Energy conservation for the Euler-Korteweg equations

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Onsager's conjecture

Consider the incompressible Euler system

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0, \qquad \operatorname{div} u = 0. \tag{1}$$

It is easy to show that for a smooth solution u the energy

$$E(t) = \int_\Omega \frac{1}{2}|u(t,x)|^2 dx \tag{2}$$

is conserved. In 1949 Lars Onsager [11] conjectured the following:

- If u is a weak solution with Hölder regualarity $\alpha > \frac{1}{3}$, then the energy is conserved.
- For any $\alpha < \frac{1}{3}$ there exists a weak solution $u \in C^\alpha$ which does not conserve the energy.

The first part of the conjecture has been proven by Constantin, E. and Titi in 1994 [3].
The second part has been resolved only recently by Isett [10] and Buckmaster, De Lellis, Székelyhidi Jr. and Vicol [2].

Euler-Korteweg equations

We consider the isothermal Euler-Korteweg system

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) &= -\rho \nabla_x \left(h'(\rho) + \frac{\kappa'(\rho)}{2}|\nabla_x \rho|^2 - \operatorname{div}(\kappa(\rho)\nabla_x \rho \right), \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0, \end{aligned} \tag{3}$$

where $\rho \geq 0$ is the scalar density of a fluid, u is its velocity, $h = h(\rho)$ is the energy density and $\kappa = \kappa(\rho) > 0$ is the coefficient of capillarity.
In conservative form

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) &= \operatorname{div} S, \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0, \end{aligned} \tag{4}$$

where S is the Korteweg stress tensor

$$S = [-p(\rho) - \frac{\rho\kappa'(\rho) + \kappa(\rho)}{2}|\nabla_x \rho|^2 + \operatorname{div}(\rho\kappa(\rho)\nabla_x \rho)]\mathbb{I} - \kappa(\rho)\nabla_x \rho \otimes \nabla_x \rho$$

$\mathbb{I}$ denotes the d-dimensional identity matrix and the local pressure is defined as $p(\rho) = \rho h'(\rho) - h(\rho)$.

- The terms on the righ-hand side model capillary effects.
- Known model to describe liquid-vapor flows. In particular the behaviour of a mixture at the interface.
- Contains as a special case the equations of quantum hydrodynamics (QHD).
- In [1] well-posedness and stability of local-in-time smooth solutions is shown. Existence and non-uniqueness of global weak solutions was considered in [5]. In [8] weak-strong uniqueness property is established.

Conservation of energy

It can be easily shown that a classical solution (ρ, u) of the Euler-Korteweg equations satisfies the additional law of conservation of total (intarnal and kinetic) energy

$$\begin{aligned} &\partial_t \left(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{\kappa(\rho)}{2}|\nabla_x \rho|^2 \right) \\ &+ \operatorname{div}\left(\rho u \left(\frac{1}{2}|u|^2 + h'(\rho) + \frac{\kappa'(\rho)}{2}|\nabla_x \rho|^2 - \operatorname{div}(\kappa(\rho)\nabla_x \rho) \right) - \kappa(\rho)\partial_t \rho \nabla \rho \right) = 0. \end{aligned}$$

Our aim is to find a sufficient regularity condition so that a weak solution also satisfies this equality.

Main result

Theorem. *Let (ρ, u) be a solution of* (3) *with constant capillarity c_k in the sense of distributions. Assume*

$$u, \nabla_x u \in B_3^{\alpha,\infty}((0,T) \times \mathbb{T}^d), \quad \rho, \rho u, \nabla_x \rho \in B_3^{\beta,\infty}((0,T) \times \mathbb{T}^d), \tag{5}$$

where $0 \leq \alpha, \beta \leq 1$ *such that* $\min(2\alpha + \beta, \alpha + 2\beta) > 1$.
Then the energy is locally conserved, i.e.

$$\partial_t(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{c_k}{2}|\nabla_x \rho|^2) + \operatorname{div}(\frac{1}{2}\rho u|u|^2 + \rho^2 u - c_k \rho u \Delta\rho - c_k \partial_t \rho \nabla \rho) = 0 \tag{6}$$

in the sense of distributions on $(0,T) \times \mathbb{T}^d$.

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