Stability of Perpetuities

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For a series of randomly discounted terms we give an integral criterion to distinguish between almost-sure absolute convergence and divergence in probability to ∞, these being the only possible forms of asymptotic behaviour. This solves the existence problem for a one-dimensional perpetuity that remains from a 1979 study by Vervaat, and yields a complete characterization of the existence of distributional fixed points of a random affine map in dimension one.

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1. Introduction

Let Ψ be a random affine map from $\mathbb{R}$ to itself, given by

$$\Psi(t,\omega) := Q(\omega) + M(\omega)t \qquad (\omega \in \Omega, t \in \mathbb{R}),$$

where Q and M are r.v.s (random variables) on a probability space $(\Omega, \mathcal{A}, P)$. Of course we normally omit the argument ω and write just $\Psi(t) := Q + Mt$. A (distributional) *fixed point* of Ψ is a probability law on $\mathbb{R}$, of a r.v. R, say, such that

$$R \overset{\mathrm{L}}{=} Q + MR, \quad R \text{ independent of } (Q, M), \tag{1.1}$$

where $\overset{\mathrm{L}}{=}$ denotes equality of probability laws. In §3 of this paper we solve the problem of finding necessary and sufficient conditions for existence of a fixed point.

It is natural to attempt to approach a fixed point by iteration, and for that purpose we suppose the existence on the probability space of a sequence of random affine maps $(\Psi_n)_{n\in\mathbb{N}}$, mutually independent and identically distributed with Ψ. Thus for each n, $\Psi_n(t) := Q_n + M_n t$ for all $t \in \mathbb{R}$, and

$$\binom{Q}{M},\ \binom{Q_1}{M_1},\ \binom{Q_2}{M_2},\ \ldots$$

are to be i.i.d. (independent, identically distributed) random elements of $\mathbb{R}^2$.

Outer iteration starts from a r.v. R_0 independent of the sequence (Ψ_n), and often taken to be a non-random constant r, or just 0, and forms successively $R_1, R_2, \ldots$ by

$$R_{n+1} := \Psi_{n+1}(R_n) = Q_{n+1} + M_{n+1}R_n \qquad (n = 0, 1, 2, \ldots). \tag{1.2}$$

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We write R_n as $R_n(R_0)$ to bring out the dependence on R_0. Then

$$R_n(R_0) = \sum_{i=1}^{n} Q_i \prod_{j=i+1}^{n} M_j + R_0 \prod_{j=1}^{n} M_j .$$

We have that $(R_n(R_0))_{n=0,1,\ldots}$ is a Markov sequence, and we may ask about its recurrence, transience, etc. Our results lead to a complete characterization of its positive recurrence.

Inner iteration, by contrast, starts from an affine map $Z_0(t) := t$ for all t, and forms successively the random affine maps $Z_1(\cdot)$, $Z_2(\cdot)$, ... by

$$Z_{n+1}(t) := Z_n \circ \Psi_{n+1}(t) = Z_n(Q_{n+1} + M_{n+1}t) \qquad (n = 0, 1, \ldots).$$

We then have $Z_n(t) = \sum_{k=1}^{n} \Pi_{k-1} Q_k + \Pi_n t$, where throughout the paper we write

$$\Pi_n := \begin{cases} \prod_{k=1}^{n} M_k & (n = 1, 2, \ldots), \\ 1 & (n = 0). \end{cases} \tag{1.3}$$

Often we are interested just in

$$Z_n := Z_n(0) = \sum_{k=1}^{n} \Pi_{k-1} Q_k \qquad (n = 0, 1, \ldots).$$

However it is useful also to allow for a randomized initial value, as with outer iteration discussed above, so we replace t in $Z_n(t)$ by a r.v. Z_0 independent of the sequence (Ψ_n), and then have

$$Z_n(Z_0) = \sum_{k=1}^{n} \Pi_{k-1} Q_k + \Pi_n Z_0 \qquad (n = 0, 1, \ldots). \tag{1.4}$$

The sequence (Z_n) is not Markov but we have the possibility of its converging in some weak or strong sense, the candidate limit being a r.v.

$$Z_\infty := \sum_{k=1}^{\infty} \Pi_{k-1} Q_k . \tag{1.5}$$

Now Z_∞, if it exists, is the probabilistic formulation of the actuarial notion of a *perpetuity*, which represents the present value of a permanent commitment to make a payment at regular intervals, say annually, into the future forever. The Q_n represent annual payments, the M_n cumulative discount factors. Both are subject to random fluctuation. Our model allows the amount Q_n in a given year and the discount factor M_n for that year to be dependent, but requires independence between different years.

In our main result, Theorem 2.1, under a side condition that excludes degeneracies, we give a necessary and sufficient condition for $Z_n(Z_0) \xrightarrow{a.s.} Z_\infty$, and prove that the series for Z_∞ is a.s. absolutely convergent when the condition holds, whereas $|Z_n(Z_0)| \xrightarrow{P} \infty$ when it does not.

Our results on the r.v.s $Z_n(Z_0)$ and Z_∞ are in §2, and consequences for fixed points and positive recurrence are in §3. We specialize our results to certain important particular cases in §4, and present our proofs in §5.

The current state of knowledge about convergence of iterations and existence and uniqueness of fixed points is largely as left by [Vervaat 1979], at least in the one-dimensional case that we treat. On convergence, Vervaat dealt mainly with the case when $E \log|M|$ exists. His results on fixed points are subsumed in our Theorem 3.1.

Two important later results are in [Grincevičius 1981] and [Grincevičius 1980]. In the 1981 paper a proof is sketched that, under the non-degeneracy assumption $P(Q+Mc = c) < 1$ for all c, either $|Z_n| \xrightarrow{P} \infty$ or the series (1.5) converges a.s. The proof is both technically brilliant and not wholly clear in a number of respects. In the course of giving an integral criterion for the a.s. absolute convergence of the series we re-prove Grincevičius's result.

In [Grincevičius 1980] the random elements (Q_n, M_n) are assumed independent but not necessarily identically distributed, and a necessary and sufficient condition is found for the law of Z_∞ to have an atom. A corollary of this result is that when the (Q_n, M_n) are i.i.d., as assumed throughout the present paper, Z_∞ is in general a continuous r.v.

Both before and after the above papers, particular cases have been discussed in various parts of the literature, usually as a prior issue before a proof of some property of fixed points or iteration schemes. The picture up to 1994 is surveyed in [Embrechts & Goldie 1994]. Questions about recurrence and transience are treated in [Kellerer 1992]. Extensions of Vervaat's results to the case of ergodic (Ψ_n) sequences are discussed in [Brandt 1986] and, multi-dimensionally, in [Bougerol & Picard 1992]. We mention the result in the latter reference that existence of a fixed point (see (1.1)) implies $\Pi_n \xrightarrow{a.s.} 0$. Stability results for random recursions that are not affine but only approximately so are treated in [Letac 1986], [Goldie 1991] and [Glasserman & Yao 1995]. Numerous applications of random affine maps in many diverse fields are given in the above references and in further references therein. We mention here a new application in the analysis of algorithms; see [Grübel & Rösler 1996], [Goldie & Grübel 1996] and [Grübel 1998].

Notation to be used throughout the paper will be, in addition to the Π_n defined in (1.3),

$$X := -\log|M|, \quad X_n := -\log|M_n|, \quad Y := \log|Q,| \quad Y_n := \log|Q_n| \quad (n = 1, 2, \dots). \tag{1.6}$$

We allow at this stage the possibility that $M = 0$, corresponding to $X = +\infty$, with positive probability, though this case turns out to be trivial (see Remark 2.4 below), and also the possibility that $Q = 0$ with positive probability. The X_n form the steps of the random walk (S_n), where

$$S_n := \sum_{k=1}^{n} X_k = -\log|\Pi_n| \qquad (n = 0, 1, 2, \dots). \tag{1.7}$$

We need the following truncated mean of X:

$$A_M(y) := E(X^+ \wedge y) = \int_0^y P(X > x)\,dx \qquad (y > 0). \tag{1.8}$$

Any sum, maximum or minimum {respectively product} over an empty range is, by convention, 0 {1}. Thus S_0 is identically zero. We denote the indicator r.v. of an event A by $\mathbf{1}_A$ or $\mathbf{1}\{A\}$. We write $x^+ := x \vee 0$ and $x^- := -(x \wedge 0)$. Convergence in distribution is denoted by $\xrightarrow{\mathrm{L}}$.

2. Convergence and divergence.

We now state our results on the $Z_n(Z_0)$ defined in (1.4). The main result is Theorem 2.1.

THEOREM 2.1. *Suppose $P(Q = 0) < 1$ and $P(M = 0) = 0$. The following are equivalent:*

$$\Pi_n \xrightarrow{a.s.} 0 \ (n \to \infty) \quad \textit{and} \quad \int_{(1,\infty)} \left(\frac{\log q}{A_M(\log q)}\right) dP(|Q| \le q) < \infty; \tag{2.1}$$

$$P(|M| = 1) < 1 \textit{ and } \sup_{n \in \mathbb{N}} |\Pi_{n-1}Q_n| < \infty \quad \text{a.s.}; \tag{2.2}$$

$$\Pi_{n-1}Q_n \xrightarrow{a.s.} 0 \qquad (n \to \infty); \tag{2.3}$$

$$\sum_{n=1}^{\infty} |\Pi_{n-1}Q_n| < \infty \quad \text{a.s.}; \tag{2.4}$$

$$\sum_{n \ge 1} P\Big(\min_{1 \le j \le n-1} |\Pi_j||Q_n| \ge e^{-x}\Big) < \infty \textit{ for all } x > 0. \tag{2.5}$$

Each of these implies

$$Z_n(Z_0) \xrightarrow{a.s.} Z_\infty \qquad (n \to \infty), \tag{2.6}$$

where Z_∞ is given by (1.5), and the series in (1.5) is a.s. absolutely convergent. Conversely, assuming

$$P(Q + Mc = c) < 1 \textit{ for all } c \in \mathbb{R}, \tag{2.7}$$

if (2.1) does not hold (that is, if Π_n does not converge to 0 a.s. as $n \to \infty$ or if the integral in (2.1) diverges) then $|Z_n(Z_0)| \xrightarrow{P} \infty$ as $n \to \infty$.

Remark 2.2. Relevant properties for the function A_M in (1.8) and (2.1) are that it is non-decreasing and concave, with $A_M(0) = 0$ and $A_M(\infty) = EX^+ \le \infty$. Also $A_M(y) > 0$ for some (hence all) $y > 0$ if and only if $P(X > 0) = P(|M| < 1) > 0$.

To bring out further properties of the integrand in (2.1) we write

$$\frac{A_M(y)}{y} = E\left(\frac{X^+}{y} \wedge 1\right) = \int_0^1 P(X > yz)\,dz,$$

which gives that $A_M(y)/y$ is non-increasing, with limits $P(X > 0) = P(|M| < 1)$ at $0+$, and $P(M = 0)$ at $+\infty$. Note that if $\Pi_n \xrightarrow{a.s.} 0$ then clearly we must have

$P(|M| < 1) > 0$, so the integrand in (2.1) is bounded in the neighbourhood of $q = 1$ when the first part of (2.1) holds. If, on the other hand, $|M| \geq 1$ a.s., then the first part of (2.1) fails so we do not need to evaluate the integral (in which the integrand is identically $+\infty$). We will take $A_M(y)/y$ to have the value $P(|M| < 1)$ at 0, and with that convention we could thus equivalently integrate over $[1, \infty)$ rather than $(1, \infty)$ in (2.1).

Remark 2.3. (2.7) is a *non-degeneracy* condition in the following sense. When $Q + Mc = c$ a.s. for some real constant c, we call the random affine map Ψ a *tied-down line*: it is constrained to pass through the point (c, c) in $\mathbb{R}^2$, and only its slope is random. Observe that when $Q_k = c - M_k c$, (1.4) reduces to

$$Z_n(Z_0) = c + (Z_0 - c)\Pi_n \qquad (n = 1, 2, \ldots). \tag{2.8}$$

So when Z_0 is degenerate at c we have $Z_n(c) = c$ a.s., whereas otherwise the asymptotic behaviour of $Z_n(Z_0)$ reduces to that of Π_n. Failure of (2.1) does not then imply $|Z_n(Z_0)| \xrightarrow{P} \infty$, so (2.7) is needed for the converse assertion in Theorem 2.1.

Remark 2.4. We rule out the case when $P(M = 0) > 0$ in Theorem 2.1, which is trivial in the following sense. When it occurs, the integrand in (2.1) is bounded by $1/P(M = 0)$ (see Remark 2.2), so the integral is finite regardless of Q. Also, then, $N := \min\{n : M_n = 0\} < \infty$ a.s. and $\Pi_n = 0$ for all $n \geq N$, hence all of (2.1)–(2.6) hold, where in (2.6), $Z_n(Z_0) = \sum_{k=1}^{N} \Pi_{k-1} Q_k$ for all $n \geq N$. There is in particular no content in the converse assertion of Theorem 2.1, for (2.1) cannot fail to hold in this case.

We also rule out the case when $Q = 0$ a.s. in Theorem 2.1. When this occurs, (1.4) gives $Z_n(Z_0) = \Pi_n Z_0$ a.s., and so $Z_n(Z_0)$ converges a.s. if and only if Π_n converges a.s., as long as Z_0 is not degenerate at 0. Now Π_n converges a.s. if and only if Π_n converges to 0 a.s., since the random walk S_n can only drift to $+\infty$, to $-\infty$, or oscillate, and only the first of these corresponds to convergence of Π_n (see (1.7)). Thus the behaviour of $Z_n(Z_0)$ is well understood in this case. The converse in Theorem 2.1, that $|Z_n(Z_0)| \xrightarrow{P} \infty$ as $n \to \infty$ if (2.1) does not hold, is not true when $Q = 0$ a.s., but this is ruled out by (2.7) (with $c = 0$) in any case.

Remark 2.5. We noted above the equivalence that, when $P(M = 0) = 0$, $\Pi_n \xrightarrow{a.s.} 0$ if and only if $S_n \xrightarrow{a.s.} \infty$. For the property $S_n \xrightarrow{a.s.} \infty$ we quote a necessary and sufficient condition which comes ultimately from [Erickson 1973], and which we give in the form derived in [Kesten & Maller 1996, Lemma 1.1]. In terms of

$$J_- := \int_{[0,\infty)} \left(\frac{y}{A_M(y)} \right) |dP(X \leq -y)|, \tag{2.9}$$

we have:

PROPOSITION 2.6. *When $P(M = 0) = 0$, for the assertion $\Pi_n \xrightarrow{a.s.} 0$ that forms the first half of (2.1) it is necessary and sufficient that*

$$J_- < E(X^+) = \infty \quad \textit{or} \quad 0 < EX \leq E|X| < \infty. \tag{2.10}$$

We could even consider the case $P(M=0)>0$ to be covered by $J_- < E(X^+) = \infty$, for it corresponds to $X = -\log|M| = +\infty$ with positive probability, hence $E(X^+) = \infty$, and it makes the J_- integral finite because the integrand is bounded by $1/P(M = 0)$ (see Remark 2.4). In this sense Proposition 2.6 remains true when $P(M=0)>0$, as do the conditions (2.1)–(2.6) of Theorem 2.1.

Remark 2.7. In answer to a question put to us by Rainer Wittmann, we note that neither half of (2.1) is superfluous. Lemma 5.5 will show that finiteness of the integral is needed. On the other hand, Theorem 2.1 shows that $\Pi_n \xrightarrow{a.s.} 0$ is certainly needed for the other assertions of the theorem to hold, and $\Pi_n \xrightarrow{a.s.} 0$ is not implied by finiteness of the integral in (2.1), even if Q has unbounded support, by (2.10).

Remark 2.8. Under the non-degeneracy condition (2.7), Theorem 2.1 establishes that $Z_n(Z_0)$ has one of two extremely contrasted forms of asymptotic behaviour: either $Z_n(Z_0) \xrightarrow{a.s.} Z_\infty$, or $|Z_n(Z_0)| \xrightarrow{P} \infty$. Consequently any infringement of one form is enough to establish the other: for instance, if $P(|Z_n(Z_0)| \le A)$ is bounded away from 0 for some constant A along some infinite sequence of n, then $Z_n(Z_0)$ converges a.s.

Theorem 2.1 is also remarkable in that the joint distribution of Q and M plays no role except in the non-degeneracy condition (2.7): thus (2.1) involves only the marginal laws of Q and M, and changing their joint law in a way consistent with these marginals has no effect on the convergence behaviour of the $Z_n(Z_0)$.

It is even possible to weaken the independence assumed of the (Q_k, M_k). So long as the M_k remain i.i.d. and the Q_k remain identically distributed, the Q_k can have arbitrary dependence on the M_k and each other for all except the converse assertion in Theorem 2.1. This will be clear from the proof.

Remark 2.9. Convergence of the perpetuity (1.5) is unaffected by taking an arbitrary positive power of the Q_k, and a different arbitrary positive power of the Π_{k-1}. That is, we have the following.

Corollary 2.10. *Assume (2.1). Then with probability 1,* $\sum_{k=1}^{\infty} |\Pi_{k-1}|^a |Q_k|^b < \infty$ *for each* $a>0$, $b>0$.

Remark 2.11. We can replace $A_M(y)$ in (2.1) by

$$\hat{A}_M(y) := \int_0^y P(|X|>x)\,dx = \int_0^y (P(X>x) + P(X<-x))\,dx.$$

That is, we have the following.

Proposition 2.12. *Suppose* $P(Q=0)<1$ *and* $P(M=0)=0$. *Then (2.1) is equivalent to*

$$\Pi_n \xrightarrow{a.s.} 0 \ (n\to\infty) \quad \textit{and} \quad \int_{(1,\infty)} \left(\frac{\log q}{\hat{A}_M(\log q)}\right) dP(|Q|\le q) < \infty. \tag{2.11}$$

Again we can replace $(1,\infty)$ by $[1,\infty)$ in the integral if desired, by taking $\hat{A}_M(y)/y$ to have its limiting value $P(|M|\neq 1)$ at 0, which value is positive if $\Pi_n \to 0$ a.s.

Remark 2.13. The forward part of Theorem 2.1 extends to a multi-dimensional setting. Consider the case where M is a $d \times d$ random matrix and $\mathbf{Q}$ a random column d-vector, and $(\mathbf{Q}, M)$, $(\mathbf{Q}_1, M_1)$, $(\mathbf{Q}_2, M_2)$, $\ldots$ are i.i.d. Set $\Pi_0 := I$, $\Pi_k := M_1 M_2 \cdots M_k$ for $k = 1, 2, \ldots$, and

$$\mathbf{Z}_n(\mathbf{Z}_0) := \sum_{k=1}^{n} \Pi_{k-1}\mathbf{Q}_k + \Pi_n \mathbf{Z}_0 \qquad (n = 1, 2, \ldots),$$

where $\mathbf{Z}_0$ is independent of the $(\mathbf{Q}_k, M_k)$. On $\mathbb{R}^d$ use the Euclidean norm $|\cdot|$ and on $d \times d$ matrices the operator norm $\|A\| := \sup_{|\mathbf{x}|=1} |A\mathbf{x}|$. Then $\|AB\| \leq \|A\|\|B\|$ so, for $m < n$,

$$\begin{aligned} |\mathbf{Z}_n(\mathbf{Z}_0) - \mathbf{Z}_m(\mathbf{Z}_0)| &\leq \sum_{k=m+1}^{n} |\Pi_{k-1}\mathbf{Q}_k| \\ &\leq \sum_{k=m+1}^{n} \|\Pi_{k-1}\||\mathbf{Q}_k| \\ &\leq \sum_{k=m+1}^{n} \left(\prod_{j=1}^{k-1} \|M_j\| \right) |\mathbf{Q}_k|. \end{aligned}$$

Apply Theorem 2.1 to the i.i.d. sequence $(|\mathbf{Q}_k|, \|M_k\|)$ to get the following, in which $A_{\|M\|}(y) := \int_0^y P(-\log\|M\| > x)\, dx$.

Corollary 2.14. *Suppose $P(|\mathbf{Q}| = 0) < 1$ and $P(\|M\| = 0) = 0$. If*

$$\prod_{k=1}^{n} \|M_k\| \xrightarrow{a.s.} 0 \ (n \to \infty) \quad \text{and} \quad \int_{(1,\infty)} \left(\frac{\log q}{A_{\|M\|}(\log q)} \right) dP(|\mathbf{Q}| \leq q) < \infty,$$

then $\mathbf{Z}_\infty := \sum_{k=1}^{\infty} \Pi_{k-1}\mathbf{Q}_k$ is a.s. absolutely convergent in norm, and $\mathbf{Z}(\mathbf{Z}_0) \xrightarrow{a.s.} \mathbf{Z}_\infty$ as $n \to \infty$.

Proof of a converse looks harder. From, for instance, tightness of $\{\mathbf{Z}_n\}$ we can immediately deduce $\{\Pi_{n-1}\mathbf{Q}_n\}$ tight. But to bring in a suitable non-degeneracy condition and deduce $\Pi_n \xrightarrow{a.s.} 0$ we would need detailed study of the consequences of our conditions on the random walk (Π_n) on the group $GL(d)$, rather than just norm calculations as above.

3. Fixed points and positive recurrence.

In [Vervaat 1979, Lemma 1.1] it is proved that if $R_n(R_0) \xrightarrow{\mathrm{L}} R$ then the distribution of R is a fixed point of (1.1), and Theorem 1.5 in the same paper characterizes fixed points when they exist. We assemble these results together with our convergence results of the previous section into the following complete characterization.

THEOREM 3.1.

(a) $P(M=0)>0$. *Then there is an integer-valued r.v.* N *such that* $Z_n(Z_0)=Z_N$ *for all* $n\geq N$, *and the distribution* ν *of* Z_N *is the unique fixed point of (1.1); also* $R_n(R_0)\xrightarrow{\mathrm{L}}\nu$ *whatever the distribution of* R_0.

(b) $P(M=0)=0$ *and* $P(Q+Mc=c)=1$ *for some* $c\in\mathbb{R}$. *Then*

$$R_n(R_0)=c+(R_0-c)\Pi_n \text{ a.s. for all } n=0,\,1,\,\ldots\,,$$

and (2.8) holds, so that $Z_n(Z_0)$ *reduces to* $R_n(Z_0)$.

(i) $M=1$ a.s. *In this case* $R_n(R_0)=R_0$ *a.s., so all distributions for* R_0 *are fixed points.*

(ii) $M=-1$ a.s. *Then*

$$R_n(R_0)=c+(-1)^n(R_0-c)\qquad(n=0,1,\ldots).$$

The fixed points consist of all distributions for R *with* $R-c\stackrel{\mathrm{L}}{=}-(R-c)$, *i.e. for which* R *is symmetric about* c, *and* $R_n(R_0)\xrightarrow{\mathrm{L}}R$ *if and only if* $R_0\stackrel{\mathrm{L}}{=}R$, *in which case* $R_n(R_0)\stackrel{\mathrm{L}}{=}R$ *for all* n.

(iii) $|M|=1$ a.s. *but* $0<P(M=1)<1$. *The fixed points consist of all distributions for* R *with* $R-c\stackrel{\mathrm{L}}{=}-(R-c)$, *and* $R_n(R_0)\xrightarrow{\mathrm{L}}R$ *whatever the distribution of* R_0, *the limit law being that of* $c+(R_0-c)U$ *where* $U=\pm1$ *with probabilities* $\frac{1}{2}$, $\frac{1}{2}$, *independent of* R_0.

(iv) $\Pi_n\xrightarrow{P}0$. *Then* $R=c$ *a.s. is the only fixed point, and* $R_n(R_0)\xrightarrow{P}c$ *whatever the distribution of* R_0.

(v) $P(|M|=1)<1$ *and* Π_n *does not tend to 0 in probability. Here* $R=c$ *a.s. is the only fixed point, and* $R_n(R_0)\xrightarrow{P}c$ *if and only if* $R_0=c$ *a.s.*

(c) $P(M=0)=0$ *and* $P(Q+Mc=c)<1$ *for all* $c\in\mathbb{R}$.

(i) *(2.1) holds. Then the distribution* ν *of* Z_∞ *is non-degenerate and is the only fixed point, and* $R_n(R_0)\xrightarrow{\mathrm{L}}\nu$ *whatever the distribution of* R_0.

(ii) *(2.1) fails. Then there exists no fixed point, and* $|R_n(R_0)|\xrightarrow{P}\infty$ *whatever the distribution of* R_0.

A fixed-point distribution for (1.1) is a normalized invariant measure for the Markov sequence $(R_n(R_0))$, so existence of the fixed-point distribution says that this Markov sequence is "positive recurrent" by the usual definition. It is natural then to ask about ergodicity of $(R_n(R_0))$, and this is proved in the positive recurrent case in [Kellerer 1992, Theorem 9.3]. The papers [Kellerer 1992] treat the special case of (1.1) and (1.2) restricted to $\mathbb{R}_+$, but the proof (via mixing) of Kellerer's Theorem 9.3 holds for the general case. We quote it as follows.

THEOREM 3.2 [KELLERER 1992]. *Assume* $P(M=0)=0$, $P(Q=0)<1$, *and (2.1), and let* ν *denote the fixed-point law, i.e. the distribution of* Z_∞. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be such that* $E|f(Z_\infty)|<\infty$ *and either (a)* f *is bounded and* ν*-a.e. continuous, or (b)* f *is uniformly continuous. Then, whatever the distribution of* R_0,

$$\frac{1}{n}\sum_{k=1}^{n}f(R_n(R_0))\to Ef(Z_\infty)\ a.s.\ (n\to\infty).$$

Another way of saying this is that with probability 1 the empirical laws $n^{-1}\sum_{k=1}^{n}\delta_{R_n}$ converge weakly to ν. The reason for the continuity requirements on f are that without special assumptions on the law of (Q,M) it is quite unclear in what subset of $\mathbb{R}$ the limit law ν "really" lives. A simple example shows this: let $M := \frac{1}{2}$ a.s. and $P(Q=1) := p$, $P(Q=0) := 1-p$, where $0 < p \leq 1$. Then the three cases $p := \frac{1}{2}$, $p \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$, $p := 1$ make ν respectively an absolutely continuous distribution, a continuous singular distribution, and a degenerate distribution.

4. Special cases.

We first relate our main result to moment conditions on M and Q. To have well-defined moments of $-\log|M|$ we assume throughout this section that $P(M=0)=0$, and for a simple formulation we assume the non-degeneracy condition (2.7) (which also rules out $P(Q=0)=1$). Let us agree to call the conclusions that $Z_n(Z_0) \xrightarrow{a.s.} Z_\infty$ given by (1.5), and that (2.2)–(2.4) hold, "convergence". As we know from Theorem 2.1, precisely when convergence fails we have "divergence" in the sense that $|Z_n(Z_0)| \xrightarrow{P} \infty$. Furthermore, by Theorem 3.1 there is a fixed point of (1.1), namely the distribution of Z_∞, precisely when convergence occurs.

We write $\log^+ x := \log(x \vee 1)$, $\log^- x := -\log(x \wedge 1)$ for $x > 0$. Then J_- (see Remark 2.5) can be expressed as

$$J_- = E\left(\frac{\log^+|M|}{A_M(\log^+|M|)}\right).$$

Corollary 4.1. *Assume $P(M=0)=0$ and the non-degeneracy condition (2.7).*

(a) $0 \leq E\log|M| \leq \infty$. *Convergence does not occur.*
(b) $-\infty < E\log|M| < 0$. *Convergence occurs if and only if $E\log^+|Q| < \infty$.*
(c) $E\log|M| = -\infty$. *Convergence occurs if and only if the integral in (2.1) is finite, and in particular if $E\log^+|Q| < \infty$.*
(d) $E\log|M|$ *does not exist, i.e.* $E\log^+|M| = \infty = E\log^-|M|$.
   (i) $J_- < \infty$. *Convergence occurs if and only if the integral in (2.1) is finite, and in particular if $E\log^+|Q| < \infty$.*
   (ii) $J_- = \infty$. *Convergence does not occur.*

Remark 4.2. (b) is due to [Grincevičius 1974], while (a) and the sufficiency parts of (c) and (d) are due to [Vervaat 1979]. The rest is new. We derive the whole result from our main Theorem 2.1 to demonstrate that the latter includes it.

Our second special case is when M is degenerate at $m \neq 0$. If $|m| \geq 1$ both parts of (2.1) fail, the function A_M being identically zero. Otherwise, when $0 < |m| < 1$, the first part of (2.1) holds, and for the second we have $A_M(y) = x \wedge y$ where $x := -\log|m| > 0$; thus finiteness of the integral is equivalent to $E\log^+|Q| < \infty$. We thus deduce the next result.

Corollary 4.3. *Consider the partial sum $Z_n := \sum_{k=1}^{n} m^{k-1}Q_k$ of a random power series built from a constant $m \neq 0$ and i.i.d. r.v.s $Q, Q_1, Q_2, \ldots$, where $P(Q=0) < 1$. If*

$$|m| < 1 \quad \text{and} \quad E\log^+|Q| < \infty$$

then $\sum_{k=1}^{\infty} |m|^{k-1}|Q_k| < \infty$ *a.s., and otherwise* $|Z_n| \xrightarrow{P} \infty$ *as* $n \to \infty$*. In the former case the distribution of* $Z_\infty := \sum_{k=1}^{\infty} m^{k-1}Q_k$ *is the unique solution of*

$$R \stackrel{\mathrm{L}}{=} Q + mR, \quad R \text{ independent of } Q, \tag{4.1}$$

while otherwise no solution exists.

Remark 4.4. The observation seems to be due to [Zakusilo 1975] that, either for all $|m| < 1$ in the case $E\log^+|Q| < \infty$, or for no $m \neq 0$ in the opposite case, $Z_\infty := \sum_{k=1}^{\infty} m^{k-1}Q_k$ converges a.s. and solves (4.1).

Remark 4.5. From Corollary 4.3 we can deduce a corresponding result for $Z_n := \sum_{k=1}^{n} M_0^{k-1}Q_k$, where Q, Q_1, Q_2, ... are conditionally i.i.d. given a r.v. M_0, with $P(M_0 = 0) = 0$ and $P(Q = 0|M_0) < 1$ a.s. In fact, the tightness of $\{Z_n\}$ implies

$$P(|M_0| < 1) = 1 \quad \text{and} \quad E(\log^+|Q||M_0) < \infty \text{ a.s.}, \tag{4.2}$$

and, conversely, (4.2) implies $\sum_{k=1}^{\infty} |M_0|^{k-1}|Q_k| < \infty$ a.s. We will omit the proof.

Our final special case is when Q is degenerate at some non-zero value, which without loss of generality we take to be 1. This case is important for the analysis of algorithms. We can satisfy the non-degeneracy condition (2.7) simply by assuming M non-degenerate. The following is then immediate from Theorems 2.1 and 3.1.

Corollary 4.6. *Assume* M *is non-degenerate. If* $\Pi_n := \prod_{j=1}^{n} M_j \xrightarrow{a.s.} 0$ *as* $n \to \infty$ *then the series* $Z_\infty := \sum_{k=1}^{\infty} \Pi_k$ *is a.s. absolutely convergent. Further, the distribution* ν *of* Z_∞ *is non-degenerate and is the only solution of*

$$R \stackrel{\mathrm{L}}{=} 1 + MR, \quad R \text{ independent of } M, \tag{4.3}$$

and the sequence $R_{n+1}(R_0) := 1 + M_{n+1}R_n(R_0)$ *converges in distribution to* ν*, whatever the distribution of* R_0*.*

Conversely, if it is not the case that $\Pi_n \xrightarrow{a.s.} 0$*, then*

$$Z_n(Z_0) := \sum_{k=1}^{n} \Pi_{k-1} + \Pi_n Z_0$$

satisfies $|Z_n(Z_0)| \xrightarrow{P} \infty$*; also there is no solution to (4.3), and* $|R_n(R_0)| \xrightarrow{P} \infty$ *whatever the distribution of* R_0*.*

5. Proofs.

We carry out most of the proof of Theorem 2.1 in the sequence of lemmas 5.1–5.8 below. Our use of the function A_M that appears in our main condition (2.1) is a consequence of an estimate due to Erickson, [Erickson 1973, Lemma 1]. Similar working in another context appears in [Chow & Zhang 1986] and in [Klass & Wittmann 1993]. We will need another generalization of Erickson's result, Lemma 5.1, which is easily deduced from Erickson's lemma and [Kesten & Maller 1996].

LEMMA 5.1. *Let $V_1, V_2, \ldots$ be i.i.d. r.v.s, not degenerate at 0, and suppose $V_1 + \ldots + V_n \to \infty$ a.s. as $n \to \infty$. Then there is a finite positive constant c_+, depending on the distribution of V_1, such that for all $y \geq 0$,*

$$\frac{y}{E(V_1^+ \wedge y)} \leq \sum_{n=0}^{\infty} P\Big(\max_{1 \leq j \leq n}(V_1 + \ldots + V_j) \leq y\Big) \leq \frac{c_+ y}{E(V_1^+ \wedge y)}, \tag{5.1}$$

where $y/E(V_1^+ \wedge y)$ is interpreted to have value $1/P(V_1 > 0) < \infty$ at $y = 0$.

PROOF: When $V_1 \geq 0$ a.s., $\max_{1 \leq j \leq n}(V_1 + \ldots + V_j) = V_1 + \ldots + V_n$, and (5.1) follows from [Erickson 1973, Lemma 1] with $c_+ = 2$. Since for all $y \geq 0$

$$P(V_1^+ + \ldots + V_n^+ \leq y) \leq P\Big(\max_{1 \leq j \leq n}(V_1 + \ldots + V_j) \leq y\Big),$$

the left-hand inequality in (5.1) is true as stated for general V_j. By [Kesten & Maller 1996] (see their (4.3), (4.2) and (4.5)), $V_1 + \ldots + V_n \to \infty$ a.s. implies that the right-hand inequality of (5.1) holds for all large enough y. The limit of the non-increasing function $E(V_1^+ \wedge y)/y$ as $y \downarrow 0$ is $P(V_1 > 0)$, which must be positive since $V_1 + \ldots + V_n \to \infty$ a.s. So by choosing c_+ larger if necessary we can assume (5.1) holds for all $y \geq 0$. $\square$

LEMMA 5.2. *Suppose $P(Q = 0) < 1$ and $P(M = 0) = 0$. If (2.1) holds, then for some $c > 0$,*

$$\Pi_{n-1} Q_n = o(e^{-cn}), \quad (n \to \infty), \text{ a.s.} \tag{5.2}$$

Consequently, (2.1) implies (2.4) and (2.6).

PROOF: The hypotheses imply that for the r.v.s $X_j = -\log|M_j|$ and $Y_j = \log|Q_j|$, defined in (1.6), we have

$$P(|X_j| < \infty) = 1 = P(Y_j^+ < \infty) \ (j = 1, 2, \ldots),$$

and we can rewrite (2.1) as

$$\Pi_n \xrightarrow{a.s.} 0 \ (n \to \infty) \quad \text{and} \quad \int_{[0,\infty)} \left(\frac{y}{E(X^+ \wedge y)}\right) dP(Y_1^+ \leq y) < \infty. \tag{5.3}$$

($\Pi_n \xrightarrow{a.s.} 0$ implies $P(|M| < 1) = P(X > 0) > 0$, and we take $y/E(X^+ \wedge y)$ to have value $1/P(X > 0) < \infty$ at $y = 0$ for the integrand in (5.3).)

With $S_n = \sum_{j=1}^n X_j$, we will now show that

$$S_{n-1} - Y_n - cn \xrightarrow{a.s.} \infty \tag{5.4}$$

for some $c > 0$, which implies (5.2). To prove (5.4), note that $\Pi_n \xrightarrow{a.s.} 0$ implies $S_n \to \infty$ under $P(M = 0) = 0$ (see Remark 2.5). Now consider some cases.

Case 1. Suppose $E(X^+) = \infty$. Then $\sum_{j=1}^{n-1} X_j^+/n \xrightarrow{a.s.} \infty$. We can obtain from the convergence of the integral in (5.3) that

$$\frac{Y_n^+}{\sum_{j=1}^{n-1} X_j^+} \xrightarrow{a.s.} 0. \tag{5.5}$$

To see this, take an ε in $(0,1)$ and write, noting the independence of Y_n and of $\{X_j\}_{1\le j\le n-1}$,

$$\begin{aligned}
&\sum_{n\ge 1} P(\varepsilon(X_1^+ + \ldots + X_{n-1}^+) \le Y_n^+) \\
&= \int_{[0,\infty)} \sum_{n\ge 1} P(X_1^+ + \ldots + X_{n-1}^+ \le y/\varepsilon)\, dP(Y^+ \le y) \\
&\le \int_{[0,\infty)} \left(\frac{c_+ y/\varepsilon}{E(X^+ \wedge (y/\varepsilon))} \right) dP(Y^+ \le y) \\
&\le \frac{c_+}{\varepsilon} \int_{[0,\infty)} \left(\frac{y}{E(X^+ \wedge y)} \right) dP(Y^+ \le y) < \infty.
\end{aligned}$$

We used the right hand inequality in (5.1) in the last estimate. An application of the Borel-Cantelli Lemma now establishes (5.5). Now $S_n \to \infty$ a.s. and $E(X^+) = \infty$ imply $J_- < \infty$ (see (2.10)), and hence $\sum_{j=1}^{n-1} X_j^- = o(\sum_{j=1}^{n-1} X_j^+)$ a.s. by [Pruitt 1981, Lemma 8.1]. This together with $n = o(\sum_{j=1}^{n-1} X_j^+)$ a.s. and (5.5) implies

$$\sum_{j=1}^{n-1} X_j^+ - n - Y_n^+ - \sum_{j=1}^{n-1} X_j^- = \left(\sum_{j=1}^{n-1} X_j^+ \right) (1 - o(1)) \xrightarrow{a.s.} \infty, \tag{5.6}$$

which implies (5.4), with $c = 1$.

Case 2. Suppose $E(X^+) < \infty$. Then $E(Y^+) < \infty$ by (5.3). Since $S_n \to \infty$ a.s., we must have $E(X^-) < \infty$ and $\mu := EX > 0$. Now $Y_n^+ = o(n)$ a.s., and $(S_{n-1} - n\mu/2)/n \to \mu/2 > 0$ a.s., so

$$\frac{Y_n^+}{S_{n-1} - n\mu/2} \xrightarrow{a.s.} 0. \tag{5.7}$$

It follows that

$$S_{n-1} - n\mu/2 - Y_n^+ = (S_{n-1} - n\mu/2)(1 - o(1)) \xrightarrow{a.s.} \infty$$

so (5.4) holds with $c = \mu/2$.

This establishes (5.2). It follows that the series (1.5), $Z_\infty = \sum_{k=1}^\infty \Pi_{k-1} Q_k$, converges absolutely a.s., which is (2.4). Z_∞ is the a.s. limit of its partial sums: $Z_n \xrightarrow{a.s.} Z_\infty$. Using $\Pi_n \xrightarrow{a.s.} 0$, which is the first part of (2.1), we have further that $Z_n(Z_0) = Z_n + \Pi_n Z_0 \xrightarrow{a.s.} Z_\infty$, which is (2.6). $\square$

LEMMA 5.3. *Assume that $P(M = 0) = 0$ and suppose $\Pi_n \xrightarrow{a.s.} 0$ as $n \to \infty$. Define $a_n := \sum_{j=1}^n X_j^- / \sum_{j=1}^n X_j^+$ and set $\tilde{M}_j := e^{-X_j^+}$ and $\tilde{\Pi}_n := \prod_{j=1}^n \tilde{M}_j$. Then $|\Pi_n| = \tilde{\Pi}_n^{1-a_n}$ for all n. Further, there exists a constant $a < 1$ such that*

$$\limsup_{n\to\infty} a_n < a < 1 \quad \text{a.s.,} \tag{5.8}$$

so that as soon as $a_n < a$ *(and hence for all large* n*, a.s.),*

$$\tilde{\Pi}_n \le |\Pi_n| \le \tilde{\Pi}_n^{1-a}. \tag{5.9}$$

PROOF: The assumption $\Pi_n \xrightarrow{a.s.} 0$ means that the random walk $S_n = -\log|\Pi_n| \xrightarrow{a.s.} \infty$, as $n \to \infty$, which is equivalent, as noted in Remark 2.5, to $J_- < E(X^+) = \infty$ or $0 < EX \le E|X| < \infty$. If $J_- < \infty = E(X^+)$ then by [Pruitt 1981, Lemma 8.1], $a_n \xrightarrow{a.s.} 0$ as $n \to \infty$. If on the other hand $0 < EX \le E|X| < \infty$ then $a_n \xrightarrow{a.s.} E(X^-)/E(X^+) < 1$. Thus in either case there is an $a < 1$, which by the Hewitt-Savage Law we can take to be a constant, such that (5.8) holds.

We have

$$S_n = \sum_{j=1}^{n}(X_j^+ - X_j^-) = (1-a_n)\sum_{j=1}^{n} X_j^+, \tag{5.10}$$

and consequently

$$|\Pi_n| = e^{-S_n} = \left(\prod_{j=1}^{n} e^{-X_j^+}\right)^{1-a_n} = \tilde{\Pi}_n^{1-a_n}.$$

Finally, the left-hand inequality of (5.9) holds as soon as $a_n < 1$, and the right-hand as soon as $a_n < a$. □

LEMMA 5.4. *Assume that* $P(M=0)=0$, $P(0 \le M \le 1) = 1$, $P(0 < M < 1) > 0$ *and* $P(Q=0) < 1 = P(Q \ge 0)$. *Suppose also that* $\limsup_{n\in\mathbb{N}} \Pi_{n-1}Q_n < \infty$ *a.s. Then the integral in (2.1) converges.*

PROOF: Define events $E_n(u) := \{\Pi_{n-1}Q_n \ge u\}$ for $n = 1, 2, \dots$. By the Hewitt-Savage Law the lim sup in the statement of the Lemma is degenerate, so we have that for some constant $u > 0$,

$$P(E_n(u) \text{ i.o.}) = 0. \tag{5.11}$$

Now $E_i(u)$ implies, since $Q \ge 0$ a.s. and $0 \le M \le 1$ a.s., that $Q_i \prod_{j=k+1}^{i-1} M_j \ge \Pi_{i-1}Q_i \ge u$ when $i > k$. Thus, for $i > k$,

$$\begin{aligned} P(E_k(u) \cap E_i(u)) &\le P(E_k(u))P\left(Q_i \prod_{j=k+1}^{i-1} M_j \ge u\right) \\ &= P(E_k(u))P(E_{i-k}(u)). \end{aligned}$$

This allows us to apply the generalized Borel-Cantelli lemma of [Spitzer 1976, p. 317] to deduce from (5.11) that $\sum_n P(E_n(u)) < \infty$, for the given value of u. Since $\sum_n P(E_n(u))$ is non-increasing in u, we can further assume that $u \ge 1$.

Since we are assuming $P(0 < M < 1) > 0$, we have $A_M(y) = \int_0^y P(-\log M > x)\,dx > 0$ for all $y > 0$, and since $P(0 \le M \le 1) = 1$, $\Pi_n = e^{-S_n}$. We apply (5.1), noting that $X = -\log M \geq 0$ a.s. here, and X is not degenerate at 0. Fix $x_0 > 0$ and set $C_0 := A_M(x_0) > 0$. Then we calculate

$$\begin{aligned}
\infty > \sum_{n=1}^{\infty} P(E_n(u)) &\geq \sum_{n=1}^{\infty} P(\Pi_{n-1} Q_n \geq u,\ Q_n > ue^{x_0}) \\
&= \int_{(ue^{x_0},\infty)} \sum_{n=1}^{\infty} P\left(S_{n-1} \leq \log \frac{q}{u}\right) dP(Q \leq q) \\
&\geq \int_{(ue^{x_0},\infty)} \left(\frac{\log(q/u)}{A_M(\log(q/u))}\right) dP(Q \leq q) \\
&\geq \int_{(ue^{x_0},\infty)} \left(\frac{\log q}{A_M(\log q)}\right) dP(Q \leq q) - \frac{\log u}{C_0}.
\end{aligned}$$

Here we used the facts that $u \geq 1$ and $A_M(y)$ is non-decreasing in y. Since $(\log q)/A_M(\log q)$ is finite on $(1, ue^{x_0})$ we see that the integral in (2.1) converges in this special case. $\square$

Lemma 5.5. *Suppose $P(Q = 0) < 1$ and $P(M = 0) = 0$. If $\Pi_n \to 0$ a.s., and $|Z_n(Z_0)|$ does not tend in probability to ∞, then $Z_n(Z_0)$ converges in distribution to a proper r.v. Z and the integral in (2.1) is finite.*

Proof: Assume the conditions of the lemma. Then, since $Z_n(Z_0) = Z_n + \Pi_n Z_0$ and $\Pi_n Z_0 \xrightarrow{a.s.} 0$, we may without loss of generality assume $Z_0 = 0$ a.s. We first show that Z_n converges in distribution.

Since $|Z_n|$ does not tend to ∞ in probability, we can find a non-random sequence (n_s) such that, as $s \to \infty$, $n_s \uparrow \infty$ and Z_{n_s} converges in law to a (possibly improper) d.f. (distribution function) F which has positive mass on $(-\infty, \infty)$. That is, with $F(\infty) := \lim_{x\uparrow\infty} F(x)$ and $F(-\infty) := \lim_{x\downarrow -\infty} F(x)$, we have $F(\infty) - F(-\infty) > 0$.

Now $Z_{n_s+1} = Z_{n_s} + Q_{n_s+1}\Pi_{n_s}$, and the term $Q_{n_s+1}\Pi_{n_s}$ tends to 0 in probability, so Z_{n_s+1} tends in law to F too. Note that, for $1 \leq m < n$,

$$Z_n = Z_m + \Pi_m \sum_{j=1}^{n-m} \left(\prod_{k=1}^{j-1} M_{k+m}\right) Q_{j+m} = Z_m + \Pi_m (Z_{n-m} \circ \theta^m),$$

where θ denotes the shift operator that adds 1 to the indices of the Q_j and the M_j. Take $n := n_s + 1$, $m := 1$, to write

$$Z_{n_s+1} = Q_1 + M_1 \bar{Z}_{n_s}, \quad \bar{Z}_{n_s} \text{ independent of } (Q_1, M_1), \quad \bar{Z}_{n_s} \stackrel{\text{L}}{=} Z_{n_s}.$$

If Z' is a (possibly improper) r.v. with d.f. F, this tells us that, on $\{|Z'| < \infty\}$,

$$Z' \stackrel{\text{L}}{=} Q_1 + M_1 Z', \qquad Z' \text{ independent of } (Q_1, M_1).$$

Now let Z have the distribution of Z' conditional on $|Z'| < \infty$, i.e., Z is a proper r.v. with d.f. G satisfying $G(x) := (F(x) - F(-\infty))/(F(\infty) - F(-\infty))$ for all real x. Since $\{|Z'| < \infty\} = \{|Q_1 + M_1 Z'| < \infty\}$, we have

$$Z \stackrel{\text{L}}{=} Q + MZ, \qquad Z \text{ independent of } (Q, M).$$

This says that G is a fixed-point distribution. Iterate this to get

$$Z \stackrel{\mathrm{L}}{=} \sum_{k=1}^{n} Q_k \Pi_{k-1} + \Pi_n Z, \qquad Z \text{ independent of } \left((Q_k, M_k)\right)_{k=1,\ldots,n}.$$

The right-hand side is just $Z_n + \Pi_n Z$. Since $\Pi_n \to 0$ a.s. and Z is proper, $\Pi_n Z$ tends in probability to 0. It follows that Z_n tends in law to Z, i.e. to the distribution G.

Now we will deduce that the integral in (2.1) is finite. We need to extend a method from [Goldie 1991, pp. 136, 157], employing a maximal inequality of [Grincevičius 1980]. Define

$$\Pi_{j,n} := \prod_{k=j+1}^{n} M_k, \quad Z_{j,n} := \sum_{k=j+1}^{n} \Pi_{j,k-1} Q_k,$$

and let "med" denote a median. The inequality is

$$P\Big(\max_{j=1,\ldots,n} (Z_j + \Pi_j \operatorname{med}(Z_{j,n} + \Pi_{j,n} y)) > x\Big) \le 2P(Z_n + \Pi_n y > x) \qquad (x, y \in \mathbb{R}),$$

valid for $n = 1, 2, \ldots$. We need only the $y = 0$ case. We note also that $Z_{j,n} \stackrel{\mathrm{L}}{=} Z_{n-j}$. Thus

$$P\Big(\max_{j=1,\ldots,n} (Z_j + \Pi_j \operatorname{med} Z_{n-j}) > x\Big) \le 2P(Z_n > x) \qquad (x \in \mathbb{R}).$$

By applying this to the pairs $(-Q_1, M_1), \ldots, (-Q_n, M_n)$ we deduce a similar inequality with $-Z_j$ in place of Z_j, so

$$P\Big(\max_{j=1,\ldots,n} |Z_j + \Pi_j \operatorname{med} Z_{n-j}| > x\Big) \le 2P(|Z_n| > x) \qquad (x \ge 0).$$

Write $m_n := \operatorname{med} Z_n$. The above implies, *a fortiori*, that for $n \ge k \ge 1$,

$$P\Big(\max_{j=1,\ldots,k} |Z_j + \Pi_j m_{n-j}| > x\Big) \le 2P(|Z_n| > x) \qquad (x \ge 0). \tag{5.12}$$

Let $n \to \infty$ with k fixed. The right-hand side converges, at least for $x \ge 0$ in the set C_Z of continuity points of Z, to $2P(|Z| > x)$. Because $Z_n \xrightarrow{\mathrm{L}} Z$ we can assume that $m_n \to m_0$, a median of Z. So (5.12) yields

$$P\Big(\max_{j=1,\ldots,k} |Z_j + \Pi_j m_0| > x\Big) \le 2P(|Z| > x) \qquad (x \in [0,\infty) \cap C_Z).$$

Now let $k \to \infty$, to conclude that $\sup_{j \in \mathbb{N}} |Z_j + \Pi_j m_0|$ is a.s. finite.

The a.s. finiteness of $\sup_{j\in\mathbb{N}} |Z_j + \Pi_j m_0|$ together with $\Pi_n \xrightarrow{a.s.} 0$ yields that $\limsup_{n\to\infty} |Z_n| < \infty$ a.s. We then have $\limsup_{n\to\infty} |\Pi_{n-1}||Q_n| < \infty$ a.s., because $\Pi_{n-1}Q_n = Z_n - Z_{n-1}$. With $\tilde{\Pi}_n$ defined as in Lemma 5.3, we have the left-hand inequality of (5.9), and therefore $\limsup_{n\to\infty} \tilde{\Pi}_{n-1}|Q_n| < \infty$ a.s. The a.s. convergence of Π_n to 0 implies that $\tilde{M} := \min(|M|, 1)$ has $P(\tilde{M} = 1) < 1$. We now have all the assumptions of Lemma 5.4 satisfied by $(|Q|, \tilde{M})$, that is, by the i.i.d. sequence $(|Q_n|, \tilde{M}_n)$, so the lemma gives

$$\int_{(1,\infty)} \left(\frac{\log q}{\tilde{A}_M(\log q)} \right) dP(Q \le q) < \infty$$

where

$$\tilde{A}_M(y) := \int_0^y P(-\log \tilde{M} > x)\, dx = \int_0^y P(X^+ > x)\, dx = \int_0^y P(X > x)\, dx = A_M(y).$$

We thus have the finiteness of the integral in (2.1). $\square$

LEMMA 5.6. *Suppose $P(Q = 0) < 1$ and $P(M = 0) = 0$. Then (2.1) holds if and only if (2.5) holds.*

PROOF: Define X_j, Y_j and S_n as in (1.6) and (1.7). Then (2.1) is equivalent to

$$\Pi_n \xrightarrow{a.s.} 0\ (n \to \infty) \quad \text{and} \quad \int_{(0,\infty)} \left(\frac{y}{\int_0^y P(X > z) dz} \right) dP(Y^+ \le y) < \infty, \tag{5.13}$$

while (2.5) is equivalent to

$$S(x) := \sum_{n \ge 1} P\Big(\max_{1 \le j \le n-1} S_j \le x + Y_n \Big) < \infty \text{ for all } x > 0. \tag{5.14}$$

Now let (5.13) hold. We have

$$\begin{aligned} S(x) \le & \sum_{n \ge 1} P\Big(\max_{1 \le j \le n-1} S_j \le x \Big) P(Y \le 0) \\ & + \int_{(0,\infty)} \sum_{n \ge 1} P\Big(\max_{1 \le j \le n-1} S_j \le x + y \Big)\, dP(Y^+ \le y). \end{aligned} \tag{5.15}$$

We assume $S_n \to \infty$ a.s., and we must have $P(|M| \ge 1) < 1$ here, since $\Pi_n \xrightarrow{a.s.} 0$ as $n \to \infty$. So $P(X > 0) > 0$, and we can apply Lemma 5.1 to get, for all $x \ge 0$,

$$\frac{x}{E(X^+ \wedge x)} \le \sum_{n \ge 1} P\Big(\max_{1 \le j \le n-1} S_j \le x \Big) \le \frac{c_+ x}{E(X^+ \wedge x)}. \tag{5.16}$$

But then by (5.15)

$$S(x) \le \frac{c_+ x}{E(X^+ \wedge x)} + \int_{(0,\infty)} \left(\frac{c_+(x + y)}{\int_0^{x+y} P(X > z) dz} \right) dP(Y^+ \le y),$$

and the integral may be written as the sum of two integrals involving, respectively, x and y in the numerator, which are then seen to converge on applying (5.13). Hence (5.14) holds.

Conversely suppose (5.14) holds. Then for any $x > 0$ and $y_0 > 0$

$$\infty > \int_{[-y_0,y_0]} \sum_{n\geq 1} P\Big(\max_{1\leq j\leq n-1} S_j \leq x+y\Big)\, dP(Y\leq y)$$
$$\geq P(|Y|\leq y_0) \sum_{n\geq 1} P\Big(\max_{1\leq j\leq n-1} S_j \leq x-y_0\Big). \tag{5.17}$$

Since $P(|Y|<\infty)>0$ (not all of the mass of Y can be at $-\infty$ because $P(Q=0)<1$) we can choose y_0 so large that $P(|Y|\leq y_0)>0$. Then, with the choice $x=y_0$, (5.17) gives

$$\sum_{n\geq 1} P\Big(\max_{1\leq j\leq n-1} S_j \leq 0\Big) < \infty.$$

Then $S_n \to \infty$ a.s. by the case $\alpha = 0$ of [Kesten & Maller 1996, Theorem 2.1] (see also (1.4) of [Kesten & Maller 1996]), and consequently $\Pi_n \xrightarrow{a.s.} 0$. Thus we again have (5.16). Going back to (5.14) we then get

$$\infty > \int_{(0,\infty)} \sum_{n\geq 1} P\Big(\max_{1\leq j\leq n-1} S_j \leq x+y\Big)\, dP(Y^+\leq y)$$
$$\geq \int_{(0,\infty)} \left(\frac{x+y}{\int_0^{x+y} P(X>z)dz}\right) dP(Y^+\leq y)$$
$$\geq \int_{(0,\infty)} \left(\frac{y}{\int_0^{y} P(X>z)dz}\right) dP(Y^+\leq y).$$

The last step follows since $x/\int_0^x P(X>z)dz$ increases in x. Hence (5.13) holds. $\square$

For the next Lemma we need the following inequality due to Esseen, to be found in [Petrov 1995, Theorem 2.15, (2.54)]. It employs the *Lévy concentration function*

$$Q(X;\lambda) := \sup_{-\infty<x<\infty} P(x\leq X\leq x+\lambda),$$

of a r.v. X.

PROPOSITION 5.7. *Let $X_1, \ldots, X_n$ be independent r.v.s, with symmetrized versions $X_1^s, \ldots, X_n^s$, and set $S_n := \sum_{j=1}^n X_j$. Let $\lambda_1, \ldots, \lambda_n$ be positive numbers and $\lambda \geq \max\{\lambda_1,\ldots,\lambda_n\}$. Then*

$$Q(S_n;\lambda) \leq \frac{A\lambda}{\sqrt{\sum_{j=1}^n \lambda_j^2 P(|X_j^s|\geq \frac{1}{2}\lambda_j)}},$$

where A is an absolute constant.

We shall use this in the following way. Since $Q(X^s;\lambda) \le Q(X;\lambda)$ for all $\lambda \ge 0$ (cf. [Petrov 1995, Lemma 1.11]) we may replace S_n by S_n^s in the left-hand side of the inequality. Then we pick $z > 0$ and put $\lambda_1 = \cdots = \lambda_n = \lambda := 2z$, and deduce that

$$P(|S_n^s| \le z) \le \frac{A}{\sqrt{\sum_{j=1}^n P(|X_j^s| \ge z)}}. \tag{5.18}$$

Lemma 5.8. *Suppose that (2.7) holds, $P(M = 0) = 0$, and Π_n does not converge to 0 a.s. Then $|Z_n(Z_0)| \xrightarrow{P} \infty$.*

Proof: (2.7) implies $P(Q = 0) < 1$. Assume also that Π_n does not converge to 0 a.s. We need the following construction. Since Q and M are real random variables there exists a regular conditional distribution for Q given M, so we may define $F_m(q) := P(Q \le q|M = m)$, such that for each $q \in \mathbb{R}$, $F_m(q)$ as a function of m is a version of the conditional probability $P(Q \le q|M = m)$, while for each $m \in \mathbb{R}$, $F_m(q)$ as a function of q is a proper distribution function. For each m let $F_m^{-1}(\cdot)$ be a left-continuous inverse of the function $q \mapsto F_m(q)$. By augmenting the probability space if necessary we may suppose it supports a sequence of independent r.v.s $(U_1, U_2, \ldots)$, independent of the sequence $((Q_1, M_1), (Q_2, M_2), \ldots)$, with each U_j uniformly distributed on $(0, 1)$. For each $j = 1, 2, \ldots$ we let $Q_j' := F_M^{-1}(U_j)$. Thus we have constructed (Q_j', M_j) which are i.i.d. with the same distribution as (Q_j, M_j), and such that each Q_j' is conditionally independent of Q_j, given M_j. Now define conditionally symmetrized r.v.s by

$$Q_j^s := Q_j - Q_j', \quad Z_n^s := \sum_{j=1}^n \Pi_{j-1} Q_j^s.$$

Note that the Q_j^s are degenerate at 0 only if Q is a Borel function of M, $Q = f(M)$, say. Assume for now that that is not the case. Let $\mathcal{G}_n := \sigma\{M_1, \ldots, M_n\}$. Applying (5.18) to the conditional distribution of Z_n^s given $\mathcal{G}_{n-1}$ we obtain, for all $z > 0$,

$$P(|Z_n^s| \le z|\mathcal{G}_{n-1}) \le \frac{A}{\sqrt{\sum_{j=1}^n P(|\Pi_{j-1}Q_j^s| \ge z|\mathcal{G}_{n-1})}} \quad \text{a.s.} \tag{5.19}$$

The denominator on the right-hand side is, everywhere on the probability space, non-zero for $z > 0$ sufficiently small. To see this, note that under the conditioning on $\mathcal{G}_{n-1}$ we may regard $|\Pi_{n-1}|$ as fixed at $m > 0$ say; then since Q_n^s is independent of $\mathcal{G}_{n-1}$ and $P(|Q_n^s| \ge z_0) > 0$ for some $z_0 > 0$ we have $P(|\Pi_{n-1}Q_n^s| \ge z|\mathcal{G}_{n-1}) > 0$ for $0 < z < z_0/m$. From (5.19) we thus have, a.s.,

$$\begin{aligned} P^2(|Z_n^s| \le z|\mathcal{G}_{n-1}) &\le \frac{A^2}{E\left(\sum_{j=1}^n \mathbf{1}(|\Pi_{j-1}Q_j^s| > z)\middle|\mathcal{G}_{n-1}\right)} \\ &\le A^2 E\Big(\frac{1}{\sum_{j=1}^n \mathbf{1}(|\Pi_{j-1}Q_j^s| > z)}\Big|\mathcal{G}_{n-1}\Big) \end{aligned}$$

by Jensen's inequality. Thus

$$\begin{aligned}(P(|Z_n^s| \le z))^2 &= E^2\left(P(|Z_n^s| \le z|\mathcal{G}_{n-1})\right) \le E\left(P^2(|Z_n^s| \le z|\mathcal{G}_{n-1})\right)\\ &\le A^2 E\left(\frac{1}{\sum_{j=1}^n \mathbf{1}(|\Pi_{j-1}Q_j^s| > z)}\right). \end{aligned} \tag{5.20}$$

Suppose that for some $z > 0$ the series in the denominator of (5.20) converges, with positive probability and hence a.s. Then $P(|\Pi_{n-1}Q_n^s| > z \text{ i.o.}) = 0$. This implies $\Pi_n \to 0$ a.s. as follows. As we have assumed, the Q_j^s are not degenerate at 0, so we may choose $\delta > 0$ so small that $P(|Q_n^s| > \delta) > 0$. Define events $A_n := \{|\Pi_{n-1}| > z/\delta\}$ and $B_n := \{|Q_n^s| > \delta\}$. B_n is independent of A_n, A_{n-1}, ..., A_1, so we have by the Lemma for Events [Loève 1977, §18] that

$$\begin{aligned}P\left(\bigcup_{n=m}^\infty A_n \cap B_n\right) &\ge P\left(\bigcup_{n=m}^\infty A_n\right) \inf_{n=m,m+1,\ldots} P(B_n)\\ &= P(|Q_1^s| > \delta) P\left(\bigcup_{n=m}^\infty A_n\right).\end{aligned}$$

This shows that if $P(A_n \text{ i.o.}) > 0$ then $A_n \cap B_n$ occurs i.o. with positive probability, hence $|\Pi_{n-1}||Q_n^s| > z$ i.o. with positive probability, a contradiction. So we have $P(|\Pi_{n-1}| > z/\delta \text{ i.o.}) = 0$, and thus $\Pi_n \xrightarrow{a.s.} 0$ since the only alternative left for the random walk (S_n) is to diverge to $+\infty$ a.s. But $\Pi_n \to 0$ a.s. contradicts the initial hypothesis of the lemma, so it must be the case that the series in the denominator of (5.20) diverges a.s. for all $z > 0$. Then by monotone convergence we get $|Z_n^s| \xrightarrow{P} \infty$, as $n \to \infty$. But then

$$\begin{aligned}&P(|Z_n^s| \le z) = E(P(|Z_n^s| \le z|\mathcal{G}_n, Z_0)) \ge\\ &E\left(P\Big(|Z_n^s| \le z,\ \Big|\sum_{k=1}^n \Pi_{k-1}Q_k' + \Pi_n Z_0\Big| \le z/2 \,\Big|\, \mathcal{G}_n, Z_0\Big)\right) \ge\\ &E\Bigg(P\Big(\Big|\sum_{k=1}^n \Pi_{k-1}Q_k + \Pi_n Z_0\Big| \le z/2 \,\Big|\, \mathcal{G}_n, Z_0\Big) \times\\ &\qquad P\Big(\Big|\sum_{k=1}^n \Pi_{k-1}Q_k' + \Pi_n Z_0\Big| \le z/2 \,\Big|\, \mathcal{G}_n, Z_0\Big)\Bigg)\\ &\quad = E\left(P^2(|Z_n(Z_0)| \le z/2|\mathcal{G}_n, Z_0)\right) \ge P^2(|Z_n(Z_0)| \le z/2),\end{aligned} \tag{5.21}$$

which shows that $|Z_n(Z_0)| \xrightarrow{P} \infty$ for this case.

Now consider the case when $Q = f(M)$ is a Borel function of M. Then

$$\begin{aligned} Z_{2n}(Z_0) &= \sum_{i=1}^{2n} \Pi_{i-1} Q_i + \Pi_{2n} Z_0 \\ &= \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} \tilde{M}_j \right) \tilde{Q}_i + \left(\prod_{j=1}^{n} \tilde{M}_j \right) Z_0 \\ &= \sum_{i=1}^{n} \tilde{\Pi}_{i-1} \tilde{Q}_i + \tilde{\Pi}_n Z_0, \text{ say,} \end{aligned}$$

where

$$(\tilde{Q}_i, \tilde{M}_i) := (Q_{2i-1} + M_{2i-1} Q_{2i}, M_{2i-1} M_{2i})$$

are i.i.d. and $\tilde{\Pi}_n = \Pi_{2n}$. Note that if $\Pi_{2n} \xrightarrow{a.s.} 0$ then $\Pi_{2n+1} = M_1(\Pi_{2n} \circ \theta) \xrightarrow{a.s.} 0$, so $\Pi_n \xrightarrow{a.s.} 0$. Since we are assuming that Π_n does not converge to 0 a.s. it follows that also $\tilde{\Pi}_n$ does not converge to 0 a.s. If $\tilde{Q}_i$ is a Borel function of $\tilde{M}_i$ (which includes the possibility that $\tilde{Q}_i = c(1 - \tilde{M}_i)$ for some c), $\tilde{Q}_i = g(\tilde{M}_i)$, say, then $Q_1 + M_1 Q_2 = g(M_1 M_2)$ and Proposition 1 of [Grincevičius 1981] shows that either $Q + cM = c$ a.s. for some c or $(Q, M) = (c_1, 1)$ for some c_1. The first of these contradicts (2.7). If $(Q, M) = (c_1, 1)$ we have $c_1 \neq 0$ since $P(Q = 0) < 1$. Then by (1.4), $|Z_n(Z_0)| = |nc_1 + Z_0| \xrightarrow{P} \infty$. Alternatively, $\tilde{Q}_i$ is not a Borel function of $\tilde{M}_i$, and the part of the present proof up to (5.21) then gives $|Z_{2n}(Z_0)| = |\sum_{i=1}^{n} \tilde{\Pi}_{i-1} \tilde{Q}_i + \tilde{\Pi}_n Z_0| \xrightarrow{P} \infty$. But the fact that $Z_{2n+1}(Z_0) = Q_1 + M_1(Z_{2n} \circ \theta)(Z_0)$ together with $P(M = 0) = 0$ then shows that $|Z_{2n+1}(Z_0)| \xrightarrow{P} \infty$, hence $|Z_n(Z_0)| \xrightarrow{P} \infty$. □

Proof of Theorem 2.1: We have $P(Q = 0) < 1$ and $P(M = 0) = 0$.

In Lemma 5.2 we proved that (2.1) implies (2.4) and (2.6). In turn, (2.4) implies (2.3), which implies (2.2).

Suppose (2.2) holds, so that $\limsup_{n\to\infty} |\Pi_{n-1}||Q_n| < \infty$ a.s. By the Hewitt-Savage Law this r.v. is degenerate, hence $P(|\Pi_{n-1}||Q_n| > z \text{ i.o.}) = 0$ for some $z > 0$. Exactly as in Lemma 5.8, using the Lemma for Events, we deduce that $\Pi_n \xrightarrow{a.s.} 0$. Now we can use Lemma 5.3, and specifically the left-hand inequality in (5.9), to get $\sup_{n\in\mathbb{N}} \tilde{\Pi}_{n-1}|Q_n| < \infty$ a.s. Again, $\Pi_n \xrightarrow{a.s.} 0$ implies that $\tilde{M} := \min(|M|, 1)$ has $P(\tilde{M} = 1) < 1$, so we can use Lemma 5.4 on the pair $(|Q|, \tilde{M})$, that is, on the sequence $(|Q_n|, \tilde{M}_n)$. It gives, as at the end of the proof of Lemma 5.5, that the integral in (2.1) converges. So we have shown that (2.2) implies (2.1). Thus (2.1)–(2.4) are equivalent, and imply (2.6).

The equivalence of (2.1) and (2.5) follows directly from Lemma 5.6.

For the converse, assume (2.7), and divide the failure of (2.1) into two cases: firstly when $\Pi_n \xrightarrow{a.s.} 0$ but the integral in (2.1) is infinite, secondly when it is not the case that $\Pi_n \xrightarrow{a.s.} 0$. In both cases we have proved that $|Z_n(Z_0)| \xrightarrow{P} \infty$, namely in Lemmas 5.5 and 5.8, respectively. □

Proof of Corollary 2.10: If $Q = 0$ a.s. or $P(M = 0) > 0$ the series always converges, so assume $P(Q = 0) < 1$ and $P(M = 0) = 0$. Then (2.1) gives $\Pi_n \xrightarrow{a.s.} 0$ so

$A_M(y) > 0$ when $y > 0$ (see Remark 2.2), and by monotonicity and the definition of $A_M(\cdot)$,

$$A_M(y) \le cA_M(y/c) \le cA_M(y) \qquad (y > 0)$$

for all $c \ge 1$. Hence replacing $|Q|$ by $|Q|^b$ in the integral in (2.1) does no more than multiply the integral by a factor between 1 and b, if $b > 1$, and so has no effect on finiteness. Similarly if $0 < b < 1$.

The effect of replacing $|M|$ by $|M|^a$ is to replace X by aX in (1.8), and this simply multiplies $A_M(y)$ by a factor between 1 and a, so again leaves the finiteness of the integral in (2.1) unaltered. □

Proof of Proposition 2.12: Assume $P(Q = 0) < 1$ and $P(M = 0) = 0$. Then $A_M(y) > 0$ for all $y > 0$ (Remark 2.2), and $\hat{A}_M(y) \ge A_M(y)$, so (2.1) implies (2.11). Conversely, let (2.11) hold. Instead of (5.10) we write

$$S_n = \sum_{j=1}^{n}(X_j^+ - X_j^-) = (1 - \hat{a}_n)\sum_{j=1}^{n}|X_j|,$$

where $\hat{a}_n := 2\sum_{j=1}^n X_j^- / \sum_{j=1}^n |X_j|$. By the same reasoning as for (5.8) we have $\limsup_{n\to\infty} \hat{a}_n < \hat{a}$ with probability 1, for some constant $\hat{a} < 1$. Thus

$$|\Pi_n| = \left(\prod_{j=1}^{n} e^{-|X_j|}\right)^{1-\hat{a}_n} = \left(\prod_{j=1}^{n} \hat{M}_j\right)^{1-\hat{a}_n},$$

where $\hat{M}_j := e^{-|X_j|} \le 1$ a.s. So if we set $\hat{\Pi}_n := \prod_{j=1}^n \hat{M}_j$, then we have $\hat{\Pi}_n \le |\Pi_n| \le \hat{\Pi}_n^{1-\hat{a}}$ once $\hat{a}_n < \hat{a}$. Thus (5.3) holds with Π_n replaced by $\hat{\Pi}_n$ and X^+ replaced by $|X|$. The working of Lemma 5.2 then gives, for Case 1, (5.5) with $\sum_{j=1}^{n-1} X_j^+$ replaced by $\sum_{j=1}^{n-1}|X_j|$. But for Case 1, $\sum_{j=1}^{n-1} X_j^- = o\big(\sum_{j=1}^{n-1} X_j^+\big)$ a.s., so (5.5) holds as stated, and hence (5.6). In Case 2, (5.7) holds again. Thus (5.4) holds as stated and we again get (5.2). □

Proof of Theorem 3.1: For (a) the existence of N is as noted in Remark 2.4, and the remaining conclusions follow by elementary deduction or as a special case of [Vervaat 1979, Theorem 1.5]. The whole of (b) is also given in the latter theorem.

For (c)(i), since $R_n(R_0) \stackrel{\mathrm{L}}{=} Z_n(R_0)$ marginally, and by our Theorem 2.1 the right-hand side converges to Z_∞ a.s., it follows that $R_n(R_0) \xrightarrow{\mathrm{L}} \nu$ and so ν is a fixed point. By [Vervaat 1979, Theorem 1.5] it is the only fixed point. The condition $P(Q + Mc = c) < 1$ for all c prevents degeneracy.

Finally, in case (c)(ii), if there is a fixed point distribution ν then on letting R_0 have that distribution we find that $R_n(R_0)$ has distribution ν for all n. But Theorem 2.1 gives that $|Z_n(R_0)| \xrightarrow{P} \infty$. Since $Z_n(R_0) \stackrel{\mathrm{L}}{=} R_n(R_0)$ we arrive at a contradiction. Thus there is no fixed point. Theorem 2.1 and $Z_n(R_0) \stackrel{\mathrm{L}}{=} R_n(R_0)$ also give $|R_n(R_0)| \xrightarrow{P} \infty$. □

Proof of Corollary 4.1: Recall that $P(Q = 0) < 1$ under (2.7) (see Remark 2.4).

(a) The condition says that $-\infty \le EX \le 0$, which prevents $S_n \to \infty$; thus the first part of (2.1) does not hold and so Theorem 2.1 gives divergence.

(b) $-\infty < E\log|M| < 0$ translates as $0 < EX < \infty$, so $|\Pi_n| = e^{-S_n} \xrightarrow{a.s.} 0$ and we have the first part of (2.1). By Theorem 2.1, convergence occurs if and only if the integral in (2.1) is finite. Since $A_M(\infty) = EX^+ < \infty$, this is equivalent to $\int_{(C,\infty)} \log q \, dP(|Q| \le q) < \infty$ for all $C > 1$, i.e. to $E\log^+|Q| < \infty$.

(c) Here, $E\log|M| = -\infty$ means that $EX^+ = \infty > EX^-$, so we have $|\Pi_n| = e^{-S_n} \xrightarrow{a.s.} 0$ and again, by Theorem 2.1, convergence occurs if and only if the integral in (2.1) is finite. Since $A_M(\infty) = EX^+ = \infty$, the integrand is $o(\log q)$ as $q \to \infty$, so it suffices for convergence that $E\log^+|Q| < \infty$ but this is no longer necessary.

(d) Here we are given that $EX^+ = \infty = EX^-$. By the theorem quoted in Remark 2.5, $\Pi_n \xrightarrow{a.s.} 0$ is in this case equivalent to $J_- < \infty$. Parts (i) and (ii) now are seen to be re-statements of the possibilities for (2.1) to be satisfied, hence we complete (d) by applying Theorem 2.1. □

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