Some comments on Cohen et al. (2019): A general spectral decomposition of causal influences applied to integrated information

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We shall consider the definition and interpretation of the causal measure *ci* of Cohen et al. (2019) under a Granger causality-type "disconnection".

We consider given a VAR(p) model for the stable, invertible and stationary vector stochastic process $\boldsymbol{u}_t$ of the form

$$A(z)\boldsymbol{u}_t = \boldsymbol{\varepsilon}_t \tag{1}$$

with $\boldsymbol{\varepsilon}_t$ a white noise process with covariance matrix $\Sigma = \mathbf{E}\left[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t^\mathsf{T}\right]$ and AR operator[1]

$$A(z) = I - \sum_{k=1}^{p} A_k z^k \tag{2}$$

We assume the process is purely nondeterministic, so that Σ is positive-definite. We partition the process as $\boldsymbol{u}_t = \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{y}_t \end{bmatrix}$, and $\boldsymbol{\varepsilon}_t$, $A(z)$, Σ, etc., are partitioned accordingly, so the process satisfies

$$\boldsymbol{x}_t = \sum_{k=1}^{p} A_{k,xx}\boldsymbol{x}_{t-k} + \sum_{k=1}^{p} A_{k,xy}\boldsymbol{y}_{t-k} + \boldsymbol{\varepsilon}_{xt} \tag{3}$$

$$\boldsymbol{y}_t = \sum_{k=1}^{p} A_{k,yx}\boldsymbol{x}_{t-k} + \sum_{k=1}^{p} A_{k,yy}\boldsymbol{y}_{t-k} + \boldsymbol{\varepsilon}_{yt} \tag{4}$$

Note that we do not assume $\Sigma_{xy} = \mathbf{E}\left[\boldsymbol{\varepsilon}_{xt}\boldsymbol{\varepsilon}_{yt}^\mathsf{T}\right] = 0$; that is, $\boldsymbol{\varepsilon}_{xt}$ and $\boldsymbol{\varepsilon}_{yt}$ may be correlated. We now consider the null hypothesis of zero Granger causality in the $\boldsymbol{y} \to \boldsymbol{x}$ direction, namely $A_{xy}(z) \equiv 0$. Following Cohen et al. (2019), we seek a disconnected model of the form

$$A'(z)\boldsymbol{u}_t = \boldsymbol{\varepsilon}'_t \tag{5}$$

with AR operator

$$A'(z) = I - \sum_{k=1}^{\infty} A'_k z^k \tag{6}$$

where $A'_{xy}(z) \equiv 0$, enforcing the null condition, and residuals covariance matrix $\Sigma' = \mathbf{E}\left[\boldsymbol{\varepsilon}'_t\boldsymbol{\varepsilon}'^\mathsf{T}_t\right]$. Note that we allow for the possibility that the model may not be of finite order (see Note 1 below). We thus have

$$\boldsymbol{x}_t = \sum_{k=1}^{\infty} A'_{k,xx}\boldsymbol{x}_{t-k} + \boldsymbol{\varepsilon}'_{xt} \tag{7}$$

$$\boldsymbol{y}_t = \sum_{k=1}^{\infty} A'_{k,yx}\boldsymbol{x}_{t-k} + \sum_{k=1}^{\infty} A'_{k,yy}\boldsymbol{y}_{t-k} + \boldsymbol{\varepsilon}'_{yt} \tag{8}$$

[1] Here, in the time domain, z may be interpreted as the backshift (lag) operator, while in the frequency domain $z = e^{-i\omega}$ lies on the unit circle in complex plane, where ω is the phase angle, measured in radians.

Following Cohen et al. (2019), the model (5) is defined to be the least-squares solution for the remaining AR operators $A'_{xx}(z), A'_{yx}(z), A'_{yy}(z)$; but in fact the least-squares solution for $A'_{yx}(z), A'_{yy}(z)$ is identical to that for $A_{yx}(z), A_{yy}(z)$ in the original VAR(p) model (1), so that $A'_{yx}(z) = A_{yx}(z)$, $A'_{yy}(z) = A_{yy}(z)$, $\boldsymbol{\varepsilon}'_{yt} = \boldsymbol{\varepsilon}_{yt}$ and $\Sigma'_{yy} = \Sigma_{yy}$. The disconnected model is thus of the form

$$\boldsymbol{x}_t = \sum_{k=1}^{\infty} A'_{k,xx}\boldsymbol{x}_{t-k} \qquad\qquad\qquad + \boldsymbol{\varepsilon}'_{xt} \tag{9}$$

$$\boldsymbol{y}_t = \sum_{k=1}^{p} A_{k,yx}\boldsymbol{x}_{t-k} + \sum_{k=1}^{p} A_{k,yy}\boldsymbol{y}_{t-k} + \boldsymbol{\varepsilon}_{yt} \tag{10}$$

The first equation above is a VAR model for $\boldsymbol{x}_t$

$$A'_{xx}(z)\boldsymbol{x}_t = \boldsymbol{\varepsilon}'_{xt} \tag{11}$$

but in general of infinite VAR order[2]; the reduced AR operator is

$$A'_{xx}(z) = I - \sum_{k=1}^{\infty} A'_{k,xx} z^k \tag{12}$$

(11) is the same reduced VAR model which appears in the standard Granger-Geweke causality formalism (Geweke, 1982, 1984). We note the following:

> Unlike (1), the disconnected model (5) *is not in general a VAR model for* $\boldsymbol{u}_t$. This may easily be seen since, under "the usual" regularity conditions [see e.g., Geweke (1982), Sec. 2], if a stochastic process admits a VAR representation, then that representation is *unique* (it is the unique solution of the Yule-Walker equations for the VAR coefficients). The implication is that, in general, the residuals process $\boldsymbol{\varepsilon}'_t$ for the disconnected model cannot be a white noise; it must be serially correlated[3]; more specifically, since the individual residuals $\boldsymbol{\varepsilon}'_{xt}$ and $\boldsymbol{\varepsilon}_{yt}$ are (by definitions) white, there must, in general, be non-vanishing lagged cross-correlations between them.

An exception is when $A_{xy}(z) \equiv 0$, in which case $A'_{xx}(z) = A_{xx}(z)$; that is, precisely when the null hypothesis is satisfied, and $F_{\boldsymbol{y}\to\boldsymbol{x}} = 0$.

We now state the following:

Proposition 1: We have:

$$\Sigma'_{xy} = \mathbf{E}\big[\boldsymbol{\varepsilon}'_{xt}\boldsymbol{\varepsilon}^{\mathsf{T}}_{yt}\big] = \mathbf{E}\big[\boldsymbol{\varepsilon}_{xt}\boldsymbol{\varepsilon}^{\mathsf{T}}_{yt}\big] = \Sigma_{xy} \tag{13}$$

Proof: $\Sigma'_{xy} = \mathbf{E}\big[\boldsymbol{\varepsilon}'_{xt}\boldsymbol{\varepsilon}^{\mathsf{T}}_{yt}\big]$ follows from $\boldsymbol{\varepsilon}'_{yt} = \boldsymbol{\varepsilon}_{yt}$, as noted above, while $\Sigma_{xy} = \mathbf{E}\big[\boldsymbol{\varepsilon}_{xt}\boldsymbol{\varepsilon}^{\mathsf{T}}_{yt}\big]$ is just by definition. Now note that from (9) $\boldsymbol{\varepsilon}'_{xt} = \boldsymbol{x}_t - \sum_{k=1}^{\infty} A'_{k,xx}\boldsymbol{x}_{t-k}$. But $\boldsymbol{\varepsilon}_{yt}$ is (by definition) uncorrelated with all the lags $\boldsymbol{x}_{t-k}$, $k = 1, 2, \ldots$, so that $\mathbf{E}\big[\boldsymbol{\varepsilon}'_{xt}\boldsymbol{\varepsilon}^{\mathsf{T}}_{yt}\big] = \mathbf{E}\big[\boldsymbol{x}_t\boldsymbol{\varepsilon}^{\mathsf{T}}_{yt}\big]$. But from (3), since again $\boldsymbol{\varepsilon}_{yt}$ is uncorrelated with all lags $\boldsymbol{x}_{t-k}$ and also $\boldsymbol{y}_{t-k}$ for $k = 1, 2, \ldots$, $\mathbf{E}\big[\boldsymbol{x}_t\boldsymbol{\varepsilon}^{\mathsf{T}}_{yt}\big] = \mathbf{E}\big[\boldsymbol{\varepsilon}_{xt}\boldsymbol{\varepsilon}^{\mathsf{T}}_{yt}\big]$, and the proposition is established.

[2] It will, however, be finite-order VARMA (Hannan and Deistler, 2012); see also the worked example later.

[3] We note that, in consequence, $S'(z) = H'(z)\Sigma' H'(z)^*$ as specified in Cohen et al. (2019), eq. (17), does not specify a "spectral decomposition" in the usual sense. In theory, since $S'(z) = H'(z)\Sigma' H'(z)^*$ is clearly a Hermitian decomposition, $S'(z)$ must be the spectral process of *some* process, but that process is not $\boldsymbol{u}_t$!

Now by Proposition 1 and the block-determinant formula, we have

$$|\Sigma| = |\Sigma_{xx}|\,|\Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}| \tag{14}$$

$$|\Sigma'| = |\Sigma'_{xx}|\,|\Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}'^{-1}\Sigma_{xy}| \tag{15}$$

so that

$$ci_{\boldsymbol{y}\to\boldsymbol{x}} = F_{\boldsymbol{y}\to\boldsymbol{x}} \Leftrightarrow |\Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}'^{-1}\Sigma_{xy}| = |\Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}| \tag{16}$$

As an immediate corrollary, we have:

$$\Sigma_{xy} = 0 \text{ or } F_{\boldsymbol{y}\to\boldsymbol{x}} = 0 \Rightarrow ci_{\boldsymbol{y}\to\boldsymbol{x}} = F_{\boldsymbol{y}\to\boldsymbol{x}} \tag{17}$$

since $F_{\boldsymbol{y}\to\boldsymbol{x}} = 0 \Leftrightarrow |\Sigma'_{xx}| = |\Sigma_{xx}|$. However, if $\Sigma_{xy} \neq 0$ and $F_{\boldsymbol{y}\to\boldsymbol{x}} > 0$, then in general $\dfrac{|\Sigma'|}{|\Sigma|} \neq \dfrac{|\Sigma'_{xx}|}{|\Sigma_{xx}|}$, contrary to Cohen et al. (2019), eq. (24), and $ci_{\boldsymbol{y}\to\boldsymbol{x}} \neq F_{\boldsymbol{y}\to\boldsymbol{x}}$. More specifically, the RHS equality in (16) only holds, at best, on a lower-dimensional (measure zero) sub-manifold of the manifold of all VAR models.

Discussion

We have shown that eq. (24) in Cohen et al. (2019)—and consequently the identification of ci in this context with Granger causailty—applies only under the condition of vanishing residuals correlation, vanishing Granger causality, or more generally, on a measure-zero subset of VAR models. As a consequence of the general identity (Szegö's Theorem), eq. (16) in Cohen et al. (2019), this is true too for the spectral case.

While this in no way invalidates ci as a measure in its own right (see, e.g., Note 2 below), it does raise questions about interpretation. If we lose the identification with Granger causality, in what sense should ci be described as "causal"? Note too that, while Granger causality has (under Gaussian assumptions) a clear information-theoretic interpretation as transfer entropy (Barnett et al., 2009; Barnett and Bossomaier, 2013), it is not clear whether ci has an information-theoretic interpretation in a comparable sense (i.e., expressible in terms of conditional/lagged mutual information).

The issue of non-vanishing residuals correlation is an important one in particular for analysis of neurophysiological time series. While it has been argued that on the analogue, continuous-time scale (Barnett and Seth, 2017) residuals correlation may not be physically plausible, discrete-time subsampling (as in all neuroimaging technologies), as well as spatial aggregation, invariably induce residuals correlation; indeed this is routinely evidenced by empirical analysis of neuroimaging data.

Note 1

We remarked earlier that the disconnected model (5), and indeed the reduced model (11) for $\boldsymbol{x}_t$ alone, should not be assumed of finite VAR model order. The reason for this is well-known in the Granger causality literature: firstly, if the same model order p is chosen for the reduced model (11), then (i) (11) fails to specify a VAR; the residuals $\boldsymbol{\varepsilon}'_{xt}$ will generally be serially correlated, and (ii) the resulting

Granger-Geweke statistic (or equivalently the transfer entropy) fails to control in full for the predective contribution of the past of $\boldsymbol{x}_t$ itself on its current value, leading to over-estimation of the Granger causality; see e.g., Bossomaier et al. (2016), Chap. 4.2.1. This was raised again more recently in Stokes and Purdon (2017), although the authors there make several erroneous claims, and in particular do not seem to have been aware that the issue had already been addressed as much as a decade earlier (Barnett et al., 2018a,b; Dhamala et al., 2018; Faes et al., 2017), in particular by Dhamala et al. (2008); Barnett and Seth (2014, 2015).

We believe that the same principles apply for the *ci* measure. Furthermore, for our worked example, we find (not shown) that Cohen et al. (2019), eq. (24) in any case fails if the reduced model order is taken as $p = 1$.

Note 2

We note that the caveats on interpretation of *ci* do not apply to the "instantaneous" measure of Cohen et al. (2019), eq. 31; as remarked there, this is just Geweke's "instantaneous linear feedback" measure (Geweke, 1982, Sec. 3), which has a well-defined information-theoretic equivalence under Gaussian assumptions as $\mathrm{I}(\boldsymbol{x}_t : \boldsymbol{y}_t \mid \boldsymbol{X}_{t-1}, \boldsymbol{Y}_{t-1})$.

A worked example

Here, we "brute force" a simple, analytically-solvable example to illustrate the above analysis. We specialise to the minimal bivariate VAR(1) model:

$$x_t = ax_{t-1} + cy_{t-1} + \varepsilon_{xt} \tag{18}$$

$$y_t = \qquad\qquad by_{t-1} + \varepsilon_{yt} \tag{19}$$

or

$$A(z)\boldsymbol{u}_t = \boldsymbol{\varepsilon}_t \tag{20}$$

where $\boldsymbol{u}_t \equiv \begin{bmatrix} x_t \\ y_t \end{bmatrix}$ is the bivariate vector process, $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ the VAR coefficients matrix, $\Sigma \equiv \mathbf{E}\left[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t^\intercal\right] = \begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ the residuals covariance matrix (κ is the residuals correlation), and the AR operator $A(z)$ is

$$A(z) = I - Az = \begin{bmatrix} 1-az & -cz \\ 0 & 1-bz \end{bmatrix} \tag{21}$$

The MA operator (transfer function) is

$$H(z) \equiv A(z)^{-1} = \frac{1}{(1-az)(1-bz)} \begin{bmatrix} 1-bz & cz \\ 0 & 1-az \end{bmatrix} \tag{22}$$

and the (Hermitian) spectral matrix $S(z)$ may be calculated from the spectral factorisation

$$S(z) = H(z)\Sigma H(z)^* \tag{23}$$

('$*$' denotes complex conjugate transpose) as

$$S(z) = \frac{1}{|1-az|^2|1-bz|^2} \begin{bmatrix} |1-(b-\kappa c)z|^2 + (1-\kappa^2)\,c^2 & [\kappa + (c-\kappa b)z](1-a\bar{z}) \\ [\kappa + (c-\kappa b)\bar{z}](1-az) & |1-az|^2 \end{bmatrix}, \qquad |z| = 1 \tag{24}$$

where $\bar{z}$ denotes complex conjugate.

The disconnected model

$$A'(z)\boldsymbol{u}_t = \boldsymbol{\varepsilon}'_t \tag{25}$$

takes the form:

$$x_t = \sum_{k=1}^{\infty} \alpha_k x_{t-k} + \varepsilon'_{xt} \tag{26}$$

$$y_t = b y_{t-1} \qquad + \varepsilon_{yt} \tag{27}$$

The x-component of the decomposed model may be written $A'_{xx}(z)x_t = \varepsilon'_{xt}$, where $A'_{xx}(z) \equiv 1 - \sum_{k=1}^{\infty} \alpha_k z^k$, and the decomposed model AR and MA operators are, respectively

$$A'(z) = \begin{bmatrix} A'_{xx}(z) & 0 \\ 0 & 1-bz \end{bmatrix} \tag{28}$$

$$H'(z) = \begin{bmatrix} H'_{xx}(z) & 0 \\ 0 & (1-bz)^{-1} \end{bmatrix} \tag{29}$$

with $H'_{xx}(z) = A'_{xx}(z)^{-1} = \left(1 - \sum_{k=1}^{\infty} \alpha_k z^k\right)^{-1}$.

We may use (24) to calculate the reduced x-transfer function and residuals covariance via spectral factorisation. For convenience, we set $h(z) \equiv H'_{xx}(z)$ and $v \equiv \Sigma'_{xx}$. From (24), the power spectral density of the sub-process x_t is

$$S_{xx}(z) = \frac{|1-(b-\kappa c)z|^2 + (1-\kappa^2)\,c^2}{|1-az|^2|1-bz|^2}, \qquad |z| = 1 \tag{30}$$

We factor this as

$$S_{xx}(z) = H'_{xx}(z)\Sigma'_{xx}H'_{xx}(z)^* = v|h(z)|^2, \qquad |z| = 1 \tag{31}$$

By inspection, we try[4]

$$h(z) = \frac{1-hz}{(1-az)(1-bz)}, \tag{32}$$

which yields

$$v|1-hz|^2 = |1-(b-\kappa c)z|^2 + \left(1-\kappa^2\right)c^2, \qquad |z| = 1, \tag{33}$$

leading to

$$v(1+h^2) = 2\Delta, \qquad \Delta \equiv \tfrac{1}{2}\left(1 + b^2 + c^2 - 2\kappa bc\right) \tag{34}$$

$$vh = b - \kappa c \tag{35}$$

This yields the quadratic equation

$$v^2 - 2\Delta v + (b-\kappa c)^2 = 0 \tag{36}$$

[4]This implies a VARMA$(2,1)$ model; *cf.* previous footnote.

for v, so that

$$v = \Delta + \sqrt{\Delta^2 - (b - \kappa c)^2} \tag{37}$$

$$h = \frac{\Delta - \sqrt{\Delta^2 - (b - \kappa c)^2}}{b - \kappa c} \tag{38}$$

(the signs on the square roots correctly yield $v = 1$ when $c = 0$). Note that the $y \to x$ Granger causality is just

$$F_{y \to x} = \log \frac{\Sigma'_{xx}}{\Sigma_{xx}} = \log v \tag{39}$$

We now calculate the autocovariance sequence $\Gamma_k \equiv \mathbf{E}\left[\boldsymbol{u}_t \boldsymbol{u}_{t-k}^\intercal\right]$ of the process for $k = 0, 1, 2, \ldots$. From the Yule-Walker equations, we may easily calculate

$$\Gamma_0 = A \Gamma_0 A^\intercal + \Sigma \tag{40}$$

$$\Gamma_k = A^k \Gamma_0 \,, \qquad k = 1, 2, \ldots \tag{41}$$

Setting $\Gamma_0 = \begin{bmatrix} p & r \\ r & q \end{bmatrix}$ and solving (40) (a discrete-time Lyapunov equation), we find

$$\left(1 - a^2\right) p = 2acr + c^2 q + 1 \tag{42}$$

$$(1 - ab)\, r = bcq + \kappa \tag{43}$$

$$\left(1 - b^2\right) q = 1 \tag{44}$$

which may be solved explicitly for p, q, r. We may also calculate that

$$A^k = \begin{bmatrix} a^k & \theta\left(b^k - a^k\right) \\ 0 & b^k \end{bmatrix}, \qquad \theta \equiv \frac{c}{b - a} \tag{45}$$

We already have $\Sigma'_{xx} = v$ and $\Sigma'_{yy} = 1$. The only entry left to calculate is $w \equiv \Sigma'_{xy} = \mathbf{E}\left[\varepsilon'_{xt} \varepsilon'_{yt}\right] = \mathbf{E}[\varepsilon'_{xt} \varepsilon_{yt}]$. Below we show (this follows from Proposition 1, but here we do it the long way) that $w = \kappa$; i.e., $\mathbf{E}[\varepsilon'_{xt} \varepsilon_{yt}] = \mathbf{E}[\varepsilon_{xt} \varepsilon_{yt}]$. Thus:

$$|\Sigma| = \begin{vmatrix} 1 & \kappa \\ \kappa & 1 \end{vmatrix} = 1 - \kappa^2 \tag{46}$$

$$|\Sigma'| = \begin{vmatrix} v & w \\ w & 1 \end{vmatrix} = v - \kappa^2 \tag{47}$$

and we see that $\dfrac{|\Sigma'|}{|\Sigma|} = \dfrac{|\Sigma'_{xx}|}{|\Sigma_{xx}|}$ [*cf.* Cohen et al. (2019), eq. (24)] only holds when $\kappa = 0$ (or in the trivial case where $c = 0 \implies v = 1$, so that $F_{y \to x} = 0$).

Proof of $w = \kappa$

We note that

$$\varepsilon'_{xt} = h(z)^{-1} x_t \quad = x_t - \sum_{k=1}^{\infty} \alpha_k x_{t-k} \tag{48}$$

$$\varepsilon_{yt} = (1 - bz) y_t = y_t - b y_{t-1} \tag{49}$$

Firstly,

$$\begin{aligned}
\mathbf{E}\left[\left(x_t - \sum_{k=1}^{\infty} \alpha_k x_{t-k}\right) y_t\right] &= \mathbf{E}[x_t y_t] - \sum_{k=1}^{\infty} \alpha_k \mathbf{E}[x_{t-k} y_t] \\
&= \Gamma_{0,xy} - \sum_{k=1}^{\infty} \alpha_k \Gamma_{-k,xy} \\
&= \Gamma_{0,yx} - \sum_{k=1}^{\infty} \alpha_k \Gamma_{k,yx} \\
&= \Gamma_{0,yx} - \sum_{k=1}^{\infty} \alpha_k (A^k \Gamma_0)_{yx} && \text{from (41)} \\
&= r - \sum_{k=1}^{\infty} \alpha_k b^k r && \text{from (45)} \\
&= r\left(1 - \sum_{k=1}^{\infty} \alpha_k b^k\right) \\
&= rh(b)^{-1}
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{E}\left[\left(x_t - \sum_{k=1}^{\infty} \alpha_k x_{t-k}\right) y_{t-1}\right] &= \mathbf{E}[x_t y_{t-1}] - \sum_{k=1}^{\infty} \alpha_k \mathbf{E}[x_{t-k} y_{t-1}] \\
&= \Gamma_{1,xy} - \sum_{k=1}^{\infty} \alpha_k \Gamma_{-(k-1),xy} \\
&= \Gamma_{1,xy} - \alpha_1 \Gamma_{0,yx} - \sum_{k=2}^{\infty} \alpha_k \Gamma_{k-1,yx} \\
&= (A\Gamma_0)_{xy} - \alpha_1 \Gamma_{0,yx} - \sum_{k=2}^{\infty} \alpha_k (A^{k-1}\Gamma_0)_{yx} && \text{from (41)} \\
&= ra + qc - r\alpha_1 - \sum_{k=2}^{\infty} \alpha_k r b^{k-1} && \text{from (45)} \\
&= ra + qc - r\sum_{k=1}^{\infty} \alpha_k b^{k-1} \\
&= ra + qc - rb^{-1}[1 - h(b)^{-1}]
\end{aligned}$$

Putting this together and using (43) we find

$$w = \mathbf{E}[\varepsilon'_{xt}\varepsilon_{yt}] = r(1 - ab) - qbc = \kappa \tag{50}$$

as claimed.

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