and (B). Indeed, in the latter example the requirement that ν be a finite measure on $\mathbb{R}^2$ can be weakened to it being locally finite ('Radon') on the compactified plane $[-\infty,\infty]^2$ punctured by the removal of the point $-\boldsymbol{\infty}$.

Let C be a closed cone in $\mathbb{R}^2$ satisfying the requirements

(C0) $C \neq \mathbb{R}^2$, $C \neq \emptyset$;

(C1) $\mathbf{x} \leq \mathbf{y}$ implies $C + \mathbf{x} \supseteq C + \mathbf{y}$.

Of these, (C0) excludes trivialities while (C1) is needed to make records defined in terms of C work reasonably well. Thus, define the counting function M of *C-records* as

$$M(A) := \int_{t=0}^{\infty} \int_{\mathbf{x} \in A} \mathbf{1}\{N((0,t] \times (C + \mathbf{x})) = 1\} N(dt, d\mathbf{x}).$$

Thus a C-record in A occurs at a point of the N-process $(t, \mathbf{x})$ if $N((0,t] \times (C + \mathbf{x})) = 1$ and $\mathbf{x} \in A$. We have the following examples in mind:

(a) if $C = (-\boldsymbol{\infty}, \mathbf{0})^c$, then we get notion (i) of record considered in the previous section: $\mathbf{X}_n$ is a record iff $\mathbf{X}_k < \mathbf{X}_n$ for $k = 1, \ldots, n-1$;

(b) if $C = [\mathbf{0}, \boldsymbol{\infty})$, then we get notion (vi): $\mathbf{X}_n$ is a C-record iff $\mathbf{X}_k \not\geq \mathbf{X}_n$ for $k = 1, \ldots, n-1$.

(C0) and (C1) in fact restrict C to a very special class of cones. Note first that in terms of polar coordinates, writing the points of $\mathbb{R}^2$ as $(r\cos\theta, r\sin\theta)$ where $0 \leq r < \infty$ and $-\pi < \theta \leq \pi$, any cone C can be written as $\{(r\cos\theta, r\sin\theta) : 0 \leq r < \infty,\ \theta \in \Theta_C\}$) for some $\Theta_C \subseteq (-\pi, \pi]$. We then have the following characterization.

Propostion 2.1. *A closed cone C satisfies (C0) and (C1) if and only if $\Theta_C = [\theta^-, \theta^+]$ where $-\frac{\pi}{2} \leq \theta^- \leq 0$ and $\frac{\pi}{2} \leq \theta^+ \leq \pi$.*

Proof. C is a non-empty cone so contains $\mathbf{0}$. In (C1) take $\mathbf{x} := \mathbf{0}$ and $\mathbf{y}$ any point of $[0,\infty)^2$; thus $\mathbf{y} \in C + \mathbf{y} \subseteq C$, so C contains $[0,\infty)^2$.

Now we shall show that there cannot exist $-\pi < \theta_2 < \theta_1 < 0$ such that $\theta_1 \notin \Theta_C$ but $\theta_2 \in \Theta_C$. For suppose there exist such θ_1 and θ_2. Since $\mathbf{0} \leq (1,0)$ we know $C + (1,0) \subseteq C$. Thus C contains all points $(1 + r\cos\theta_2, r\sin\theta_2)$ for $0 \le r < \infty$. However one of these points necessarily is on the ray $\{(s\cos\theta_1, s\sin\theta_1) : 0 \le s < \infty\}$, which is disjoint from C. This contradiction means that θ_1 and θ_2 cannot exist.

We may similarly prove that there cannot exist $\frac{\pi}{2} < \theta_1 < \theta_2 \le \pi$ with $\theta_1 \notin \Theta_C$ but $\theta_2 \in \Theta_C$. Thus Θ_C is some sub-interval of $(-\pi, \pi]$. Further, Θ_C is relatively closed in $(-\pi, \pi]$ because C is closed. Since C contains $[0,\infty)^2$, Θ_C contains $[0, \frac{\pi}{2}]$.

It remains only to prove that Θ_C has no points in the third quadrant $(-\pi, -\frac{\pi}{2})$. For suppose there is such a point θ_0. Since $\mathbf{0} \le (0,1)$, we know by (C1) that $C \supseteq C + (0,1)$. So all points $(r\cos\theta_0, r\sin\theta_0) + (0,1)$, for $r \ge 0$, are in C. However the arg of these points covers all of $[\frac{\pi}{2}, \pi]$, as r ranges over $[0, -\operatorname{cosec}\theta_0]$, and covers all of $(-\pi, \theta_0)$, as r ranges over $(-\operatorname{cosec}\theta_0, \infty)$. But that makes Θ_C the whole of $\mathbb{R}^2$, contradicting (C0). So Θ_C is as claimed. $\square$

The cones C in examples (a) and (b) above are now seen to be respectively the maximal and minimal possible under the restrictions (C0) and (C1). Another special case arises when Θ_C is the interval $[\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}]$ for some $\theta \in [0, \frac{\pi}{2}]$; that makes C a half-plane and C-records are just directional records in the direction θ. In other words, the points which are C-records are those points whose orthogonal projections on a ray of direction θ yield records in the one dimensional sense.

The next result gives the ordered independent scattering property for C-records.

Theorem 2.2. *If N is satisfies assumptions (A) and (B) and if (C0) and (C1) hold, then M is ordered independently scattered.*

Proof. Let $(\mathcal{F}_t)_{t\ge 0}$ be the natural filtration of N, where $\mathcal{F}_t$ is generated by the restriction of N to $(0,t] \times \mathbb{R}^2$. We use assumption (B) at this point: it gives that for every $t \in (0, \infty)$, the strictly-prior-to-t σ-algebra $\mathcal{F}_{<t}$ and the post-t σ-algebra $\mathcal{F}_{\ge t}$ are independent.

We take the case of two subsets of $\mathbb{R}^2$, as the general case of an unspecified finite number may be dealt with wholly analogously, and the result follows from that. Thus suppose $A < B$ are Borel subsets of $\mathbb{R}^2$. Let

$$T := \inf\{t \ge 0 : N((0,t] \times (C + B) > 0\} \le \infty.$$

So T is a stopping time with respect to the natural filtration. Assumption (A) comes in at this point: it implies that $T > 0$ with probability 1. The above independence, with the strong Markov property, then gives that the strictly-prior-to-T σ-algebra $\mathcal{F}_{<T}$ and the post-T σ-algebra $\mathcal{F}_{\ge T}$ are independent. It then suffices to prove that $M(A)$ is $\mathcal{F}_{<T}$-measurable and $M(B)$ is $\mathcal{F}_{\ge T}$-measurable. For that we need the following lemma.

Lemma 2.3. *In the time interval $[T, \infty)$, no C-records fall in A. In the time interval $(0,T)$, no C-records fall in B.*

Proof. To verify the first assertion, assume $P(T < \infty) > 0$ for non-triviality. Working within the event $T < \infty$, assume $(t_k, \mathbf{j}_k)$ is a support point for N such that $t_k \ge T$ and $\mathbf{j}_k \in A$. By the preceding proposition and the assumption $A < B$, $\mathbf{j}_k$ is not in $C + B$, and so $\mathbf{j}_k$ is distinct from the point $\mathbf{x}_T \in C + B$, where $(T, \mathbf{x}_T)$ is the support point of N whose existence the definition of T ensures. But then since by (C1), $C + \mathbf{j}_k \supseteq C + B$, $(0, t_k] \times (C + \mathbf{j}_k)$ contains at least the two distinct support points $(T, \mathbf{x}_T)$ and $(t_k, \mathbf{j}_k)$ of N. So $\mathbf{j}_k$ is not a record in A since $N((0,t_k] \times (C + \mathbf{j}_k)) \ge 2$.

To verify the second assertion, note that $N((0,T) \times (C + B)) = 0$ by the definition of T. For $\mathbf{x} \in B$, $N((0,T) \times (C + \mathbf{x})) = 0$ so the second assertion holds. $\square$

Proof of Theorem 2.2, continued. Recall that $\mathcal{F}_{<T}$ has to be somewhat awkwardly defined as the σ-algebra generated by the events $A \cap \{t < T\}$ for all $t > 0$ and $A \in \mathcal{F}_t$ (Dellacherie & Meyer (1978), Definition IV–54). Let $T_k := \max\{i2^{-k} : i \in \mathbb{Z}_+, i2^{-k} < T\}$, where $\mathbb{Z}_+$ is the set of non-negative integers. By the above lemma,

$$M(A) = \int_{0<t<T} \int_{\mathbf{x}\in A} \mathbf{1}\{N((0,t] \times (C + \mathbf{x})) = 1\} N(dt, d\mathbf{x}),$$

and we see that $M(A)$ is (for each ω) the limit of

$$M_k(A) := \int_{0<t<T_k} \int_{\mathbf{x}\in A} \mathbf{1}\{N((0,t]\times(C+\mathbf{x}))=1\} N(dt, d\mathbf{x})$$

as $k\to\infty$. However,

$$M_k(A) \\ = \sum_{i=1}^{\infty} \mathbf{1}\left\{\frac{i}{2^k}<T\right\} \int_{(i-1)2^{-k}<t\le i2^{-k}} \int_{\mathbf{x}\in A} \mathbf{1}\{N((0,t]\times(C+\mathbf{x}))=1\} N(dt, d\mathbf{x}),$$

and $M_k(A)$ is thus $\mathcal{F}_{<T}$-measurable according to the latter's definition.

For the $\mathcal{F}_{\ge T}$-measurability of $M(B)$ we have

$$M(B) = \int_0^{\infty} \int_{x\in B} \mathbf{1}\{N((0,t]\times(C+\mathbf{x}))=1\} N(dt, d\mathbf{x})$$

which from the second assertion of Lemma 2.3 is

$$\int_{[T,\infty)} \int_{x\in B} \mathbf{1}\{N((0,t]\times(C+\mathbf{x}))=1\} N(dt, d\mathbf{x}).$$

Now for $\mathbf{x}\in B$,

$$0 = N((0,T]\times(C+B)) \ge N((0,T]\times(C+\mathbf{x})),$$

so

$$M(B) = \int_{[T,\infty)} \int_{x\in B} \mathbf{1}\{N([T,t]\times(C+\mathbf{x}))=1\} N(dt, d\mathbf{x}),$$

which is $\mathcal{F}_{\ge T}$-measurable. Thus $M(A)$ and $M(B)$ are independent. □

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