CHAOTIC BEHAVIOUR OF NONLINEAR WAVES AND SOLITONS OF PERTURBED KORTEWEG–DE VRIES EQUATION

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This paper considers the properties of nonlinear waves and solitons of Korteweg-de Vries equation in the presence of external perturbation. For time-periodic hamiltonian perturbation the width of the stochastic layer is calculated. The conclusions about chaotic behaviour in long-period waves and solitons are inferred. Obtained theoretical results find experimental confirmation in experiments with the propagation of ion-acoustic waves in plasma.

1. Introduction

Recently a significant attention is paid to the study of soliton equations under external perturbations. Many of these equations are by themselves completely integrable nonlinear partial differential equations, and hence cannot display chaotic behaviour. But the addition of a perturbation to an integrable equation may lead to chaotic dynamics. The nature of this external perturbation can be different and varies from one physical problem to another. As for the KdV equation, perturbations appear in it to describe, for example, solitons generated by moving pressure disturbances [1], resonant forcing in a tank of finite length [2], or traveling steady pressure distribution on water of finite depth [3]. For the case of Burgers friction and periodic forcing, steady and cyclic states were obtained in [2], while the chaotic behaviour was studied in [4, 5]. The KdV equation was also considered under simultaneous action of dissipation and instability [6]. In such a form this equation describes current-driven ion-acoustic instability in a collision-dominated plasma. For the strongly dissipative case overall evolution demonstrated irregular behaviour, therefore leading to nonstationary and irregular soliton interactions.

In this work we add periodic hamiltonian deterministic perturbation to the KdV equation and study physical consequences of this. The reduction in the form of traveling waves is made in order to turn the system into a second-order ordinary differential equation. The existence of solutions to such an equation was studied in [7], while the construction of periodic orbits can be found in [5] for nonresonant and primary resonant cases, and in [4] for secondary resonances. We will, however, perform the analysis of chaotic properties in this situation. It is known, that for any hamiltonian perturbation, which makes the system near-integrable, a stochastic layer around a separatrix appears. This layer is bounded by unbroken KAM-surfaces, and within it the system evolves with mixing. For dissipative perturbations, however, these KAM-surfaces are broken and special ideas like Melnikov method should be used to determine the conditions for chaos to appear, as it was done in [4, 5]. So, we calculate the width of a stochastic layer, which contains long-period waves and solitons. It is done on the basis of Chirikov criterion for the overlap of resonances, which is widely used for investigation of chaotic properties in hamiltonian systems (references to original papers in this field can be found in [8]). Results we obtain are as follows: solitons and nonlinear waves prove to be chaotic in the meaning that in a small distance from the peaks of solitons and long-period waves there must be a region of chaotic dynamics, where these waves acquire small irregular deviations from the smooth initial profile. This conclusion is confirmed in experiments with ion-acoustic waves in plasma [9].

The outline of this paper is the following. In the next Section, KdV equation is reduced to ODE and the unperturbed solutions of the latter are considered. In Section 3, we introduce canonically conjugated action-angle variables and obtain general expressions for the criterion of stochasticity. After that, in Section 4, this criterion is applied to KdV waves directly. Section 5 contains conclusions and summary.

2. Stationary waves

Let us consider the perturbed KdV equation, taken in the following form:

$$U_t + U \cdot U_x + \beta U_{xxx} = V(U, U_t, U_x, x - vt), \qquad x \in (-\infty, \infty), \quad t \in (0, \infty), \tag{1}$$

where x and t denote, respectively, a one-dimensional space coordinate and time; $U(x,t)$ is supposed to be differentiable with respect to x and t sufficient number of times; $V(U, U_t, U_x, x-vt)$ is a small external perturbation. The nature of this perturbation, as it was mentioned in Introduction lays in the external forces (moving pressure disturbances and so on), acting on a unperturbed KdV system. The spatial and temporal dependence in a perturbation has the form of the wave, propagating in a positive direction of the x axis with the velocity v (as it was said, V is a periodic function of a combination $x - vt$). As far as we restrict ourselves on the case of hamiltonian perturbation, it means that within the class of functions V we will consider only those, which will not contain derivatives after integration over $(x - vt) \equiv \xi$. It is easy to check that the general form of such perturbations can be represented as

$$V(U, U_t, U_x, \xi) = f_1(U, \xi) \cdot U_t(x,t) + f_2(U, \xi) \cdot U_x(x,t) + f_3(U, \xi), \tag{2}$$

with the condition $[f_2(U,\xi) - v \cdot f_1(U,\xi)]_\xi = f_3(U,\xi)_U$.

Let us search for a solution of Eq. (1) in the form of a nonlinear stationary wave $U(x,t) = f(\xi)$. Substitution of this expression in (1) gives the following ODE (prime means differentiation with respect to ξ):

$$\beta f''' = vf' - f \cdot f' + V(f, f', \xi), \tag{3}$$

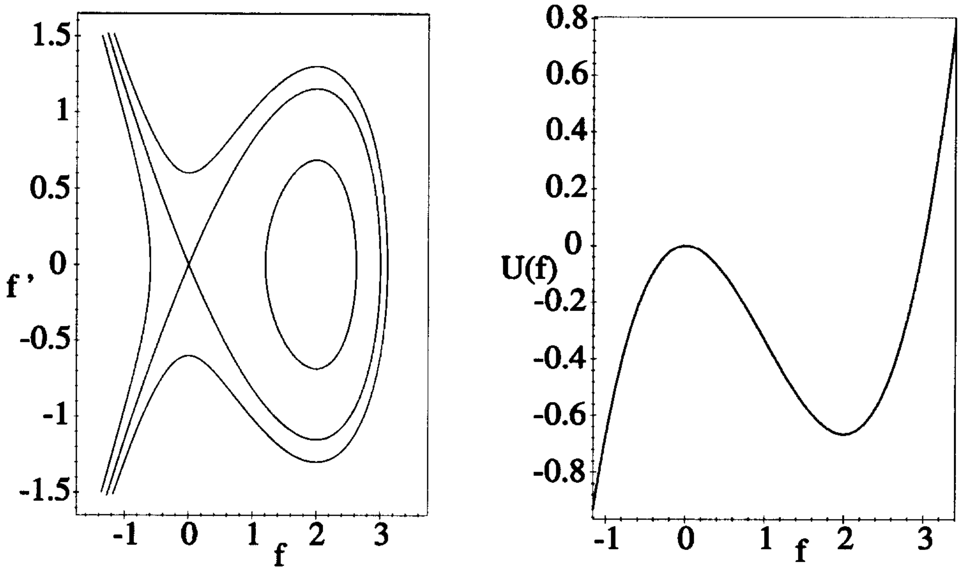

Fig. 1: (Left) Phase plane of the unperturbed system (5). (Right) Plot of the potential energy (6). Here $v = 1$ and $\beta = 1$.

which can be integrated once over ξ to yield

$$\beta f'' = vf - \frac{f^2}{2} + F(f, f', \xi). \tag{4}$$

Here $F(f, f', \xi)$ denotes the primitive function of $V(f, f', \xi)$. Integration constant is absent here because it can be turned into a zero by an appropriate choice of variables. As a second-order ODE, (4) can be treated as a dynamical system, containing a point of "mass" β, whose position is described by "coordinate" f in "time" ξ. Eq. (4) without perturbation ($F(f, f', \xi) = 0$) has the form

$$\beta f'' = vf - \frac{f^2}{2}, \tag{5}$$

which can be treated as the motion in a potential field

$$U(f) = \frac{f^3}{6} - \frac{vf^2}{2}. \tag{6}$$

Full mechanical energy of such a nonlinear oscillator is equal to

$$H = \frac{p^2}{2\beta} - \frac{vf^2}{2} + \frac{f^3}{6}, \tag{7}$$

where $p = \beta f'$ is the "momentum" of the point. Phase plane of this oscillator together with the plot of the potential energy is presented in the Fig. 1.

If $H = -\frac{2}{3}v^3$, what corresponds to the bottom of the potential pit, then the phase curve shrinks to a point. In the case $-\frac{2}{3}v^3 < H < 0$ we have closed phase curves representing conoidal waves of the form

$$f^s(\xi) = a + b \cdot \mathrm{cn}^2(k\xi \mid s), \tag{8}$$

where $\mathrm{cn}(z \mid s)$ denotes the Jacobi elliptic cosine function of modulus s ($0 \leq s \leq 1$), and a, b and k are constants, which can be expressed through s. Substitution of (8) in (5) allows one to calculate these constants and gives finally the following results:

$$\begin{cases} f^s(\xi) = v + v\left[(1-2s) + 3s \cdot \mathrm{cn}^2(k\xi \mid s)\right]/\sqrt{s^2 - s + 1}\,, \\ T^s = 4K(s)/k\,, \\ k = \frac{1}{2}\sqrt{v/\beta}(s^2 - s + 1)^{-\frac{1}{4}}. \end{cases} \tag{9}$$

Here $K(s)$ is the complete elliptic integral of the first kind, and T^s is the period of the corresponding solution. Under $H = 0$ ($s = 1$), which corresponds to a separatrix in the phase plane, solution (9) yields the homoclinic orbit

$$f(\xi) = \frac{3v}{\cosh^2\left(\frac{\xi}{\Delta}\right)}, \tag{10}$$

where $\Delta = 2\sqrt{\frac{\beta}{v}}$ [10].

3. A criterion for chaotic motion

Let us transform variables (f, f') to the canonical action-angle pair (I, θ) using standard formulae

$$I = \frac{1}{2\pi}\oint p(f,H)\,df = I(H), \qquad \theta = \frac{\partial S(f,I)}{\partial I}, \qquad S(f,I) = \int p(f,H)\,df, \tag{11}$$

where $S(f,I)$ is the reduced action and p is the momentum introduced in the previous section. The Hamiltonian of the perturbed system can be then written in the form

$$H(I,\theta,\xi) = H_0(I) + V(I,\theta,\xi), \tag{12}$$

where $H_0(I)$ is the Hamiltonian of the unperturbed system from (7), and $V(I,\theta,\xi)$ corresponds to the perturbation term in Eq. (4). An explicit expression for the perturbation in terms of (I,θ) variables cannot be obtained due to impossibility to calculate the integrals (11).

As far as V is periodic in time (as it is assumed) with frequency ν, and taking into consideration the fact that unperturbed motion is integrable, we may expand the perturbation into the following Fourier series:

$$\begin{cases} V(I,\theta,\xi) = \frac{1}{2}\sum\limits_{k,l} V_{kl}(I)\exp[i(k\theta - l\nu\xi]\,, \\ V_{kl} = V^*_{-k,-l}\,. \end{cases} \tag{13}$$

The perturbed equations of motion in (I,θ) variables take now the form

$$\begin{cases} \dot{I} = -\frac{1}{2} i \sum\limits_{k,l} k V_{kl}(I) \exp[i(k\theta - l\nu\xi], \\ \dot{\theta} = \omega(I) + \frac{1}{2} i \sum\limits_{k,l} \frac{dV_{kl}}{dI} \exp[i(k\theta - l\nu\xi)], \end{cases} \tag{14}$$

where $\omega(I) = \frac{dH_0}{dI}$ is the frequency of the unperturbed motion. Let us consider the motion in the vicinity of one particular resonance,

$$m\omega(I_{mn}) - n\nu = 0\,. \tag{15}$$

The width of this resonance on frequency is $\Omega = \sqrt{4V_0|\omega'|}$, where $V_0 \equiv |V_{mn}(I_{mn})|$ and $\omega' \equiv d\omega(I_0)/dI$. The distance between two adjacent resonances is equal to

$$\Delta\omega_{\alpha\beta} = |\omega(I_{m\pm\alpha,n\pm\beta}) - \omega(I_{m,n})|\,, \tag{16}$$

where $\alpha = 0,1$; $\beta = 0,1$. As a condition of chaotic motion, which means the appearance of mixing, we will use the Chirikov criterion for the overlap of resonances [11],

$$K = \left(\frac{\Omega}{\Delta\omega}\right)^2 \geq 1, \tag{17}$$

where $\Delta\omega$ is the minimal possible value of $\Delta\omega_{\alpha\beta}$ from (16). For the cases $(\alpha,\beta) = (1,0)$, $(0,1)$ and $(1,1)$ calculations similar to [11] give, respectively, the following conditions for the border of stochasticity:

$$\begin{cases} K_{10} = 4V_0|\omega'|m^2/\omega^2 \geq 1, & m \gg 1, \quad n \geq 1, \\ K_{01} = 4V_0|\omega'|n^2/\omega^2 \geq 1, & n \gg 1, \quad m \geq 1, \\ K_{11} = 4V_0|\omega'|m^2 n^2/\omega^2(m-n)^2 \geq 1, & m,n \gg 1, \quad |m-n| \sim 1. \end{cases} \tag{18}$$

4. Chaos of Korteweg–de Vries waves

To apply these results to the system governed by KdV equation one must first evaluate exact expressions for ω and ω', which appear in the conditions (18). The frequency ω is equal to

$$\omega = \frac{2\pi}{T} = \frac{\pi k}{2K(s)}\,. \tag{19}$$

For convenience let us rewrite it as

$$\omega = \frac{\pi}{K(z)}\sqrt{\frac{v \sin\frac{\phi}{3}}{2\beta\sqrt{3}}}\,, \tag{20}$$

where we introduced φ by the correlation $\cos\frac{\varphi}{3} = 1 + 3\frac{H}{v^3}$, and $z \equiv \frac{1}{2} - \frac{\sqrt{3}}{2}\cot\frac{\varphi}{3}$. To evaluate ω', one should represent it as

$$\omega' = \frac{d\omega}{dH}\frac{dH}{dI} = \omega\frac{d\omega}{d\varphi}\left(\frac{dH}{d\varphi}\right)^{-1}\,. \tag{21}$$

Tedious calculations give finally

$$\omega' = \frac{\pi^2\omega}{16v^{5/2}\sin\varphi K^2(z)}\sqrt{\frac{3}{2\beta\sin\frac{\varphi}{3}}}\left[-4\cos\frac{\varphi}{3}F(1.5;0.5;1;z) + \right.$$

$$\left. + \left(\sqrt{3}\sin\frac{\varphi}{3} + \cos\frac{\varphi}{3}\right) F(1.5;1.5;2;z)\right], \tag{22}$$

where $F(a,b,c,z)$ is the hypergeometric function. Substituting (20) and (22) in (18) one can make sure that with approach to separatrix $K \to \infty$ under arbitrary small V_0. This means that for any external perturbation, however small it is, Chirikov criterion is executed starting with some energies, and this leads to the formation of the corresponding stochastic layer. The transition from regular to chaotic motion can be found approximately from the condition $K \approx 1$. This bound on energy is defined by the inequality

$$H_{\min} \le H \le 0, \qquad H_{\min} = \frac{v^3}{3}(\cos\varphi_{\min} - 1). \tag{23}$$

Here the value of $\varphi_{\min}$ is found from

$$\frac{4v^3\sin\varphi_{\min}\tan(\varphi_{\min}/3)K(z_{\min})}{-4F(1.5;0.5;1;z_{\min}) + \left(\sqrt{3}\tan(\varphi_{\min}/3)+1\right)F(1.5;1.5;2;z_{\min})} = \pi V_0\zeta(m,n), \tag{24}$$

in which $z_{\min} = \frac{1}{2} - \frac{\sqrt{3}}{2}\cot\frac{\varphi_{\min}}{3}$ and $\zeta(m,n) = m^2,\ n^2,\ \frac{m^2n^2}{(m-n)^2}$ for three cases in (18). Certainly, there is only half-width of stochastic layer, but the phase curves, laying outside separatrix are physically meaningless and thus uninteresting (they are unbounded at infinity).

5. Summary and conclusions

We have calculated the width of the stochastic layer around a separatrix, corresponding to a soliton solution of Korteweg–de Vries equation under small time-periodic hamiltonian perturbations. As far as the motion in this stochastic layer is chaotic (in the meaning of mixing), so nonlinear wave solutions corresponding to the phase curves within this stochastic layer will also possess certain chaotic properties. Let us now consider the question about the spatio-temporal region, where this chaotic behaviour can be registered.

The time scale, after which stochasticity in a nonlinear oscillator can be found, is defined by $\tau_c \ll t$, where τ_c is the time of the decay of correlations. In [11] the following estimation of τ_c for a nonlinear oscillator is obtained:

$$\tau_c \sim \frac{1}{\ln K}, \tag{25}$$

where K is the coefficient for the overlap of resonances (17). As far as in our consideration, the variable ξ plays the role of time t, so the region where chaotic regime can be detected for the waves is

$$\frac{1}{\ln K} \ll \xi \tag{26}$$

or

$$\frac{1}{\ln K} \ll (x - vt) \tag{27}$$

in old denotions. Therefore, we can conclude that this region represents wave formation, which outstrips nonlinear wave or soliton, and propagates in the same direction with the same velocity. So, the chaos is not a spatial or temporal one, but their mixture, realized in a wave. Under $t = 0$ one obtains

$$\frac{1}{\ln K} \ll x. \tag{28}$$

As far as with approach to a separatrix $K \to \infty$, so for solitons the minimal distance on which chaotic behaviour should appear $x_c \to 0$, and thus this effect is realized in the close vicinity of a soliton peak. The chaotic behaviour we have found, is manifested in the irregular small deflections from a smooth initial soliton profile. So, as a result of this paper it can be inferred that long-period nonlinear waves, and especially solitons of Korteweg–de Vries equation, obtain chaotic properties in the presence of time-periodic hamiltonian external perturbations.

Exactly the same result was observed in an experiment with ion-acoustic and Langmuir waves in a nonmagnetized plasma [9]. There was registered a faint splash directly before the soliton peak. From the concepts developed in this paper this phenomenon can be explained in the following way. Together with a soliton, a wave generator also produces a group of waves of small amplitude and almost zero frequencies. These waves are produced constantly with the generator on, and therefore can be considered as a small deterministic periodic external perturbation. Stochastic destruction of soliton due to these waves must realize itself, as it was described above, just before the soliton peak, exactly as it was registered in the experiment.

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