Outflow Boundary Condition Leading to Minimal Energy Dissipation for an Incompressible Flow

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Introduction

We consider a stationary flow of an incompressible fluid through a region with an artificial boundary (for example the outflow), such as the pipe flow. In order to complete the corresponding system of equations, one needs to introduce some boundary condition at the outflow, which is a priori unknown. The use of the popular "do-nothing" boundary condition has its downsides - see [H] for details.

We, on the other hand, study the possibility of selecting the outflow boundary condition in such a way that the resulting flow minimizes given functional. This functional may represent the dissipation of energy, for example. Then, once the existence of that minimum is proved, one automatically gets boundary condition which is both physically and mathematically reasonable. The boundary conditions obtained in this way are implicit in general but at least for the Stokes system we show that they imply some familiar explicit boundary conditions.

Mathematical formulation

Let $\Omega \subset \mathbb{R}^d$. Then let $\Gamma \subset \partial\Omega$ be the part of the boundary, where the Dirichlet boundary condition is prescribed. The remaining part will be denoted by Γ_a and it may represent artificial boundaries such as inflow or outflow.

Let $\mathbb{D}\boldsymbol{v} := \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)$. The stationary flow of an incompressible fluid is described by

$$\begin{aligned} \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathbb{D}\boldsymbol{v} &= -\nabla p && \text{in } \Omega \\ \operatorname{div}\boldsymbol{v} &= 0 && \text{in } \Omega \\ \boldsymbol{v} &= \boldsymbol{v}_0 && \text{on } \Gamma, \end{aligned} \tag{1}$$

where $\boldsymbol{v}_\mathrm{in}$ is such that there exists its divergence free extension in $\boldsymbol{H}^1_\Gamma(\Omega)$ (the construction can be found in [L, Chapter I, Problem 2.1]).

Instead of prescribing a boundary condition on Γ_a, we require that $\boldsymbol{v}$ minimizes

$$F(\boldsymbol{v}) := \int_\Omega |\mathbb{D}\boldsymbol{v}|^2, \quad \text{or} \quad G(\boldsymbol{v}) := \int_\Omega |\nabla\boldsymbol{v}|^2$$

over the (non-empty) set of solutions to (1). Then the following questions arise:

1) Is there a solution to this problem? (Does the minimum exist?)
2) Does this requirement imply some explicit boundary condition on Γ_a?

Existence of the minimum

We are able to prove the existence of minimum for system (1) and also for more general systems describing a flow of some non-Newtonian fluids defined via an implicit relation $G(\mathbb{T}^d, (\mathbb{D}\boldsymbol{v})^d)$, where $A^d = A - \frac{1}{3}\operatorname{tr} A\,\mathbb{I}$. We need coercivity of G to prove the boundedness of a minimizer, monotonicity of G to identify the weak limits and finally we use the weak lower semi-continuity of F and G to show that the infimum is attained in the set of solutions to (1). Also, due to the presence of $\mathbb{D}\boldsymbol{v}$ in F, one needs to prove an appropriate version of Korn's inequality, namely

$$\|\boldsymbol{\varphi}\|_{1,2} \le c\,\|\mathbb{D}\boldsymbol{\varphi}\|_2 \quad \forall\boldsymbol{\varphi} \in \boldsymbol{H}^1_\Gamma(\Omega).$$

Implicit boundary condition

Suppose that $\boldsymbol{v}$ solves (1) and that it is also a minimum of F. We want to find $\boldsymbol{\varphi}_\varepsilon$ such that $\boldsymbol{v} + \varepsilon\boldsymbol{\varphi}_\varepsilon$ solves (1). This means that $\boldsymbol{\varphi}_\varepsilon$ has to satisfy $\boldsymbol{\varphi}_\varepsilon \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$ and

$$\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) + \varepsilon\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi}_\varepsilon + \boldsymbol{\varphi}_\varepsilon\otimes\boldsymbol{v}) + \varepsilon^2\operatorname{div}(\boldsymbol{\varphi}_\varepsilon\otimes\boldsymbol{\varphi}_\varepsilon) - \operatorname{div}\mathbb{D}\boldsymbol{v} - \varepsilon\operatorname{div}\mathbb{D}\boldsymbol{\varphi}_\varepsilon = -\nabla\pi_\varepsilon. \tag{2}$$

Since $\boldsymbol{v}$ solves (1), by dividing by $\varepsilon \neq 0$ and redefining π_ε, we obtain that (2) is equivalent to

$$\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi}_\varepsilon + \boldsymbol{\varphi}_\varepsilon\otimes\boldsymbol{v}) + \varepsilon\operatorname{div}(\boldsymbol{\varphi}_\varepsilon\otimes\boldsymbol{\varphi}_\varepsilon) - \operatorname{div}\mathbb{D}\boldsymbol{\varphi}_\varepsilon = -\nabla\pi_\varepsilon. \tag{3}$$

Then, if $\boldsymbol{\varphi}_\varepsilon \rightharpoonup \boldsymbol{\varphi} \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$ for a subsequence, we eventually get

$$\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi} + \boldsymbol{\varphi}\otimes\boldsymbol{v}) - \operatorname{div}\mathbb{D}\boldsymbol{\varphi} = -\nabla\pi \tag{4}$$

for certain pressure π.

The optimality condition can be written as

$$0 \le F(\boldsymbol{v} + \varepsilon\boldsymbol{\varphi}_\varepsilon) - F(\boldsymbol{v}) = 2\varepsilon\int_\Omega \mathbb{D}\boldsymbol{v}\cdot\mathbb{D}\boldsymbol{\varphi}_\varepsilon + \varepsilon^2\int_\Omega |\mathbb{D}\boldsymbol{\varphi}_\varepsilon|^2.$$

If we divide by $\varepsilon \neq 0$ and then we let $\varepsilon \to 0_\pm$, we finally obtain

$$\int_\Omega \mathbb{D}\boldsymbol{v}\cdot\mathbb{D}\boldsymbol{\varphi} = 0 \quad \text{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega) \quad \text{satisfying (4).} \tag{5}$$

Explicit boundary condition

For the Stokes system, we can show that implicit condition (5) can be reduced to an explicit one. Indeed, in this case (5) reads as

$$\int_\Omega \mathbb{D}\boldsymbol{v}\cdot\mathbb{D}\boldsymbol{\varphi} = 0 \quad \text{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega) \quad \text{such that} \quad -\operatorname{div}\mathbb{D}\boldsymbol{\varphi} = -\nabla\pi. \tag{6}$$

Therefore, for all such $\boldsymbol{\varphi}$, we get

$$\int_{\Gamma_\mathrm{a}} \mathbb{T}\boldsymbol{n}\cdot\boldsymbol{\varphi} = \int_\Omega \operatorname{div}\mathbb{T}\cdot\boldsymbol{\varphi} + \int_\Omega \mathbb{T}\cdot\mathbb{D}\boldsymbol{\varphi} = 0,$$

where $\mathbb{T} := -p\mathbb{I} + \mathbb{D}\boldsymbol{v}$ (the trace of $\mathbb{T}\boldsymbol{n}$ exists in the distribution sense since $\mathbb{T}$ is an integrable solenoidal function). Since there is no restriction on the trace of $\boldsymbol{\varphi}$ at Γ_a (there is a solution to the Stokes system for any boundary condition $\boldsymbol{w}$ satisfying $\int_{\partial\Omega}\boldsymbol{w}\cdot\boldsymbol{n} = 0$, see [T, Chapter I, Theorem 2.4]), we obtain

$$\int_{\Gamma_\mathrm{a}} \mathbb{T}\boldsymbol{n}\cdot\boldsymbol{w} = 0 \quad \text{for all} \quad \boldsymbol{w} \in C_0^\infty(\Gamma_\mathrm{a}), \quad \int_{\Gamma_\mathrm{a}} \boldsymbol{w}\cdot\boldsymbol{n} = 0. \tag{7}$$

From that, by considering the normal and tangential part of $\mathbb{T}\boldsymbol{n}$ separately, we finally obtain that

$$\mathbb{T}\boldsymbol{n} = c\,\boldsymbol{n} \quad \text{on } \Gamma_\mathrm{a} \tag{8}$$

for some constant $c \in \mathbb{R}$. Thus, we have shown that for the Stokes system, the implicit boundary condition (6) reduces to the modification of the "do-nothing" boundary condition with the symmetric velocity gradient. If we consider the functional G instead of F, we can proceed as before and obtain

$$-2p\boldsymbol{n} + (\nabla\boldsymbol{v})\boldsymbol{n} = c\,\boldsymbol{n} \quad \text{on } \Gamma_\mathrm{a}. \tag{9}$$

If Γ_a has multiple components with prescribed flow rates, the constant c may vary on each of these (to allow non-trivial flow).

For the Navier-Stokes system, so far I was able to deduce certain explicit boundary condition only under additional hypotheses. Namely, if $\nabla\boldsymbol{n} = 0$ on the components of Γ_a, then the flow minimizing the energy dissipation over the set of all solutions to the NS system with zero tangential part of the velocity on Γ_a satisfies

$$\begin{aligned} \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathbb{D}\boldsymbol{v} &= -\nabla p && \text{in } \Omega \\ \operatorname{div}\boldsymbol{v} &= 0 && \text{in } \Omega \\ \boldsymbol{v} &= \boldsymbol{v}_0 && \text{on } \Gamma \\ \boldsymbol{v}_\tau &= 0 && \text{on } \Gamma_\mathrm{a} \\ 2(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\cdot\boldsymbol{n} &= p + \tfrac{3}{2}|\boldsymbol{v}|^2 + c_i && \text{on } \Gamma_\mathrm{a}. \end{aligned}$$

References

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[L] Ladyzhenskaya, O. A., *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, English translation, Second Edition, 1969.

[T] Temam, R., *Navier-Stokes equations: theory and numerical analysis*, North-Holland Pub. Co., 1979.