Profile decomposition and its applications to Navier-Stokes system

Jean-Yves Chemin
Laboratoire J.-L. Lions, Case 187
Université Pierre et Marie Curie, 75 230 Paris Cedex 05, France
chemin@ann.jussieu.fr

Introduction

This text is notes of a series of lectures given first in the Morningside Center in February and March 2014 in Beijing and then at Jacques-Louis Lions laboratory in May and June 2014. I thank very warmly the audience of this two series of lectures for their interest and for their remarks. I want to thank espaccially Professor Ping Zhang for a rereading of the notes and a lot of suggestions of improvements.

The purpose of theses lectures was the description, up to an extraction, of lack of compacness of the Sobolev embeddings as acieved in the work [18] of P. Gérard. We apply this result to the describtion of the possible blow up in the three dimensionnal incompressible Navier-Stokes.

More precisely, the first chapter will be devoted to the proof of a refined Sobolev inequality which is in particular invariant under translation in Fourier space (i.e. multiplication by an oscillating function). This involves some particular classes of Besov spaces we shall define and study.

The second chapter is devoted to the statement and the detailled proof of the P. Gérard's result which provides a precise describtion, up an extraction, of a sequence of function which is bounded in the homogenenous Sobolev $\dot{H}^s(\mathbb{R}^d)$. It claims that in some sense, it is the sum of dilation and translation of some given fuction in $\dot{H}^s(\mathbb{R}^d)$. For a given sequence $(\lambda_n)_{n\in\mathbb{N}}$, the concept of $(\lambda_n)_{n\in\mathbb{N}}$-oscillating sequence is defined. Together with some particular class of Besov spaces, this tunrs out to be the crucial tool of the proof.

In the third chapter, we first recall some basic results about incomressible Navier-Stokes system. Then we prove some result about bounded sequences of initial data in the spirit of the work by I. Gallagher (see [14]. And we apply this resulst to prove the celebrated result by L. Escauriaza, G. Serëgin, V. Sverak (see [12]) which claims that if a solution u to the incompressible three dimensionnal Navier-Stokes equation develops a singularity at time $T^\star$, then we have

$$\limsup_{t\to T^\star} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} = \infty.$$

We follow the approach developed by C. Kenig and G. Koch in [20].

Contents

Chapter 1

Sobolev embeddings revisited

1.1 Sobolev spaces and Sobolev embedding

Definition 1.1.1. *Let s be a real number. The homogeneous Sobolev space $\dot H^s(\mathbb{R}^d)$ is the set of tempered distributions u the Fourier transform of which $\widehat{u}$ belongs to $L^1_{loc}(\mathbb{R}^d)$ and satisfies*

$$\|u\|^2_{\dot H^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

The following interpolation inequality is of constant use. We left as an exercise to the reader the proof that $\dot H^s(\mathbb{R}^d)$ is a Hilbert space in the case when s is less than $d/2$.

Proposition 1.1.1. *If u belongs to $\dot H^{s_1} \cap \dot H^{s_2}$, for any s between s_1 and s_2, u belongs on $\dot H^s$ and*

$$\|u\|_{\dot H^s} \leq \|u\|^{\theta}_{\dot H^{s_1}} \|u\|^{1-\theta}_{\dot H^{s_2}} \quad \text{with} \quad s = \theta s_1 + (1-\theta) s_2.$$

Proof. Let us simply apply Hölder inequality for the fonction $|\xi|^{2\theta s_1}$ and $|\xi|^{2(1-\theta)s_2}$ for the measure $|\widehat{u}(\xi)|^2 d\xi$ and this gives the result. □

Let us state the classical Sobolev inequality and its dual version.

Theorem 1.1.1. *If s belongs to $[0, \frac{d}{2}[$, then the space $\dot H^s(\mathbb{R}^d)$ is continuously embedded in $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$. If p belongs to $]1, 2]$, then the space $L^p(\mathbb{R}^d)$ is continuusly included in the space $\dot H^{d\left(\frac{1}{2}-\frac{1}{p}\right)}$.*

Proof. The second part is easily deduced from the first one proceeding by duality. Let us write that

$$\|a\|_{\dot H^s} = \sup_{\|\varphi\|_{\dot H^{-s}(\mathbb{R}^d)} \leq 1} \langle a, \varphi \rangle.$$

As $s = d\left(\frac{1}{2} - \frac{1}{p}\right) = d\left(1 - \frac{1}{p} - \frac{1}{2}\right)$, we have by the first part

$$\|\varphi\|_{L^{\overline{p}}} \leq C \|\varphi\|_{\dot H^{-s}}$$

where $\overline{p}$ is the conjugate of p defined by $\frac{1}{p} + \frac{1}{\overline{p}} = 1$ and thus

$$\begin{aligned} \|a\|_{\dot H^s} &\leq C \sup_{\|\varphi\|_{L^{\overline{p}}} \leq 1} \langle a, \varphi \rangle \\ &\leq C \|a\|_{L^p}. \end{aligned}$$

This concludes the proof of the second part.

There are many different proofs of the first part. We shall use a frequency cut off argument which gives for free a refined version of this inequality which will be crucial in the second chapter. Let us introduce the following definition.

Definition 1.1.2. *Let θ be a function of $\mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\theta}$ be compactly supported, has value* 1 *near* 0 *and satisfies $0 \leq \widehat{\theta} \leq 1$. For u in $\mathcal{S}'(\mathbb{R}^d)$ and $\sigma > 0$, we set*

$$\|u\|_{\dot{B}^{-\sigma}} \stackrel{def}{=} \sup_{A>0} A^{d-\sigma} \|\theta(A\cdot) \star u\|_{L^\infty}.$$

The fact that $B^{-\sigma}$ is a Banach space is an exercice left to the reader. We shall see later on that the space is independant of the choice of the function θ. Let us observe that if u belongs to $\dot{H}^s$, then $\widehat{u}$ is locally in L^1 and the function $\widehat{\theta}(A^{-1}\cdot)\widehat{u}$ is in L^1. The inverse Fourier theorem implies that

$$\begin{aligned} \|A^d\theta(A\cdot) \star u\|_{L^\infty} &\leq (2\pi)^{-d} \|\widehat{\theta}(A^{-1}\cdot)\widehat{u}\|_{L^1} \\ &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{\theta}(A^{-1}\xi)|\xi|^{-s}|\xi|^s|\widehat{u}(\xi)|\, d\xi. \end{aligned}$$

Using the fact that $\widehat{\theta}$ is compactly supported, Cauchy-Schwarz inequality implies that

$$\|A^d\theta(A\cdot) \star u\|_{L^\infty} \leq \frac{C}{\left(\frac{d}{2} - s\right)^{\frac{1}{2}}} A^{\frac{d}{2}-s} \|u\|_{\dot{H}^s}.$$

This means exactly that the space $\dot{H}^s$ is continuously included in $\dot{B}^{s-\frac{d}{2}}$. By multiplication, we can assume that $\|u\|_{\dot{B}^{s-\frac{d}{2}}} = 1$. Then let us estimate $\|u\|_{L^p}$. We decompose the function u in low and high frequencies. More precisely, let us write

$$u = u_{\ell,A} + u_{h,A} \quad \text{with} \quad u_{\ell,A} = \mathcal{F}^{-1}(\widehat{\theta}(A^{-1}\cdot)\widehat{u}) \tag{1.1}$$

where θ is the function of Definition 1.1.2. The triangle inequality implies that

$$\big(|u| > \lambda\big) \subset \big(|u_{\ell,A}| > \lambda/2\big) \cup \big(|u_{h,A}| > \lambda/2\big)\cdot$$

By definition of $\|\cdot\|_{\dot{B}^{s-\frac{d}{2}}}$, we have $\|u_{\ell,A}\|_{L^\infty} \leq A^{\frac{d}{2}-s}$. From this, we deduce that

$$A = A_\lambda \stackrel{\text{def}}{=} \left(\frac{\lambda}{2}\right)^{\frac{p}{d}} \Longrightarrow \mu\big(|u_{\ell,A}| > \lambda/2\big) = 0.$$

We deduce that

$$\|u\|_{L^p}^p \leq p \int_0^\infty \lambda^{p-1} \mu\big(|u_{h,A_\lambda}| > \lambda/2\big)\, d\lambda.$$

Using that

$$\mu\big(|u_{h,A_\lambda}| > \lambda/2\big) \leq 4\frac{\|u_{h,A_\lambda}\|_{L^2}^2}{\lambda^2},$$

we get

$$\|u\|_{L^p}^p \leq 4p \int_0^\infty \lambda^{p-3} \|u_{h,A_\lambda}\|_{L^2}^2\, d\lambda.$$

Because the Fourier transform is (up to a constant) an isometry on $L^2(\mathbb{R}^d)$ and the function $\widehat{\theta}$ has value 1 near 0, we thus get for some $c > 0$ depending only on $\widehat{\theta}$,

$$\|u\|_{L^p}^p \leq 4p\,(2\pi)^{-d} \int_0^\infty \lambda^{p-3} \int_{(|\xi| \geq cA_\lambda)} |\widehat{u}(\xi)|^2 \, d\xi \, d\lambda \tag{1.2}$$

for some positive constant c. Now, by definition of A_λ, we have

$$|\xi| \geq cA_\lambda \Longleftrightarrow \lambda \leq C_\xi \stackrel{\text{def}}{=} 2\Big(\frac{|\xi|}{c}\Big)^{\frac{d}{p}}.$$

Fubini's theorem thus implies that

$$\begin{aligned} \|u\|_{L^p}^p &\leq 4p\,(2\pi)^{-d} \int_{\mathbb{R}^d} \Big(\int_0^{C_\xi} \lambda^{p-3} d\lambda\Big) |\widehat{u}(\xi)|^2 \, d\xi \\ &\leq (2\pi)^{-d} \frac{p2^p}{p-2} \int_{\mathbb{R}^d} \Big(\frac{|\xi|}{c}\Big)^{\frac{d(p-2)}{p}} |\widehat{u}(\xi)|^2 \, d\xi. \end{aligned}$$

As $s = d\Big(\dfrac{1}{2} - \dfrac{1}{p}\Big)$, the theorem is proved. □

1.2 Interpretation in terms of Besov spaces and oscillations

In fact the above proof tells more than the classical Sobolev theorem, namely the following theorem.

Theorem 1.2.1. *Let s be in $]0, d/2[$. There exists a constant C depending only on d and $\widehat{\theta}$ such that*

$$\|u\|_{L^p} \leq \frac{C}{(p-2)^{\frac{1}{p}}} \|u\|_{\dot{B}^{s-\frac{d}{2}}}^{1-\frac{2}{p}} \|u\|_{\dot{H}^s}^{\frac{2}{p}} \quad \textit{with} \quad p = \frac{2d}{d-2s} \cdot$$

Let us see what type of improvement it is compared with the classical inequality. Let φ be a given function in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and ω a unit vector of $\mathbb{R}^d$. Let us consider the family of functions defined by

$$u_\varepsilon(x) = e^{i\frac{(x|\omega)}{\varepsilon}} \varphi(x).$$

Let us prove that for any σ in $]0, d]$ we have

$$\|u_\varepsilon\|_{\dot{B}^{-\sigma}} \lesssim \varepsilon^\sigma. \tag{1.3}$$

By Hölder's inequality, we have

$$A^d \|\theta(A\cdot) \star \phi_\varepsilon\|_{L^\infty} \leq \|\theta\|_{L^1} \|\phi\|_{L^\infty}.$$

From this we deduce that, if $A\varepsilon \geq 1$ then we have

$$A^{d-\sigma} \|\theta(A\cdot) \star \phi_\varepsilon\|_{L^\infty} \leq \varepsilon^\sigma \|\theta\|_{L^1} \|\phi\|_{L^\infty}. \tag{1.4}$$

If $A\varepsilon \leq 1$, we perform integrations by parts. More precisely, using that

$$(-i\varepsilon\partial_1)^d e^{i\frac{x_1}{\varepsilon}} = e^{i\frac{x_1}{\varepsilon}}$$

and Leibniz formula, we get

$$\begin{aligned} A^d(\theta(A\cdot)\star\phi_\varepsilon)(x) &= (iA\varepsilon)^d\int_{\mathbb{R}^d}\partial^d_{y_1}\left(\theta(A(x-y))\phi(y)\right)e^{i\frac{y_1}{\varepsilon}}\,dy\\ &= (iA\varepsilon)^d\sum_{k\le d}\binom{d}{k}A^k((-\partial_1)^k\theta)(A\cdot)\star(e^{i\frac{y_1}{\varepsilon}}\partial_1^{d-k}\phi)(x). \end{aligned}$$

Using Hölder's inequalities, we get that

$$A^k\left\|((-\partial_1)^k\theta)(A\cdot)\star(e^{i\frac{y_1}{\varepsilon}}\partial_1^{d-k}\phi)\right\|_{L^\infty}\le\|\partial_1^k\theta\|_{L^{\frac{d}{k}}}\|\partial_1^{d-k}\phi\|_{L^{(\frac{d}{k})'}}.$$

Thus, we get $A^d\|\theta(A\cdot)\star\phi_\varepsilon\|_{L^\infty}\le C(A\varepsilon)^d$. As we are in the case when $A\varepsilon\le1$, we get, for any $\sigma\le d$,

$$A^d\|\theta(A\cdot)\star\phi_\varepsilon\|_{L^\infty}\le C(A\varepsilon)^\sigma.$$

Together with (1.4), this concludes the proof of Inequality (1.3).

Considering that $\|u_\varepsilon\|_{\dot H^s}\lesssim\varepsilon^{-s}$, then we can check that

$$\|u_\varepsilon\|^{1-\frac{2}{p}}_{\dot B^{s-\frac{d}{2}}}\|u_\varepsilon\|^{\frac{2}{p}}_{\dot H^s}\lesssim\varepsilon^{\left(\frac{d}{2}-s\right)\left(1-\frac{2}{p}\right)-\frac{2s}{p}}\sim1.$$

This shows that the refined inequality of Theorem 1.2.1 is invariant under translation in Fourier spaces (i.e. multiplication by oscillating functions).

The spaces defined in Definition 1.1.2 have a universal property: they are the biggest normed spaces which are translation invariant and which have the same scaling. More precisely we have the following proposition.

Proposition 1.2.1. *Let E be a norm space continuously included in the space of tempered distribution. Let us assume that the space E is globally invariant under dilations and translations and that a constant C and a positive real number σ exists such that*

$$\|u(\lambda\cdot-\overrightarrow{a})\|_E\le C\lambda^{-\sigma}\|u\|_E.$$

Then the space E is continuously embedded in $B^{-\sigma}$.

Proof. As E is continuously included in $\mathcal{S}'$, then we have

$$|\langle u,\theta\rangle|\le C\|u\|_E.$$

Because of the hypothesis on E, we get

$$|\langle u(A^{-1}\cdot+x),\theta\rangle|\le\|u(A^{-1}\cdot+x)\|_E\le CA^\sigma\|u\|_E.$$

As we have

$$\langle u(A^{-1}\cdot+x),\theta\rangle=A^d(\theta(A\cdot)\star u)(x),$$

we get the result. □

1.3 The link with Besov norms

The following proposition is an exercise about partition of unity.

Proposition 1.3.1. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d \,/\, 3/4 \leq |\xi| \leq 8/3\}$. There exist two radial functions χ and φ valued in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B(0,4/3))$ and to $\mathcal{D}(\mathcal{C})$, and such that*

$$\forall \xi \in \mathbb{R}^d\,,\ \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \tag{1.5}$$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}\,,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{1.6}$$

$$|j-j'| \geq 2 \Rightarrow \text{Supp }\ \varphi(2^{-j}\cdot) \cap \text{Supp }\ \varphi(2^{-j'}\cdot) = \emptyset, \tag{1.7}$$

$$j \geq 1 \Rightarrow \text{Supp }\ \chi \cap \text{Supp }\ \varphi(2^{-j}\cdot) = \emptyset, \tag{1.8}$$

the set $\widetilde{\mathcal{C}} \stackrel{\text{def}}{=} B(0,2/3) + \mathcal{C}$ is an annulus and we have

$$|j-j'| \geq 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} = \emptyset. \tag{1.9}$$

Besides, we have

$$\forall \xi \in \mathbb{R}^d\,,\ \frac{1}{2} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1, \tag{1.10}$$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}\,,\ \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{1.11}$$

Let us state the following definition.

Definition 1.3.1. *Let s be a real number, and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is the subset of distributions u of $\mathcal{S}'_h$ such that*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{rjs} \|\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}} < \infty.$$

Let us point out that in the case when $p=r=2$, this is homogeneous Sobolev spaces and in the case when $r=\infty$, this definition coincides with Definition 1.1.2.

Proposition 1.3.2. *The $\dot{H}^s$ norm and the $\dot{H}^s$ norm are equivalent. For any positive σ, the two norms $\dot{B}^{-\sigma}$ and $\dot{B}^{-\sigma}_{\infty,\infty}$ norm are equivalent. Moreover, we have, for s less than $d/2$,*

$$\|a\|_{\dot{B}^s_{2,\infty}} \lesssim \|a\|_{\dot{B}^{s-\frac{d}{2}}} \lesssim \|a\|_{\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}}.$$

Proof. It is possible to use Proposition 1.2.1. Let us give here a direct proof of this inequality. Let us write that

$$A^d \|\theta(A\cdot) \star u\|_{L^\infty} \leq (2\pi)^{-d} \|\widehat{\theta}(A^{-1}\cdot)\widehat{u}\|_{L^1}.$$

Because the Fourier transform of θ is compacly supported, we have

$$\widehat{\theta}(A^{-1}\cdot)\widehat{u} = \sum_{2^j \lesssim A} \widehat{\theta}(A^{-1}\cdot)\mathcal{F}(\Delta_j u).$$

Thus, we get

$$
\begin{aligned}
A^d\|\theta(A\cdot)\star u\|_{L^\infty} &\lesssim \sum_{2^j\lesssim A}\|\theta(A^{-1}\cdot)\mathcal{F}(\Delta_j u)\|_{L^1}\\
&\lesssim \sum_{2^j\lesssim A} 2^{j\frac{d}{2}}\|\Delta_j u\|_{L^2}\\
&\lesssim \biggl(\sum_{2^j\lesssim A} 2^{j\left(\frac{d}{2}-s\right)}\biggr)\|u\|_{\dot B^s_{2,\infty}}\\
&\lesssim A^{\frac{d}{2}-s}\|u\|_{\dot B^s_{2,\infty}}.
\end{aligned}
$$

This proves the proposition. □

In order to figure out the difference between the norm $\dot H^s=\dot B^s_{2,2}$ and the norm $\dot B^s_{2,\infty}$, let us consider the following example which based on the idea of lacunar series. Let us consider a function χ in $\mathcal{S}$ such that its Fourier transform is supported in a (small) ball of center 0 and radius ε_0. Let us consider, for a vector ω of $\mathbb{R}^d$ of Euclidian norm 3/2, the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined by

$$
f_n(x)=\sqrt{n}\sum_{j\geq n}2^{-js}\frac{1}{j+1}e^{i2^j(x|\omega)}\chi(x).
$$

As the support of the Fourier transform of the function $e^{i2^j(\cdot|\omega)}\chi$ is included in $2^j\omega+B(0,\varepsilon_0)$, we get

$$
\|f_n\|^2_{H^s}\sim n\sum_{j\geq n}\frac{1}{(j+1)^2}\sim 1 \quad\text{and}\quad \|f_n\|_{\dot B^s_{2,\infty}}\lesssim\frac{1}{\sqrt{n}}\,\cdot
$$

Chapter 2

The theory of profiles

2.1 The fundamental theorem about bounded sequences

Let us state the following definition

Definition 2.1.1. *We say that a sequence* $(\lambda_n^j, x_n^j)_{(j,n)\in\mathbb{N}^2}$ *of* $]0,\infty[\times\mathbb{R}^3$ *is a sequence of scales and cores if it satisfies*

$$j \neq k \Longrightarrow \quad \text{either} \ \lim_{n\to\infty}\left|\log\Big(\frac{\lambda_n^j}{\lambda_n^k}\Big)\right| = +\infty \quad \text{or} \quad \lambda_n^j = \lambda_n^k \ \text{ and } \lim_{n\to\infty}\frac{|x_n^j - x_n^k|}{\lambda_n^j} = \infty. \tag{2.1}$$

The following theorem has been proved by P. Gérard in [18] and describes, up to extraction, the defect of compactness of Sobolev embeddings.

Theorem 2.1.1. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a bounded sequence of* $\dot H^s$ *with* s *less than* $d/2$. *Then there exists a sequence of scales and cores* $(\lambda_n^j, x_n^j)_{(j,n)\in\mathbb{N}^2}$ *in the sense of Definition 2.1.1, a sequence* $(\varphi^j)_{j\in\mathbb{N}}$ *in* $\dot H^s$ *and a sequence* $(r_n^j)_{(j,n)\in\mathbb{N}^2}$ *of functions which satisfies, up to an extraction on* $(u_n)_{n\in\mathbb{N}}$, *the following properties:*

$$\forall J\in\mathbb{N}\,,\ u_{\phi(n)}(x) = \sum_{j=0}^{J}\frac{1}{(\lambda_n^j)^{\frac d2 - s}}\varphi^j\Big(\frac{x-x_n^j}{\lambda_n^j}\Big) + r_n^J(x)\,,$$

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|r_n^J\|_{\dot B^{s-\frac d2}_{\infty,\infty}} = 0 \quad \text{and}$$

$$\forall J\in\mathbb{N}\,,\ \lim_{n\to\infty}\bigg(\|u_n\|_{\dot H^s}^2 - \sum_{j=0}^{J}\|\varphi^j\|_{\dot H^s}^2 - \|r_n^J\|_{\dot H^s}^2\bigg) = 0.$$

Remarks The functions φ^j are called the profiles and they satisfies

$$\lim_{n\to\infty}(\lambda_n^j)^{\frac d2 - s}u_n(\lambda_n^j\cdot + x_n^j) \rightharpoonup \varphi^j$$

in the sense of distributions. In particular φ^0 is the weak limit of $(u_n)_{n\in\mathbb{N}}$.

Moreover, the refined Sobolev embeddings proved in Theorem 1.2.1 on page 9 implies that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|r_n^J\|_{L^p} = 0 \quad \text{with} \quad p = \frac{2d}{d-2s}\cdot$$

The above theorem has the following corollary.

Corollary 2.1.1. *Let us consider a bounded sequence $(u_n)_{n\in\mathbb{N}}$ of $\dot H^s$ such that for any sequence $(\lambda_n, x_n)_{n\in\mathbb{N}}$ of $]0,\infty[\times\mathbb{R}^d$, we have*

$$\lambda_n^{\frac{d}{2}-s} u_n(\lambda_n\cdot + x_n) \rightharpoonup 0$$

Then the infimum limit of $\|u_n\|_{L^p}$ is 0 for $p = \dfrac{2d}{d-2s}\cdot$

2.2 The extraction of the scales

Let us state some definitions.

Definition 2.2.1. *A scale is simply a sequence of positive real numbers. We say that two scales $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ are orthogonal (denotes by $(\lambda_n)_{n\in\mathbb{N}} \perp (\lambda'_n)_{n\in\mathbb{N}}$) if*

$$\lim_{n\to\infty}\Big|\log\Big(\frac{\lambda_n}{\lambda'_n}\Big)\Big| = +\infty.$$

The following proposition describes some effects of orthogonality or not orthogonality of scales.

Proposition 2.2.1. *Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ be two scales. If they are orthogonal, then*

$$\forall R>1\,,\ \exists n_R\,/\ \ n\geq n_R \Longrightarrow (\lambda'_n)^{-1}\mathcal{C}_R \subset \lambda_n^{-1}\mathcal{C}_R^c$$

where, as in all that follows, $\mathcal{C}_R \stackrel{\text{def}}{=} \{\xi\in\mathbb{R}^d\ /\ |\xi|\in[R^{-1},R]\}$.

If $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ are not orthogonal, an extraction ϕ exists such that

$$\frac{1}{C} \leq \frac{\lambda_{\phi(n)}}{\lambda'_{\phi(n)}} \leq C.$$

Proof. Let us observe, for any R greater than 1 and any n,

$$\frac{1}{R} \leq \lambda'_n|\xi| \leq R \Longleftrightarrow \frac{\lambda_n}{\lambda'_n R} \leq \lambda_n|\xi| \leq \frac{\lambda_n R}{\lambda'_n}\cdot \tag{2.2}$$

As the scales $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ are assumed to be orthogonal, it exists n_R such that

$$n\geq n_R \Longrightarrow \max\Big\{\frac{\lambda_n}{\lambda'_n},\frac{\lambda'_n}{\lambda_n}\Big\} \geq R^2.$$

Using (2.2), we observe that $\dfrac{\lambda_n}{\lambda'_n} \geq R^2$, then

$$\frac{1}{R} \leq \lambda'_n|\xi| \Longrightarrow \lambda_n|\xi| \geq R$$

and if $\dfrac{\lambda_n}{\lambda'_n} \geq R^2$, then

$$\lambda'_n|\xi| \leq R \Longrightarrow \lambda_n|\xi| \leq \frac{1}{R}$$

and the first assertion is proved.

The fact that the scales $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ are not orthogonal exactly means that

$$\liminf_{n\to\infty}\Big|\log\Big(\frac{\lambda_n}{\lambda'_n}\Big)\Big| = C<\infty.$$

Thus an extraction ϕ, exists such that, for any n large enough

$$C^{-\frac12}\leq \frac{\lambda_{\phi(n)}}{\lambda'_{\phi(n)}}\leq C^{\frac12}.$$

The proposition is proved. □

Let us define the concept of sequence which is oscillatory or unrelated with respect to a scale.

Definition 2.2.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in L^2 and $(\lambda_n)_{n\in\mathbb{N}}$ a scale. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $(\lambda_n)_{n\in\mathbb{N}}$-oscillatory if*

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\lambda_n^{-1}\mathcal{C}_R^c}|\widehat{f}_n(\xi)|^2d\xi=0.$$

The sequence $(f_n)_{n\in\mathbb{N}}$ is said $(\lambda_n)_{n\in\mathbb{N}}$-unrelated if

$$\forall R>1\,,\ \lim_{n\to\infty}\int_{\lambda_n^{-1}\mathcal{C}_R}|\widehat{f}_n(\xi)|^2d\xi=0.$$

A first remark is that, if $(f_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}}$-oscillatory, then the sequence $(g_n)_{n\in\mathbb{N}}$ defined by

$$g_n(x)\stackrel{\text{def}}{=}\lambda_n^{\frac d2}f_n(\lambda_n x)$$

is 1-oscillating. Let us notice that a typical example of a 1-oscillating sequence in a convergent sequence in $L^2(\mathbb{R}^d)$.

Now the following proposition translates in terms of orthogonality the properties of being oscillating or unrelated.

Proposition 2.2.2. *If $(f_n)_{n\in\mathbb{N}}$ is a $(\lambda_n)_{n\in\mathbb{N}}$ oscillating sequence and $(g_n)_{n\in\mathbb{N}}$ a $(\lambda_n)_{n\in\mathbb{N}}$ unrelated sequence, then*

$$\lim_{n\to\infty}(f_n|g_n)_{L^2}=0.$$

Proof. Let us write that

$$\begin{aligned}(2\pi)^d\big|(f_n|g_n)_{L^2}\big| &= \Big|\int_{\mathbb{R}^d}\widehat{f}_n(\xi)\overline{\widehat{g}}_n(\xi)d\xi\Big|\\ &\leq \Big|\int_{\lambda_n^{-1}\mathcal{C}_R^c}\widehat{f}_n(\xi)\overline{\widehat{g}}_n(\xi)d\xi\Big|+\Big|\int_{\lambda_n^{-1}\mathcal{C}_R}\widehat{f}_n(\xi)\overline{\widehat{g}}_n(\xi)d\xi\Big|\\ &\leq \|\widehat{f}_n\|_{L^2(\lambda_n^{-1}\mathcal{C}_R^c)}\|\widehat{g}_n\|_{L^2}+\|\widehat{f}_n\|_{L^2}\|\widehat{g}_n\|_{L^2(\lambda_n^{-1}\mathcal{C}_R)}.\end{aligned}$$

Let ε be a positive real number; because $(f_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}}$-oscillating, a radius R_ε exists such that

$$\limsup_{n\to\infty}\int_{\lambda_n^{-1}\mathcal{C}_{R_\varepsilon}^c}|\widehat{f}_n(\xi)|^2d\xi\leq\frac{\varepsilon}{2}\cdot$$

As $(g_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}}$-unrelated we have $\displaystyle\lim_{n\to\infty}\|\widehat{g}_n\|_{L^2(\lambda_n^{-1}\mathcal{C}_{R_\varepsilon})}=0$. Thus

$$\forall\varepsilon>0\,,\ \limsup_{n\to\infty}|(f_n|g_n)_{L^2}|\leq\varepsilon$$

and the result is proved. □

The following corollary will be useful.

Corollary 2.2.1. *Let (f_n^j, λ_n^j) be an element of $(L^2 \times]0,\infty[)^{\mathbb{N}^2}$ such that for any j, the sequence $(f_n^j)_{n\in\mathbb{N}}$ is $(\lambda_n^j)_{n\in\mathbb{N}}$-oscillating and for any j different from j' two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^{j'})_{n\in\mathbb{N}}$ are orthogonal. Then, for any J,*

$$\lim_{n\to\infty}\left(\Big\|\sum_{j=0}^{J} f_n^j\Big\|_{L^2}^2 - \sum_{j=0}^{J}\|f_n^j\|_{L^2}^2\right) = 0.$$

Proposition 2.2.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of L^2 and $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ two scales. Let us assume that $(f_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}}$ oscillating.*

If the scales $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ are orthogonal then $(f_n)_{n\in\mathbb{N}}$ is $(\lambda'_n)_{n\in\mathbb{N}}$ unrelated.

Conversely, if $\liminf\limits_{n\to\infty}\|f_n\|_{L^2}$ is positive and the sequence $(f_n)_{n\in\mathbb{N}}$ is $(\lambda'_n)_{n\in\mathbb{N}}$-unrelated, then the scales $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ are orthogonal.

Proof. Let R be a real number greater than 1 and a positive real number ε. Proposition 2.2.1 implies that

$$\forall R>1\,,\ \exists n_R\,/\ n\geq n_R \Longrightarrow \int_{(\lambda'_n)^{-1}\mathcal{C}_R}|\widehat{f}_n(\xi)|^2 d\xi \leq \int_{\lambda_n^{-1}\mathcal{C}_R^c}|\widehat{f}_n(\xi)|^2 d\xi. \tag{2.3}$$

As $(f_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}}$-oscillating, a real number R_ε (which can be choosen greater than R) exists such that

$$\limsup_{n\to\infty}\int_{\lambda_n^{-1}\mathcal{C}_R^c}|\widehat{f}_n(\xi)|^2 d\xi \leq \varepsilon.$$

This means that an interger $\widetilde{n}_\varepsilon$ (which can be choosen greater than n_R) exists such that

$$\forall n\geq \widetilde{n}_\varepsilon\,,\ \int_{\lambda_n^{-1}\mathcal{C}_{R_\varepsilon}^c}|\widehat{f}_n(\xi)|^2 d\xi \leq \varepsilon.$$

Using (2.3) allows to conclude that $(f_n)_{n\in\mathbb{N}}$ is $(\lambda'_n)_{n\in\mathbb{N}}$ -unrelated.

Now let us argue by contraposition. If $(\lambda_n)_{n\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ are not orthogonal, Proposition 2.2.1 claims that an extraction ϕ exists such that $\lambda_{\phi(n)}\equiv\lambda'_{\phi(n)}$. For the scale $(\lambda_{\phi(n)})_{n\in\mathbb{N}}$, the sequence $(f_{\phi(n)})_{n\in\mathbb{N}}$ is oscillating and unrelated. Proposition 2.2.2 implies that $\|f_{\phi(n)}\|_{L^2}$ tends to 0 when n tends to infinity.

□

The following proposition describes the relation between scales and Besov spaces.

Proposition 2.2.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of L^2. Then the sequence $(f_n)_{n\in\mathbb{N}}$ tends to 0 in $\dot{B}^0_{2,\infty}$ if and only if the sequence $(f_n)_{n\in\mathbb{N}}$ is unrelated to any scale $(\lambda_n)_{n\in\mathbb{N}}$.*

Proof. Let us assume that the sequence $(f_n)_{n\in\mathbb{N}}$ is unrelated to any scale $(\lambda_n)_{n\in\mathbb{N}}$. Let us observe that

$$\begin{aligned}\limsup_{n\to\infty}\|f_n\|_{\dot{B}^0_{2,\infty}} &= \limsup_{n\to\infty}\sup_{k\in\mathbb{Z}}\|\Delta_k f_n\|_{L^2}\\ &= \sup_{(k_n)\in\mathbb{Z}^{\mathbb{N}}}\lim_{n\to\infty}\|\Delta_{k_n} f_n\|_{L^2}.\end{aligned}$$

Let us consider any sequence $(k_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{Z}$. Let us define $\lambda_n = 2^{-k_n}$. By definition of the operator Δ_k, we have, for some R large enough,

$$\|\Delta_{k_n} f\|_{L^2}^2 \lesssim \int_{2^{k_n}\mathcal{C}_R} |\widehat{f}_n(\xi)|^2 d\xi.$$

As $(f_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}}$-unrelated then $\|\Delta_{k_n} f_n\|_{L^2}$ tends to 0.

Now let us assume that $(f_n)_{n\in\mathbb{N}}$ is not $(\lambda_n)_{n\in\mathbb{N}}$-unrelated for some scale $(\lambda_n)_{n\in\mathbb{N}}$. By rescaling, we can assume that $(f_n)_{n\in\mathbb{N}}$ is not 1-unrelated. This means that

$$\liminf_{n\to\infty} \int_{\mathcal{C}_R} |\widehat{f}_n(\xi)|^2 d\xi$$

is positive. As we have

$$\int_{\mathcal{C}_R} |\widehat{f}_n(\xi)|^2 d\xi \leq C_R \sup_{j'_R\leq j\leq j_R} \|\Delta_j f_n\|_{L^2}$$

we have $\liminf_{n\to\infty} \|f_n\|_{\dot{B}^0_{2,\infty}} > 0$ and the proposition is proved. □

The main theorem of this section of the following one.

Theorem 2.2.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of L^2. It exists an extraction ϕ, a sequence $(g_n^j, r_n^j, \lambda_n^j)_{(j,n)\in\mathbb{N}^2}$ of $L^2\times L^2\times]0,\infty[$ such that we have the following decomposition of $f_{\phi(n)}$. For any integer J, we have*

$$\forall n\geq J\,,\ f_{\phi(n)} - \sum_{j=0}^{J} g_n^j = r_n^J$$

with the following properties:

- *for any couple (j,j') with j different from j', the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^{j'})_{n\in\mathbb{N}}$ are orthogonal,*
- *for any j, the sequence $(g_n^j)_{n\in\mathbb{N}}$ is $(\lambda_n^j)_{n\in\mathbb{N}}$-oscillatory,*
- *The sequence $(r_n^j)_{(j,n)\in\mathbb{N}^2}$ tends to 0 in the sense that*

$$\lim_{j\to\infty}\limsup_{n\to\infty} \|r_n^j\|_{\dot{B}^0_{2,\infty}} = 0\,, \tag{2.4}$$

- *for any couple (j',j) such that j' is less than or equal to j, the sequence $(r_n^j)_{n\in\mathbb{N}}$ is $(\lambda_n^{j'})_{n\in\mathbb{N}}$-unrelated which implies that*

$$\forall J\in\mathbb{N}\,,\ \lim_{n\to\infty}\Big(\|f_{\phi(n)}\|_{L^2}^2 - \sum_{j=0}^{J}\|g_n^j\|_{L^2}^2 - \|r_n^J\|_{L^2}^2\Big) = 0.$$

Proof. The proof is based on the repeated application of the following lemma.

Lemma 2.2.1. *Let us consider $(f_n)_{n\in\mathbb{N}}$ a bounded sequence of L^2 such that*

$$\limsup_{n\to\infty} \|f_n\|_{\dot{B}^0_{2,\infty}} = L > 0.$$

Then a scale $(\lambda_n)_{n\in\mathbb{N}}$, a sequence $(g_n)_{n\in\mathbb{N}}$ and an extraction ϕ exist such that

- *the sequence $(g_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}}$-oscillating,*
- *the sequence $(f_{\phi(n)}-g_n)_{n\in\mathbb{N}}$ is $(\lambda_n)_{n\in\mathbb{N}}$-unrelated and for any scale $(\lambda'_n)_{n\in\mathbb{N}}$ such that the sequence $(f_n)_{n\in\mathbb{N}}$ is $(\lambda'_n)_{n\in\mathbb{N}}$-unrelated, the sequence $(f_{\phi(n)}-g_n)_{n\in\mathbb{N}}$ is also $({\lambda'}_{\phi(n)})_{n\in\mathbb{N}}$-unrelated,*
- *we have $\displaystyle\lim_{n\to\infty}\|g_n\|_{L^2}\geq\frac{L}{2}\cdot$*

Proof. By definition of the Besov norm $\|\cdot\|_{\dot B^0_{2,\infty}}$, the hypothesis implies that that a sequence of integers $(k_n)_{n\in\mathbb{N}}$ exists such that , for large enough n,

$$\|\Delta_{k_n}f_n\|_{L^2}\geq\frac{L}{2}\cdot \tag{2.5}$$

Now let us consider the scale $\lambda_n\stackrel{\rm def}{=}2^{-k_n}$and let us consider the sequence of functions $(F_n)_{n\in\mathbb{N}}$ defined by

$$F_n\left\{\begin{array}{ccc}]1,\infty[& \longrightarrow & [0,\infty[\\ R & \longmapsto & \|\mathbb{1}_{\lambda_n^{-1}\mathcal{C}_R}\widehat{f}_n\|^2_{L^2}.\end{array}\right.$$

For any n, the function F_n is non decreasing and, for any R, $F_n(R)$ is less than or equal to the supremum of $\|f_n\|_{L^2}$. Helly's lemma implies that an extraction ϕ_1 and a (of course non decreasing) function exist such that

$$\forall R>1\,,\ \lim_{n\to\infty}F_{\phi_1(n)}(R)=F(R).$$

This implies that

$$\forall n\,,\ \exists m_n\,/\ \forall m\geq m_n\,,\ |F_{\phi_1(m)}(n)-F(n)|\leq\frac{1}{n+1}\cdot$$

Let us define by induction by $\phi_2(n)\stackrel{\rm def}{=}\max\{\phi_1(n-1)+1,m_n\}$ and state $\phi\stackrel{\rm def}{=}\phi_1\circ\phi_2$. Now we can define the sequence $(g_n)_{n\in\mathbb{N}}$ by

$$\widehat{g}_n\stackrel{\rm def}{=}\mathbb{1}_{\lambda^{-1}_{\phi(n)}\mathcal{C}_n}\widehat{f}_{\phi(n)}.$$

Let us check that $(g_n)_{n\in\mathbb{N}}$ satisfies the conclusions of the lemma. Let us observe that

$$n\geq R\Longrightarrow\mathbb{1}_{\mathcal{C}^c_R}=\mathbb{1}_{\mathcal{C}_n}-\mathbb{1}_{\mathcal{C}_R}.$$

Thus by definition of ϕ and F_n, we get, for n greater than or equal to R,

$$\begin{aligned}\int_{\lambda_n^{-1}\mathcal{C}^c_R}|\widehat{g}_n(\xi)|^2d\xi &= \int_{\lambda_n^{-1}\mathcal{C}_n}|\widehat{g}_n(\xi)|^2d\xi-\int_{\lambda_n^{-1}\mathcal{C}_R}|\widehat{g}_n(\xi)|^2d\xi\\ &= F_{\phi(n)}(n)-F_{\phi(n)}(R).\end{aligned} \tag{2.6}$$

The function F is bounded and non decreasing. Let us denote by F_∞ its limits at infinity and let us write

$$|F_{\phi(n)}(n)-F_\infty|\leq|F_{\phi(n)}-F(n)|+F_\infty-F(n)\leq\frac{1}{n+1}+F_\infty-F(n).$$

thus we have $\lim_{n\to\infty} F_{\phi(n)}(n) = F_\infty$. Then we deduce from (2.6) that

$$\lim_{n\to\infty} \int_{\lambda_n^{-1}\mathcal{C}_R^c} |\widehat{g}_n(\xi)|^2 d\xi = F_\infty - F(R).$$

Thus $(g_n)_{n\in\mathbb{N}}$ is $(\lambda_{\phi(n)})_{n\in\mathbb{N}}$-oscillating.

Moreover, if n is greater or equal to R, the ring $\mathcal{C}_R$ is included in the ring $\mathcal{C}_n$. Thus we have

$$\mathbb{1}_{\lambda_{\phi(n)}^{-1}\mathcal{C}_R}(\widehat{f}_{\phi(n)} - \widehat{g}_n) = \mathbb{1}_{\lambda_{\phi(n)}^{-1}\mathcal{C}_R}\big(\mathbb{1}_{\lambda_{\phi(n)}^{-1}\mathcal{C}_n}\widehat{f}_{\phi(n)} - \widehat{g}_n\big) = 0.$$

Thus $(f_{\phi(n)} - g_n)_{n\in\mathbb{N}}$ is $(\lambda_{\phi(n)})_{n\in\mathbb{N}}$-unrelated.

By definition of g_n and λ_n, we have

$$\begin{aligned}\|\widehat{g}_n\|_{L^2}^2 &= \int_{\lambda_{\phi(n)}^{-1}\mathcal{C}_n} |\widehat{f}_{\phi(n)}(\xi)|^2 d\xi \\ &= \int_{2^{k_{\phi(n)}}\mathcal{C}_n} |\widehat{f}_{\phi(n)}(\xi)|^2 d\xi.\end{aligned}$$

If n is large enough, the ring $\mathcal{C}_n$ contains the support of the smooth cut-off function of the Littlewood-Paley theory. Thus, because of (2.5), we have, for large enough n,

$$\|g_n\|_{L^2} \geq \frac{L}{2}\cdot$$

Up the an extraction, we can assume that $\|g_n\|_{L^2}$ converges and the lemma is proved. □

Conclusion of the proof of Theorem 2.2.1. We proceed by induction. For an integer J let us define the property $\mathcal{P}_J$ as for any j, it exists a family $(\phi_j)_{0\leq j\leq J}$ of extractions, families of sequences of functions $(g_n^j)_{n\in\mathbb{N}}$ and $(r_n^j)_{n\in\mathbb{N}}$ such that

- Up to extraction, the sequence $(f_n)_{n\in\mathbb{N}}$ can be decompose as

$$f_{\phi_0\circ\phi_1\cdots\circ\phi_J(n)} = \sum_{j=0}^{J-1} g^j_{\phi_{j+1}\circ\cdots\circ\phi_J(n)} + g_n^J + r_n^J,$$

- for any couple (j', j) such that j' is different from j, the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^{j'})_{n\in\mathbb{N}}$ are orthogonal,

- for any couple (j', j) such that j' is less than or equal to j, the sequence $(r_n^j)_{n\in\mathbb{N}}$ is $({\lambda^{j'}}_{\phi_{j'}\circ\cdots\circ\phi_j(n)})_{n\in\mathbb{N}}$-unrelated,

- for any j, the sequence $(g_n^j)_{n\in\mathbb{N}}$ is $(\lambda_n^j)_{n\in\mathbb{N}}$-oscillatory,

$$\lim_{n\to\infty} \|g_n^j\|_{L^2} \geq \frac{1}{2}\limsup_{n\to\infty} \|r_n^{j-1}\|_{\dot{B}^0_{2,\infty}}. \tag{2.7}$$

Let us prove $\mathcal{P}_{J+1}$. If $\liminf_{n\to\infty} \|r_n^J\|_{\dot{B}^0_{2,\infty}} = 0$, then we choose

$$g_n^{J+1} = r_n^{J+1} = 0 \quad \text{and} \quad \phi_{J+1}(n) = n.$$

If $\liminf_{n\to\infty} \|r_n^J\|_{\dot{B}^0_{2,\infty}}$ is positive, then we apply Lemma 2.2.1 to the sequence $(r_n^J)_{n\in\mathbb{N}}$ which provides the existence of an extraction ϕ_{J+1}, and sequences $(g_n^{J+1})_{n\in\mathbb{N}}$ and $(r_n^{J+1})_{n\in\mathbb{N}}$, and a scale $(\lambda_n^{J+1})_{n\in\mathbb{N}}$ which satisfies $\mathcal{P}_{J+1}$.

Now let us prove Assertion (2.4). Because of Proposition 2.2.2, we have, for any J,

$$\lim_{n\to\infty}\left(\|f_{\phi_0\circ\phi_1\cdots\circ\phi_J(n)}\|^2_{L^2} - \sum_{j=0}^{J-1}\|g^j_{\phi_{j+1}\circ\cdots\circ\phi_J(n)}\|^2_{L^2} - \|g_n^J\|^2_{L^2} - \|r_n^J\|^2_{L^2}\right) = 0.$$

This implies that

$$\sup_n \|f_n\|^2_{L^2} \geq \sum_{j=0}^{J-1}\|g^j_{\phi_{j+1}\circ\cdots\circ\phi_J(n)}\|^2_{L^2} + \|g_n^J\|^2_{L^2} + o_n^J(1)$$

with $\lim_{n\to\infty} o_n^J(1) = 0$. Passing to the limit when n tends to infinity gives thanks to Inequality (2.7)

$$\frac{1}{4}\sum_{j=0}^{J-1}\limsup_{n\to\infty}\|r^{j-1}_{\phi_{j+1}\circ\cdots\circ\phi_J(n)}\|^2_{\dot{B}^0_{2,\infty}} \leq \sup_n \|f_n\|_{L^2}$$

which obviously implies that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|r^{j-1}_{\phi_{j+1}\circ\cdots\circ\phi_J(n)}\|_{\dot{B}^0_{2,\infty}} = 0.$$

Now we use the diagonal process and define

$$\begin{aligned}
\phi(n) &\stackrel{\text{def}}{=} \phi_0\circ\cdots\circ\phi_n(n)\\
\widetilde{g}_n^j &\stackrel{\text{def}}{=} g^j_{\phi_{j+1}\circ\cdots\circ\phi_n(n)} \text{ for } n\geq j \quad\text{and}\quad 0 \text{ if } n<j\,,\\
\widetilde{r}_n^j &\stackrel{\text{def}}{=} r^j_{\phi_{j+1}\circ\cdots\circ\phi_n(n)} \text{ for } n\geq j \quad\text{and}\quad 0 \text{ if } n<j\,.
\end{aligned}$$

Then, for any n greater than J, we have

$$\forall n\geq J\,,\ f_{\phi(n)} = \sum_{j=0}^{J}\widetilde{g}_n^j + r_n^J$$

and the theorem is proved. □

2.3 The extraction of the cores for 1-oscillating sequences

The purpose of this section is the proof of the following theorem which describes, up to an extraction, the structure of the 1-oscillating sequences.

Theorem 2.3.1. *Let $(g_n)_{n\in\mathbb{N}}$ be a 1-oscillating sequence. It exists an extraction ϕ, two sequences $(\psi^k)_{k\in\mathbb{N}}$ and $(r_n^k)_{n\in\mathbb{N}}$ of L^2 functions and a sequence $(x_n^k)_{n\in\mathbb{N}}$ of points of $\mathbb{R}^d$ such that*

- *for any integer K, $g_{\phi(n)} = \sum_{k=0}^{K}\psi^k(x-x_n^k) + R_n^K$,*

- *for any couple (k',k) such that k and k' are different, $|x_n^k - x_n^{k'}|$ tends to ∞ when n tends to infinity,*
- *we have*

$$\lim_{n\to\infty}\Bigl(\|g_{\phi(n)} - \sum_{k=0}^{K}\|\psi^k\|_{L^2}^2 - \|R_n^K\|_{L^2}^2\Bigr) = 0 \quad \textit{and} \quad \lim_{K\to\infty}\limsup_{n\to\infty}\|R_n^K\|_{\dot B^{-\frac d2}_{\infty,\infty}} = 0.$$

Proof. It relies on the repeated application of the following lemma.

Lemma 2.3.1. *A positive constant c_0 exists such that for any 1-oscillating sequence $(g_n)_{n\in\mathbb N}$, it exists an extraction ϕ, a function ψ and a sequence $(x_n)_{n\in\mathbb N}$ such that $x_n\equiv 0$ or $|x_n|$ tends to infinity when n tends to infinity which satisifies*

$$g_{\phi(n)}(x_n+\cdot)\rightharpoonup\psi \quad \textit{and} \quad \|\psi\|_{L^2}\geq c_0\limsup_{n\to\infty}\|g_n\|_{\dot B^{-\frac d2}_{\infty,\infty}}\cdot$$

Proof. If $(g_n)_{n\in\mathbb N}$ tends to 0 in the space $\dot B^{-\frac d2}_{\infty,\infty}$, there is nothing to prove because any element ψ of the weak closure of $(g_n)_{n\in\mathbb N}$ works with $x_n\equiv 0$. Let us assume that $(g_n)_{n\in\mathbb N}$ does not tend to 0 in the space $\dot B^{-\frac d2}_{\infty,\infty}$. Up to an extraction we omit to note, we can assume that

$$\lim_{n\to\infty}\|g_n\|_{\dot B^{-\frac d2}_{\infty,\infty}} = L>0.$$

By definition of the Besov norm $\dot B^{-\frac d2}_{\infty,\infty}$, it implies that a sequence $(\ell_n,\widetilde x_n)_{n\in\mathbb N}$ of elements of $\mathbb Z\times\mathbb R^d$ exists such that

$$2^{-\ell_n\frac d2}|\Delta_{\ell_n}g_n(\widetilde x_n)|\geq\frac L2 \tag{2.8}$$

if n is large enough. The fact that $(g_n)_{n\in\mathbb N}$ is 1-oscillating implies that the sequence $(\ell_n)_{n\in\mathbb N}$ takes only a finite number of values. Indeed, Bernstein inequality implies that

$$2^{-\ell_n\frac d2}\|\Delta_{\ell_n}g_n\|_{L^\infty}\leq C_0\|\Delta_{\ell_n}g_n\|_{L^2}. \tag{2.9}$$

As $(g_n)_{n\in\mathbb N}$ is 1-oscillatory, a radius R_0 greater than 1 exists such that

$$\limsup_{n\to\infty}\int_{\mathcal C^c_{R_0}}|\widehat g_n(\xi)|^2d\xi\leq\Bigl(\frac{L}{4C_0}\Bigr)^2\cdot$$

Moreover, a constant C_1 exists such that

$$|\ell|\geq C_1\log R_0\Longrightarrow\int_{\mathcal C_{R_0}}|\mathcal F(\Delta_\ell g_n)(\xi)|^2d\xi=0.$$

Thus, an integer n_R exists such that

$$\Bigl(n\geq n_R \quad\text{and}\quad |\ell_n|\geq C_1\log R_0\Bigr)\Longrightarrow 2^{-\ell_n\frac d2}\|\Delta_{\ell_n}g_n\|_{L^\infty}\leq\frac{L}{4C_0}\cdot$$

Inequality (2.9) implies the sequence $(\ell_n)_{n\in\mathbb N}$ defining by (2.8) takes only a finite number of values. Thus, up to an extraction we omit to note, we can assume that the sequence $(\ell_n)_{n\in\mathbb N}$ is constant and equal to some integer k. By definition of the operator Δ_k, we have

$$2^{-k\frac d2}\Delta_k g_n(\widetilde x_n) = 2^{-k\frac d2}2^{kd}\int_{\mathbb R^d}h(2^k(\widetilde x_n-y))g_n(y)dy.$$

As h is even, let us change variable $z = -2^k(\widetilde{x}_n - y)$ in the above integral. This gives

$$2^{-k\frac{d}{2}}\Delta_k g_n(\widetilde{x}_n) = 2^{-k\frac{d}{2}}\int_{\mathbb{R}^d} h(z) g_n(\widetilde{x}_n + 2^{-k}z)dz.$$

The sequence $\big(2^{-k\frac{d}{2}} g_n(\widetilde{x}_n + 2^{-k}\cdot)\big)_{n\in\mathbb{N}}$ is bounded in L^2. Thus an extraction ϕ exists and a function $\widetilde{\psi}$ such that the sequence $\big(2^{-k\frac{d}{2}} g_{\phi(n}(\widetilde{x}_{\phi(n} + 2^{-k}\cdot)\big)_{n\in\mathbb{N}}$ converges weakly to $\widetilde{\psi}$. Then we deduce that

$$g_{\phi(n)}(\widetilde{x}_{\phi(n)} + \cdot) \rightharpoonup \psi \quad \text{with} \quad \psi(x) \stackrel{\text{def}}{=} 2^{k\frac{d}{2}}\widetilde{\psi}(2^k x).$$

Now let us remark that by definition of k and of the sequence $(\widetilde{x}_n)_{n\in\mathbb{N}}$, we have

$$\begin{aligned} \frac{L}{2} &\leq 2^{-k\frac{d}{2}}\big|\Delta_k g_{\phi(n)}(\widetilde{x}_{\phi(n)})\big| \\ &\leq \big|\langle 2^{-k\frac{d}{2}} g_{\phi(n)}(\widetilde{x}_{\phi(n)} + 2^{-k}\cdot), h\rangle\big|. \end{aligned}$$

Passing to the limit when n tends to infinity gives

$$\frac{L}{2} \leq \big|\langle\widetilde{\psi}, h\rangle\big| \leq \|h\|_{L^2}\|\widetilde{\psi}\|_{L^2}.$$

Once observed that $\|\widetilde{\psi}\|_{L^2} = \|\psi\|_{L^2}$, we define $x_n = \widetilde{x}_{\phi(n)}$ and then get the lemma with the constant c_0 equal to $\dfrac{1}{2\|h\|_{L^2}}$ up to the fact that $x_n \equiv 0$ or $|x_n|$ tends to infinity when n tends to infinity.

If $|x_n|$ does not tend to infinity, then, up to an extraction we omit to note, we can assume that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to some point x_∞ of $\mathbb{R}^d$. Let us write that, for any test function θ in L^2, we have

$$\int_{\mathbb{R}^d}\big(g_{\phi(n)}(x_n + x) - g_{\phi(n)}(x_\infty + x)\big)\theta(x)dx = \int_{\mathbb{R}^d} g_{\phi(n)}(x_\infty + x)\big(\theta(x + x_\infty - x_n) - \theta(x)\big)dx.$$

Because $(g_n)_{n\in\mathbb{N}}$ is bounded on L^2, the fact that translation are continuous on L^2 ensures that the sequence $\big(g_{\phi(n)}(x_n + \cdot) - g_{\phi(n)}(x_\infty + \cdot)\big)_{n\in\mathbb{N}}$ tends weakly to 0. This concludes the proof of the lemma. □

Continuation of the proof of Theorem 2.3.1 By repeated application of the above lemma, let us define a sequence $(\phi_k)_{k\in\mathbb{N}}$, a sequence $(\psi^k)_{k\in\mathbb{N}}$, a sequence $(x_n^k)_{(k,n)\in\mathbb{N}^2}$ of points of $\mathbb{R}^d$, a sequence $(R_n^k)_{(k,n)\in\mathbb{N}^2}$ which satisfies $R_0 = g_n$, $\psi^0 = 0$, $x_n^0 = 0$ and $\phi^0 = \mathrm{Id}$ and

$$R_n^k = R_{\phi_k(n)}^{k-1} - \psi^k(\cdot - x_n^k)\,,\ \|\psi^k\|_{L^2} \geq c_0 \limsup_{n\to\infty}\|R_n^{k-1}\|_{\dot{B}^{-\frac{d}{2}}_{\infty,\infty}}\,,\ R_n^k(x_n^k + \cdot) \rightharpoonup 0 \qquad (2.10)$$

Let us prove that

$$\forall k' < k\,,\ \lim_{n\to\infty}\big|x_n^k - x^{k'}_{\phi_{k'+1}\circ\cdots\circ\phi_k(n)}\big| = \infty. \qquad (2.11)$$

Let us first prove it for $k' = k-1$. By definition (2.10) of the sequences, we have

$$R_{\phi_k(n)}^{k-1}(x_n^k + \cdot) \rightharpoonup \psi^k \quad \text{and} \quad R_n^{k-1}(x_n^{k-1} + \cdot) \rightharpoonup 0.$$

Property (2.11) for $k' = k-1$ will follow from the following easy lemma, the proof of which is omitted.

Lemma 2.3.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of L^2 which converges weakly to some function f which is not the zero function. Let $(y_n)_{n\in\mathbb{N}}$ be any sequence of $\mathbb{R}^d$. Then the sequence $(f_n(y_n+\cdot))_{n\in\mathbb{N}}$ tends weakly to 0 if and only if $|y_n|$ tends to infinity.*

Now let us proceed by induction assuming that

$$\forall k' \,/\, k-p \leq k' < k\,,\ \lim_{n\to\infty} \big|x_n^k - x^{k'}_{\phi_{k'+1}\circ\cdots\circ\phi_k(n)}\big| = \infty. \tag{2.12}$$

By Definition (2.10) of the sequences, we have

$$R^{k-p-1}_{\phi_{k-p}\circ\cdots\circ\phi_k(n)} = R_n^k + \sum_{p'=0}^{p-1} \psi^{k-p'}\big(\cdot - x^{k-p'}_{\phi_{k-p'}\circ\cdots\circ\phi_k(n)}\big). \tag{2.13}$$

Using that $(R_n^k(x_n^k+\cdot))_{n\in\mathbb{N}}$ tends weakly to 0, and the induction hypothesis (2.12), we get

$$R^{k-p-1}_{\phi_{k-p}\circ\cdots\circ\phi_k(n)}(x_n^k+\cdot) \rightharpoonup \psi^k \quad \text{and} \quad R^{k-p-1}_{\phi_{k-p}\circ\cdots\circ\phi_k(n)}(x_n^{k-p-1}+\cdot) \rightharpoonup 0.$$

Lemma 2.3.2 ensures (2.11).

Moreover, we have

$$\|R^{k-1}_{\phi_k(n)}\|^2_{L^2} = \|\psi^k\|^2_{L^2} + \|R_n^k\|^2_{L^2} + 2\big(\psi^k\big|R_n^k(x_n^k+\cdot)\big)_{L^2}.$$

Using that $(R_n^k(x_n^k+\cdot))_{n\in\mathbb{N}}$ tends weakly to 0, we get that

$$\lim_{n\to\infty}\big(\|R^{k-1}_{\phi_k(n)}\|^2_{L^2} - \|\psi^k\|^2_{L^2} - \|R_n^k\|^2_{L^2}\big) = 0.$$

Using Formula (2.13), we get by iteration that, for any k,

$$\lim_{n\to\infty}\Big(\|g_{\phi_1\circ\cdots\circ\phi_k(n)}\|^2_{L^2} - \sum_{k'=1}^{k}\|\psi^k\|^2_{L^2} - \|R_n^k\|^2_{L^2}\Big) = 0.$$

As the sequence $(g_n)_{n\in\mathbb{N}}$ is bounded in L^2, the series $(\|\psi^k\|^2_{L^2})_{k\in\mathbb{N}}$ is convergent. Using that

$$\|\psi^k\|_{L^2} \geq c_0 \limsup_{n\to\infty}\|R^{k-1}_{\phi_k(n)}\|_{\dot B^{-\frac{d}{2}}_{\infty,\infty}}.$$

As a term of a square convergent series, we have that,

$$\lim_{k\to\infty}\limsup_{n\to\infty}\|R^{k-1}_{\phi_k(n)}\|_{\dot B^{-\frac{d}{2}}_{\infty,\infty}} = 0.$$

Now let us argue with a diagonal argument by defining $\phi(n) \stackrel{\text{def}}{=} \phi_1\circ\cdots\circ\phi_n(n)$. For any n greater than k, we have

$$g_{\phi(n)}(x) = \sum_{k=0}^{K}\psi^k(x - x^k_{\phi_{k+1}\cdots\circ\phi_n(n)}) + R^K_{\phi_{k+1}\cdots\circ\phi_n(n)}.$$

Defining $\widetilde{x}_n^k = x^k_{\phi_{k+1}\cdots\circ\phi_n(n)}$ and $\widetilde{R}_n^K = R^K_{\phi_{k+1}\cdots\circ\phi_n(n)}$ allows to conclude the proof of Theorem 2.3.1.

Conclusion of the proof of Theorem 2.1.1 First, let us apply Theorem 2.2.1 to the sequence $(\Lambda^s f_n)_{n\in\mathbb{N}}$ where $\Lambda^s f \stackrel{\text{def}}{=} \mathcal{F}^{-1}(|\cdot|^s \widehat{f})$. This provides the existence of a sequence of an extraction ϕ_0, two by two orthogonal scales $(\widetilde{\lambda}_n^j)_{(j,n)\in\mathbb{N}^2}$, a sequence $(g_n^j)_{(j,n)\in\mathbb{N}}$ of fonctions in L^2 such that, for any j, the sequence $(g_n^j)_{n\in\mathbb{N}}$ is $(\widetilde{\lambda}_n^j)_{n\in\mathbb{N}}$-oscillating, a sequence $(r_n^j)_{(j,n)\in\mathbb{N}^2}$ such that

$$\begin{gathered}\Lambda^s f_{\phi_0(n)} = \sum_{j=0}^{J} g_n^j + r_n^J \quad \text{with} \\ \lim_{n\to\infty}\Big(\Lambda^s f_{\phi_0(n)} - \sum_{j=0}^{J}\|g_n^j\|_{L^2}^2 - \|r_n^J\|_{L^2}^2\Big) = 0 \quad \text{and} \quad \lim_{J\to\infty}\limsup_{n\to\infty}\|r_n^J\|_{\dot{B}^0_{2,\infty}} = 0.\end{gathered} \tag{2.14}$$

Theorem 2.3.1 implies that a sequence $(\phi_j)_{j\in\mathbb{N}}$ of extraction, two sequences of functions of L^2 $(\widetilde{\psi}^{j,k})_{(j,k)\in\mathbb{N}^2}$ and $(\widetilde{R}_n^k)_{(k,n)\in\mathbb{N}^2}$, a sequence $(\widetilde{x}_n^k)_{(k,n)\in\mathbb{N}^2}$ of points of $\mathbb{R}^d$ such that, for any j, we have

$$\begin{gathered}g^j_{\phi_1\circ\cdots\circ\phi_j(n)}(x) = \sum_{k=1}^{K}\frac{1}{(\widetilde{\lambda}_n^j)^{\frac{d}{2}}}\widetilde{\psi}^{j,k}\Big(\frac{x-\widetilde{x}_n^{j,k}}{\widetilde{\lambda}_n^j}\Big) + \frac{1}{(\widetilde{\lambda}_n^j)^{\frac{d}{2}}}\widetilde{R}_n^{j,K}\Big(\frac{x}{\widetilde{\lambda}_n^j}\Big) \quad \text{with} \\ \lim_{n\to\infty}\Big(\|g^j_{\phi_1\circ\cdots\circ\phi_j(n)}\|_{L^2}^2 - \sum_{k=0}^{K}\|\widetilde{\psi}^{j,k}\|_{L^2}^2 - \|\widetilde{R}_n^{j,K}\|_{L^2}^2\Big) = 0 \quad \text{and} \\ \lim_{K\to\infty}\limsup_{n\to\infty}\|\widetilde{R}_n^{j,K}\|_{\dot{B}^{-\frac{d}{2}}_{\infty,\infty}} = 0.\end{gathered} \tag{2.15}$$

Now let us define

$$\begin{aligned}\phi(n) &\stackrel{\text{def}}{=} \phi_0\circ\phi_1\cdots\circ\phi_n(n)\,,\\ \lambda_n^j &\stackrel{\text{def}}{=} \widetilde{\lambda}^j_{\phi_{j+1}\circ\cdots\circ\phi_n(n)} \quad \text{if} \quad n\geq j+1 \quad \text{and} \quad 0 \quad \text{if not}\\ x_n^{j,k} &\stackrel{\text{def}}{=} \widetilde{x}^{j,k}_{\phi_{j+1}\circ\cdots\circ\phi_n(n)} \quad \text{if} \quad n\geq j+1 \quad \text{and} \quad 0 \quad \text{if not}\\ \psi^{j,k} &= \Lambda^{-s}\widetilde{\psi}^{j,k} \quad \text{and}\\ \rho_n^{J,K} &\stackrel{\text{def}}{=} \Lambda^{-s}r_n^J + \sum_{j=0}^{J}\frac{1}{(\lambda_n^j)^{s-\frac{d}{2}}}(\Lambda^{-s}\widetilde{R}^{j,k}_{\phi_{j+1}\circ\cdots\circ\phi_n(n)})\Big(\frac{\cdot}{\lambda_n^j}\Big)\,.\end{aligned}$$

Using (2.14) and (2.15), we have, for n greater than j

$$f_{\phi(n)} = \sum_{j=0}^{J}\sum_{k=1}^{K}\frac{1}{(\lambda_n^j)^{\frac{d}{2}-s}}\psi^{j,k}\Big(\frac{\cdot - x_n^{j,k}}{\lambda_n^j}\Big) + \rho_n^{J,K}.$$

We get the theorem by reordering the sequences.

□

Chapter 3

Some basic facts about the Navier-Stokes equation

3.1 The concept of solutions and some historical results

Let us define the concept of (weak) solutions of the incompressible Navier-Stokes system. Let us first recall what the incompressible Navier Stokes system is. We consider as unknown the speed $u = (u^1, u^2, u^3)$ a time dependant divergence free vector field on $\mathbb{R}^3$ and the pressure p. We consider the system

$$
(NS) \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p + f \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0. \end{cases}
$$

Let us immediatly notice that this system has two fundamental properties. The first one is the energy inequality. Formally, and in the case when $f = 0$, let us take the L^2 scalar product with u in the equation. We get

$$
\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} (u(t) \cdot \nabla u(t) | u(t))_{L^2} - \int_{\mathbb{R}^3} (\nabla p(t) | u(t))_{L^2}.
$$

Thanks to the divergence free condition, obvious integration by parts implies that, any function vector field a

$$
(u \cdot \nabla a | a)_{L^2} = 0 = (\nabla p | u)_{L^2} = 0 \tag{3.1}
$$

This gives

$$
\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = 0. \tag{3.2}
$$

The second one is the scaling invariance. It is easy to see that if u is a (smooth) solution of (NS) on $[0, T] \times \mathbb{R}^3$ with pressure p associated with the initial data u_0, then, for any positive λ, the vector field and the pressure

$$
u_\lambda(t, x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x)
$$

is a solution of (NS) on the interval $[0, \lambda^{-2}T] \times \mathbb{R}^3$ associated with the initial data $\lambda u_0(\lambda x)$.

The notion of C^2 solution (i.e. classical solution) is not efficient because singularity can appear here and also we can be interested in rough initial data. It has been pointed out by C. Oseen in the beginning of the 20th century (see [27] and [28]) that another concept of solution must be used. This has been formalized by J. Leray in 1934 in his seminal work [24]. Let us define the notion of weak solution (that we shall denote simply solution in all that follows).

Definition 3.1.1. *A time-dependent vector field u with components in $L^2_{loc}([0,T]\times\mathbb{R}^d)$ is a weak solution (simply a solution in these notes) of (NS) if for any smooth compactly supported divergence free vector field Ψ,*

$$\begin{aligned}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u^j(t,x)\Psi^j(t,x)dx = \sum_{j=1}^{3}\int_{\mathbb{R}^3} u^j_0(x)\Psi^j(0,x)dx\\ +\sum_{j=1}^{3}\int_0^t\int_{\mathbb{R}^3} u^j(t',x)\big(\partial_t\Psi^j(t',x)+\Delta\Psi^j(t',x)\big)dt'dx\\ +\sum_{j,k}\int_0^t\int_{\mathbb{R}^3}(u^ju^k)(t',x)\partial_j\Psi^k(t',x)\,dt'dx+\sum_{j=1}^{3}\int_0^t\langle f^j(t'),u^j(t')\rangle dt'.\end{aligned}$$

This definition is too weak in the sense there is not enough constraints on the solution. In particular it ignores the fundamental concept of energy. J. Leray introduced in his seminal paper [24] the concept of turbulent solution.

Definition 3.1.2. *A turbulent solution of (NS) is a divergence free vector field u which is a weak solution, has component in $L^\infty_T(L^2)\cap L^2_T(H^1)$ and satisfies in addition the energy inequality*

$$\frac{1}{2}\|u(t)\|^2_{L^2}+\int_0^t\|\nabla u(t')\|^2_{L^2}dt'\leq\frac{1}{2}\|u_0\|^2_{L^2}+\int_0^t\int_{\mathbb{R}^3}f(t',x)\cdot u(t',x)dt'dx. \tag{3.3}$$

Remark For a turbulent solution, Definition 3.1.1 of a weak solution becomes

$$\begin{aligned}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u^j(t,x)\Psi^j(t,x)dx = \sum_{j=1}^{3}\int_{\mathbb{R}^3} u^j_0(x)\Psi^j(0,x)dx\\ +\sum_{j=1}^{3}\int_0^t\int_{\mathbb{R}^3} u^j(t',x)\partial_t\Psi^j(t',x)dt'dx-\sum_{j,k}\int_0^t\int_{\mathbb{R}^3}\partial_k u^j(t',x)\partial_k\Psi^j(t',x)\big)dt'dx\\ +\sum_{j,k}\int_0^t\int_{\mathbb{R}^3}(u^ju^k)(t',x)\partial_j\Psi^k(t',x)\,dt'dx+\sum_{j=1}^{3}\int_0^t\langle f^j(t'),u^j(t')\rangle dt'.\end{aligned}$$

In [24], J. Leray proved the following theorem.

Theorem 3.1.1. *Let u_0 be a divergence free vector field in $L^2(\mathbb{R}^d)$. Then a turbulent solution u exists on $\mathbb{R}^+\times\mathbb{R}^3$.*

The proof of this theorem relies on compactness methods and thus no uniqueness is proved.

In this text, we are going to focus on solution which are regular enough to be unique in their own class. For this type of solutions let us state a theorem of existence of solutions by J. Leray which he called semi-regular solutions.

Theorem 3.1.2. *Let u_0 be a divergence free vector field in $L^2(\mathbb{R}^3)$ such that ∇u_0 belongs to $L^2(\mathbb{R}^3)$. Then a positive time, which can be choosen greater or equal to $\rho_0\|\nabla u_0\|_{L^2}^{-4}$ for some constant ρ_0, exists such that a unique solution u exists in $C([0,T];\dot{H}^1)\cap L^2([0,T];\dot{H}^2)$. Moreover a constant c_1 exists such that if*

$$\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} \leq c_1,$$

then T can be choosen equal to ∞.

Proof. We simply prove a formal control on the $\|\nabla u(t)\|_{L^2}$ norm. By differentiation of the equation, we get

$$\partial_t\partial_j u + u\cdot\nabla\partial_j u - \Delta\partial_j u = -\nabla p_j - \partial_j u\cdot\nabla u.$$

Taking the L^2 scalar product of the equation and summing in j gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 = \sum_{j=1}\int_{\mathbb{R}^3}\partial_j u(t,x)\cdot\nabla u(t,x)\partial_j u(t,x)dx.$$

Sobolev embeddings (see Theorem 1.1.1 on page 7) implies that

$$\begin{aligned}\int_{\mathbb{R}^3}\partial_j u(t,x)\cdot\nabla u(t,x)\partial_j u(t,x)dx &\leq \|\nabla u(t)\|_{L^3}^3\\ &\lesssim \|\nabla u(t)\|_{\dot{H}^{\frac{1}{2}}}^3.\end{aligned}$$

The interpolation inequality between Sobolev spaces (see Proposition 1.1.1 on page 7) implies that

$$\int_{\mathbb{R}^3}\partial_j u(t,x)\cdot\nabla u(t,x)\partial_j u(t,x)dx \lesssim \|\nabla u(t)\|_{L^2}^{\frac{3}{2}}\|\nabla^2 u(t)\|_{L^2}^{\frac{3}{2}}.$$

We shall vey often use the familiar convexity inequality

$$ab \leq \theta a^{\frac{1}{\theta}} + (1-\theta)b^{\frac{1}{1-\theta}}.$$

Used with $\theta = 1/4$ this gives

$$\int_{\mathbb{R}^3}\partial_j u(t,x)\cdot\nabla u(t,x)\partial_j u(t,x)dx \leq \frac{1}{2}\|\nabla^2 u(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^2}^6.$$

This gives

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 \lesssim \|\nabla u(t)\|_{L^2}^6.$$

Thus, as long as $\|\nabla u(t)\|_{L^2} \leq 2\|\nabla u_0\|_{L^2}$, we have

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 + Ct\|\nabla u_0\|_{L^2}^6.$$

Thus for $t \leq \dfrac{1}{2C}\|\nabla u_0\|_{L^2}^4$, the quantity $\|\nabla u(t)\|_{L^2}^2$ remains less than or equal to $2\|\nabla u_0\|_{L^2}^2$. This proves the local part of the theorem. In order to treat the case of small initial data, we estimate the term $(\partial_j u\cdot u|\partial_j u)_{L^2}$ in a different way. By integration by parts, let us write that

$$\begin{aligned}\int_{\mathbb{R}^3}\partial_j u(t,x)\cdot\nabla u(t,x)\partial_j u(t,x)dx = &-\int_{\mathbb{R}^3}u(t,x)\cdot\nabla\partial_j u(t,x)\partial_j u(t,x)dx\\ &-\int_{\mathbb{R}^3}u(t,x)\cdot\nabla u(t,x)\partial_j^2 u(t,x)dx.\end{aligned}$$

As u is divergence free, we get that

$$\int_{\mathbb{R}^3} u(t,x) \cdot \nabla \partial_j u(t,x) \partial_j u(t,x) dx = 0.$$

Using Hölder inequality and Sobolev embeddings, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_j u(t,x) \cdot \nabla u(t,x) \partial_j u(t,x) dx &\leq \|u(t)\|_{L^3} \|\nabla u(t)\|_{L^6} \|\nabla^2 u(t)\|_{L^2} \\ &\lesssim \|u(t)\|_{L^3} \|\nabla^2 u(t)\|_{L^2}^2. \end{aligned}$$

Using again Sobolev embeddings and interpolation inequality, we infer that

$$\|u(t)\|_{L^3} \lesssim \|u(t)\|_{L^2}^{\frac{1}{2}} \|\nabla u(t)\|_{L^2}^{\frac{1}{2}}.$$

The energy inequality implies that

$$\|u(t)\|_{L^3} \leq C_0 \|u_0\|_{L^2}^{\frac{1}{2}} \|\nabla u(t)\|_{L^2}^{\frac{1}{2}}.$$

Now, as long as

$$C_0 \|u_0\|_{L^2}^{\frac{1}{2}} \|\nabla u(t)\|_{L^2}^{\frac{1}{2}} \leq \frac{1}{2},$$

we have

$$\frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 \leq 0.$$

In particular $\|\nabla u(t)\|_{L^2}^2$ is a decreasing function. Thus, if it is small enough at initial time, it remains small and the theorem is proved. □

Corollary 3.1.1. *A constant c_0 exists such that if u_0 is a divergence free vector field in $L^2(\mathbb{R}^3)$ such that if ∇u_0 belongs to $L^2(\mathbb{R}^3)$ and if the maximal time of existence $T^\star$ of the solution associated with u_0 is finite, then we have*

$$\forall t \in [0, T^\star[\,,\ \|\nabla u(t)\|_{L^2}^4 \geq c_0 (T^\star - t)^{-1} \quad \text{and} \quad T^\star \leq c_0^{-1} \|u_0\|_{L^2}^4.$$

Proof. Applying this bound on the life span with $u(t)$ as an initial data gives the first inequality. Then the energy estimate implies that

$$\begin{aligned} (T^\star)^{\frac{1}{2}} &= 2 \int_0^{T^\star} \frac{1}{(T^\star - t)^{\frac{1}{2}}} dt \\ &\leq \frac{2}{\sqrt{c_0}} \int_0^{T^\star} \|\nabla u(t)\|_{L^2}^2 dt \\ &\leq \frac{1}{\sqrt{c_0}} \|u_0\|_{L^2}^2. \end{aligned}$$

The corollary is proved. □

3.2 The Kato method; the case of $\dot{H}^{\frac{1}{2}}$ initial data

Let us first define operators which we are going to use in this chapter.

Definition 3.2.1. *Let τ be a non negative real number. We denote by $L_0^{(\tau)}$ the operator defined by the fact that $L_0^{(\tau)} f$ is the solution of*

$$\begin{cases} \partial_t L_0^{(\tau)} f - \Delta L_0^{(\tau)} f = f - \nabla p \\ \operatorname{div} L_0^{(\tau)} f = 0 \quad \text{and} \quad L_0^{(\tau)} f_{|t=\tau} = 0. \end{cases}$$

Let us also define for k in $\{1, 2, 3\}$, the operator $L_k^{(\tau)}$ by

$$\begin{cases} \partial_t L_k^{(\tau)} f - \Delta L_k^{(\tau)} f = \partial_k f - \nabla p \\ \operatorname{div} L_k^{(\tau)} f = 0 \quad \text{and} \quad L_k^{(\tau)} f_{|t=\tau} = 0. \end{cases}$$

In the case when $\tau = 0$, we simply note $L_j^{(0)} = L_j$. Let us also define the bilinear operator B by

$$B(u, v) = -\frac{1}{2} L_0 \Big(\sum_{k=1}^{3} \partial_k (v^k u + u^k v) \Big) = -\frac{1}{2} \sum_{k=1}^{3} L_k (v^k u + u^k v).$$

Let us remark that if u and v are divergence free, we also have

$$B(u, v) = -\frac{1}{2} L_0 \big(v \cdot \nabla u + u \cdot \nabla v \big) \tag{3.4}$$

It is obvious that u a solution of (NS) if and only if u satisfies

$$u = e^{t\Delta} u_0 + B(u, u).$$

Solving (NS) is equivalent to find a fixed point for the map

$$u \longmapsto e^{t\Delta} u_0 + B(u, u).$$

Now let us assume that we have a Banach space X of functions locally in L^2 on $\mathbb{R}^+ \times \mathbb{R}^3$ such that B is a bilinar map from $X \times X$ into X. Then Picard's fixed point theorem implies the existence of a unique solution. Such a space X will be called "adapted".

Let us remark there is a strong constrain on X due to the scaling property. If X adapted, it must be scaling invariant (and also translation invariant) in the sense that

$$\forall \lambda > 0\,,\ \forall \overrightarrow{a} \in \mathbb{R}^3\,,\ u \in X \Longleftrightarrow u\big(\lambda^2 t, \lambda(\cdot - \overrightarrow{a})\big) \in X \quad \text{and} \quad \|u\|_X \sim \lambda \big\| u\big(\lambda^2 t, \lambda(\cdot - \overrightarrow{a})\big) \big\|_X.$$

Let us give a first example of an adapted spaces: the space $L^4(\mathbb{R}^+; \dot{H}^1)$.

The wellposedness of (NS) for initial data in the space $\dot{H}^{\frac{1}{2}}$ is described by the following theorem.

Theorem 3.2.1. *Let u_0 be a divergence free vector field in $\dot{H}^{\frac{1}{2}}$. A positive time T exists such that the system (NS) has a unique solution u in $L^4([0, T]; \dot{H}^1)$ which also belongs to*

$$C([0, T]; \dot{H}^{\frac{1}{2}}) \cap L^2([0, T]; \dot{H}^{\frac{3}{2}}).$$

A constant ρ_0 exists if $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq \rho_0$, then the solution is global and satisfies

$$\|u\|^2_{L^\infty(\mathbb{R}^+; \dot{H}^{\frac{1}{2}})} + \|u\|^2_{L^2(\mathbb{R}^+; \dot{H}^{\frac{3}{2}})} \leq \|u_0\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Proof. It relies on the following lemma.

Lemma 3.2.1. *The operator L_0 maps continously the space $L^2([0,T];\dot H^{-\frac12})$ into the space*

$$C([0,T];\dot H^{\frac12})\cap L^2([0,T];\dot H^{\frac32}).$$

Proof. In Fourier space, we can write that

$$\mathcal{F}L_0 f(t,\xi)=\int_0^t e^{-(t-t')|\xi|^2}\,\mathbb{P}(\xi)\widehat{f}(t',\xi)dt'$$

where $\mathbb{P}(\xi)$ is the orthogonal projection in $\mathbb{R}^3$ on the orthogonal of ξ. Thus, we get

$$|\mathcal{F}L_0 f(t,\xi)|\leq \int_0^t e^{-(t-t')|\xi|^2}|\xi|^{\frac12}\theta(t',\xi)\|f(t',\cdot)\|_{\dot H^{-\frac12}}dt'$$

with $\|\theta(t',\cdot)\|_{L^2}=1$ for any t' of $\mathbb{R}^+$. Convolution inequality gives

$$|\xi|\,\|\mathcal{F}L_0 f(\cdot,\xi)\|^2_{L^\infty([0,T])}+|\xi|^3\|\mathcal{F}L_0 f(\cdot,\xi)\|^2_{L^2([0,T])}\lesssim\int_0^T\theta^2(t',\xi)\|f(t',\cdot)\|^2_{H^{-\frac12}}dt'.$$

Taking the L^2 in ξ norm in the above inequality gives the result except the continuity. Let us assume that t_1 and t_2 are two points of $[0,T]$ such that $t_1\leq t_2$. Thus, we have

$$L_0 f(t_2)=e^{(t_2-t_1)\Delta}L_0 f(t_1)+L_k^{(t_1)}f(t_2-t_1). \tag{3.5}$$

Thus if t_2 is such that $\|f\|_{L^2([t_1,t_2];\dot H^{-\frac12})}\leq\varepsilon$ is small, then

$$\|L_0 f(t_2)-e^{(t_2-t_1)\Delta}L_0 f(t_1)\|_{\dot H^{\frac12}}\lesssim\varepsilon.$$

The continuity of the heat flow on $\dot H^{\frac12}$ allows to conclude the proof. □

Conclusion of the proof of Theorem 3.2.1 Let us observe that dual Sobolev embedding and Sobolev embedddding imply that

$$\begin{aligned}\|u\cdot\nabla v+v\cdot\nabla u\|_{\dot H^{-\frac12}} &\lesssim \|u\cdot\nabla v+v\cdot\nabla u\|_{L^{\frac32}}\\ &\lesssim \|u\|_{L^6}\|\nabla v\|_{L^2}+\|v\|_{L^6}\|\nabla u\|_{L^2}\\ &\lesssim \|\nabla u\|_{L^2}\|\nabla v\|_{L^2}.\end{aligned}\tag{3.6}$$

We infer that the bilinear operator B maps the space $L^4([0,T];\dot H^1)\times L^4([0,T];\dot H^1)$ into the space $C([0,T];\dot H^{\frac12})\cap L^2([0,T];\dot H^{\frac32})$. As we have

$$\mathcal{F}(e^{t\Delta}u_0)(\xi)=e^{-t|\xi|^2}\widehat{u}_0(\xi),$$

we infer that

$$\|\mathcal{F}(e^{t\Delta}u_0)(\xi)\|_{L^4(\mathbb{R}^+)}\leq|\xi|^{-\frac12}|\widehat{u}_0(\xi)|$$

Because of Minkowski inequality, we get

$$\|e^{t\Delta}u_0\|_{L^4(\mathbb{R}^+;\dot H^1)}\lesssim\|u_0\|_{\dot H^{\frac12}}.$$

Thus if $\|u_0\|_{\dot H^{\frac12}}$ is small enough, then a unique global solution exists in $L^4(\mathbb{R}^+;\dot H^1)$. If $\|u_0\|_{\dot H^{\frac12}}$ is not small the result is only local. Indeed let us define

$$u_0=u_{0,\flat}+u_{0,\sharp}\quad\text{with}\quad u_{0,\flat}\stackrel{\text{def}}{=}\mathcal{F}^{-1}(\mathbf{1}_{B(0,\lambda_0)}\widehat{u}_0). \tag{3.7}$$

We have

$$\|e^{t\Delta}u_0\|_{L^4([0,T];\dot H^1)} \lesssim \|u_{0,\flat}\|_{\dot H^{\frac12}} + T^{\frac14}\lambda_0^{\frac12}\|u_0\|_{\dot H^{\frac12}}.$$

We can choose λ_0 such that $\|u_{0,\flat}\|_{\dot H^{\frac12}}$ small which proves the theorem up to the last inequality. In order to prove it, let us perform a $\dot H^{\frac12}$ energy estimate which gives

$$\frac12\frac{d}{dt}\|u(t)\|^2_{\dot H^{\frac12}} + \|\nabla u(t)\|^2_{\dot H^{\frac12}} = (u\cdot\nabla u|u)_{\dot H^{\frac12}}.$$

As we have

$$\begin{aligned}(a|b)_{\dot H^{\frac12}} &= \int_{\mathbb{R}^3} |\xi|^{-\frac12}\widehat a(\xi)|\xi|^{\frac32}\widehat b(-\xi)d\xi\\ &\leq \|a\|_{\dot H^{-\frac12}}\|\nabla b\|_{\dot H^{\frac12}}\,,\end{aligned}$$

Inequality (3.6) together with an interpolation argument gives that

$$\begin{aligned}(u\cdot\nabla u|u)_{\dot H^{\frac12}} &\leq C\|\nabla u\|^2_{L^2}\|\nabla u\|_{\dot H^{\frac12}}\\ &\leq C\|u\|_{\dot H^{\frac12}}\|\nabla u\|^2_{\dot H^{\frac12}}.\end{aligned}$$

Thus as long as $C\|u\|_{\dot H^{\frac12}} \leq 1/2$, we have

$$\frac{d}{dt}\|u(t)\|^2_{\dot H^{\frac12}} + \|\nabla u(t)\|^2_{\dot H^{\frac12}} \leq 0.$$

Thus if $\|u_0\|_{\dot H^{\frac12}} \leq (2C_0)^{-1}$, we get the required inequality and the theorem is proved. □

Remark Because the $\dot H^{\frac12}$ norm is scaling invariant, the life span cannot be bounded from below by a function of the norm. Let us notice that even if we assume that u_0 belongs to $L^2\cap\dot H^1$ the best known life span is the one coming from the arguments we give here in the proof of Theorem 3.1.2.

Now let us establish a criteria for blow up of the regularity. In other words, it is a necessary condition for the appearance of singularities.

Proposition 3.2.1. *If u_0 is in $\dot H^{\frac12}$ and if $T^\star$ the maximal time of existence of a solution in the space $L^\infty_{loc}[0,T^\star[;\dot H^{\frac12})\cap L^2_{loc}(T^\star;\dot H^{\frac32})$ is finite. Then we have*

$$\forall (p,q)\in[2,\infty[\times]3,\infty]\;/\;\frac2p+\frac3q=1\,,\;\int_0^{T^\star}\|u(t)\|^p_{L^q}dt=\infty. \tag{3.8}$$

Proof. Let us write that

$$\frac12\frac{d}{dt}\|u(t)\|^2_{\dot H^{\frac12}} + \|\nabla u(t)\|^2_{\dot H^{\frac12}} = -(u\cdot\nabla u|u)_{\dot H^{\frac12}}.$$

Using that, if a and b are real valued, we have

$$\begin{aligned}(a|b)_{\dot H^{\frac12}} &= \int_{\mathbb{R}^3}\widehat a(\xi)|\xi|\widehat b(-\xi)d\xi\\ &= \big(a\,\big|\,|D|b\big)_{L^2}\quad\text{with}\quad |D|b\stackrel{\text{def}}{=}\mathcal{F}^{-1}(|\cdot|\widehat b).\end{aligned}$$

Thus we have

$$(u\cdot\nabla u|u)_{\dot H^{\frac12}} = \int_{\mathbb{R}^3} u(x)\cdot\nabla u(x)(|D|u)(x)dx.$$

Let $\widetilde{q}$ defined by

$$\frac{1}{q}+\frac{2}{\widetilde{q}}=1.$$

Hölder inequality, Sobolev embedding and interpolation inequality give using the relation between p and q,

$$\begin{aligned}(u\cdot\nabla u|u)_{\dot H^{\frac12}} &\lesssim \|u\|_{L^q}\|\nabla u\|_{L^{\widetilde{q}}}\||D|u\|_{L^{\widetilde{q}}}\\ &\lesssim \|u\|_{L^q}\|\nabla u\|^2_{\dot H^{\frac{3}{2q}}}\\ &\lesssim \|u\|_{L^q}\|\nabla u\|^2_{\dot H^{\frac12-\frac1p}}\\ &\lesssim \|u\|_{L^q}\|u\|^{\frac2p}_{\dot H^{\frac12}}\|\nabla u\|^{\frac{2}{p'}}_{\dot H^{\frac12}}.\end{aligned}$$

Then using the convexity inequality and Relation (3.8), we infer that

$$(u\cdot\nabla u|u)_{\dot H^{\frac12}}\leq\frac12\|\nabla u(t)\|^2_{\dot H^{\frac12}}+C\|u(t)\|^p_{L^q}\|u(t)\|^2_{\dot H^{\frac12}}.$$

By Gronwall lemma, we deduce that

$$\|u(t)\|^2_{\dot H^{\frac12}}+\int_0^t\|\nabla u(t')\|^2_{\dot H^{\frac12}}dt'\leq\|u_0\|^2_{\dot H^{\frac12}}\exp\Big(C\int_0^t\|u(t')\|^p_{L^q}dt'\Big).$$

The theorem is proved. □

As a conclusion of this introduction, let us introduce the set of initial data which gives birth to global regular solution.

Definition 3.2.2. *Let us denote by $\mathcal{G}$ the set of initial data u_0 such that the solution u given by Theorem 3.2.1 is global i.e. belongs to $L^4_{\rm loc}(\mathbb{R}^+;\dot H^1)$.*

We denote as $\rho_{\rm c}$ the supremum of the positive real number ρ such that if $\|u_0\|_{\dot H^{\frac12}}<\rho$ implies that u_0 gives birth to a global solution in the space in $L^4_{loc}(\mathbb{R}^+;\dot H^1)$.

3.3 Global stability results in $\dot H^{\frac12}$

The basic theorem in this section is the following.

Theorem 3.3.1. *Let u be a global solution of (NS) in $L^\infty_{loc}(\mathbb{R}^+;\dot H^{\frac12})\cap L^4_{loc}(\mathbb{R}^+;\dot H^1)$. Then we have*

$$\lim_{t\to\infty}\|u(t)\|_{\dot H^{\frac12}}=0\quad\textit{and}\quad\int_0^\infty\|u(t)\|^4_{\dot H^1}dt<\infty.$$

Proof. We shall decompose the initial data u_0 as a sum of a low and a high frequency part. A positive real number ρ being given, let us state

$$u_0=u_{0,\sharp}+u_{0,\flat}\quad\text{with}\quad u_{0,\flat}\stackrel{\rm def}{=}\mathcal{F}^{-1}(\mathbf{1}_{B(0,r)}(\xi)\widehat{u}_0(\xi)).$$

Let ε be any positive real number. We can choose r such that

$$\|u_{0,\flat}\|_{\dot H^{\frac12}}\leq\min\{\rho_0,\varepsilon/2\}.$$

where ρ_0 is the constant of Theorem 3.2.1. Let us denote by $u_\flat$ the global solution of (NS) given by Theorem 3.2.1 for the initial data $u_{0,\flat}$. We have

$$\|u_\flat\|^2_{L^\infty(\mathbb{R}^+;\dot H^{\frac12})} + \|u_\flat(t)\|^2_{L^2(\mathbb{R}^+;\dot H^{\frac32})} \leq \|u_{0,\flat}\|^2_{\dot H^{\frac12}}. \tag{3.9}$$

Let us define $u_\sharp \stackrel{\text{def}}{=} u - u_\flat$. Let us notice that $u_\sharp$ is globally defined because so are u and $u_\flat$. It satisfies

$$\begin{cases} \partial_t u_\sharp - \Delta u_\sharp + (u_\sharp + u_\flat)\cdot\nabla u_\sharp + u_\sharp\cdot\nabla u_\flat = -\nabla p \\ u_{\sharp|t=0} = u_{0,\sharp}. \end{cases}$$

By energy estimate, we infer

$$\frac12\|u_\sharp(t)\|^2_{L^2} + \int_0^t \|\nabla u_\sharp(t')\|^2_{L^2}dt' = \frac12\|u_{0,\sharp}\|^2_{L^2} + \int_0^t (u_\sharp\cdot\nabla u_\flat|u_\sharp)dt'.$$

Using Sobolev embedding, we claim that

$$\begin{aligned} \big|\langle u_\sharp(t)\cdot\nabla u_\flat(t), u_\sharp(t)\rangle\big| &\leq \|u_\sharp(t)\cdot\nabla u_\flat(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \\ &\leq \|u_\sharp(t)\|_{L^6}\|u_\flat(t)\|_{L^3}\|\nabla u_\sharp(t)\|_{L^2} \\ &\leq C\|u_\flat(t)\|_{\dot H^{\frac12}}\|\nabla u_\sharp(t)\|^2_{L^2}. \end{aligned}$$

Then we deduce that

$$\frac12\|u_\sharp(t)\|^2_{L^2} + \int_0^t \|\nabla u_\sharp(t')\|^2_{L^2}dt' \leq \frac12\|u_{0,\sharp}\|^2_{L^2} + C\|u_{0,\flat}\|_{\dot H^{\frac12}}\int_0^t \|\nabla u_\sharp(t')\|^2_{L^2}dt'.$$

Choosing ε small enough ensures that

$$\|u_\sharp(t)\|^2_{L^2} + \int_0^t \|\nabla u_\sharp(t')\|^2_{L^2}dt' \leq \|u_{0,\sharp}\|^2_{L^2}.$$

This implies that a positive time t_ε exists such that $\|u_\sharp(t_\varepsilon)\|_{\dot H^{\frac12}} < \varepsilon/2$. Thus $\|u(t_\varepsilon)\|_{\dot H^{\frac12}}$ is less than ε. Then Theorem 3.2.1 allows to conclude the proof. □

Let us remark that the set $\mathcal{G}$ contains the open ball of radius ρ_c and centered at origin. Let us state the following corollary of Theorem 3.3.1.

Theorem 3.3.2. *The set $\mathcal{G}$ is an open connected subset of $\dot H^{\frac12}$.*

Proof. Let us consider u_0 in $\dot H^{\frac12}$ such that the associated solution is global. Let us consider w_0 in $\dot H^{\frac12}$ and the (a priori) local solution v associated with the initial data $v_0 \stackrel{\text{def}}{=} u_0 + w_0$. The function $w \stackrel{\text{def}}{=} v - u$ is solution of

$$\begin{cases} \partial_t w - \Delta w + u\cdot\nabla w + w\cdot\nabla u + w\cdot\nabla w = -\nabla p \\ w_{|t=0} = w_0. \end{cases}$$

Sobolev embeddings together with interpolation inequality gives

$$\begin{aligned} \big|(u\cdot\nabla w + w\cdot\nabla u|w)_{\frac12}\big| &\leq C\|u\|_{\dot H^1}\|w\|^{\frac12}_{\dot H^{\frac12}}\|\nabla w\|^{\frac32}_{\dot H^{\frac12}} \quad \text{and} \\ \big|(w\cdot\nabla w|w)_{\frac12}\big| &\leq C\|w\|_{\dot H^{\frac12}}\|\nabla w\|^2_{\dot H^{\frac12}}. \end{aligned}$$

Let us assume that $\|w_0\|_{\dot H^{\frac12}} \leq \dfrac{1}{8C}$ and define

$$T_{w_0} \stackrel{\text{def}}{=} \sup\Big\{t \,/ \max_{0\leq t'\leq t} \|w(t')\|_{\dot H^{\frac12}} \leq \frac{1}{4C}\Big\}\cdot$$

Thanks to the convexity inequality we infer that, for any $t < T_{w_0}$,

$$\|w(t)\|^2_{\dot H^{\frac12}} + \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac12}} dt' \leq \|w_0\|^2_{\dot H^{\frac12}} + C\int_0^t \|u(t')\|^4_{\dot H^1} \|w(t')\|^2_{\dot H^{\frac12}} dt'.$$

Gronwall's Lemma and Theorem 3.3.1 imply that, for any $t < T_{w_0}$,

$$\|w(t)\|^2_{\dot H^{\frac12}} + \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac12}} dt' \leq \|w_0\|^2_{\dot H^{\frac12}} \exp\Big(C\int_0^\infty \|u(t)\|^4_{\dot H^1} dt\Big).$$

If the smallness condition

$$\|w_0\|^2_{\dot H^{\frac12}} \exp\Big(C\int_0^\infty \|u(t)\|^4_{\dot H^1} dt\Big) \leq \frac{1}{16C^2}\,, \tag{3.10}$$

is satisfied, the blow up condition for v is never satisfied. Thus $\mathcal{G}$ is open.

The fact that $\mathcal{G}$ is connected is due to the fact that as $\lim_{t\to\infty} u(t) = 0$ in $\dot H^{\frac12}$, any u_0 in $\mathcal{G}$ is connnected to 0. Thus the corollary is proved. $\square$

3.4 The Kato theory in the L^p framework

The purpose of this section is the proof of the folllowing theorem.

Theorem 3.4.1. *Let u_0 be a divergence free vector field in L^3. A positive time T exists such that the system (NS) has a unique solution u in $L^5([0,T]\times\mathbb{R}^3)$ which also belongs to $C([0,T];L^3)$. Moreover, a constant ρ_1 exists if $\|u_0\|_{L^3} \leq \rho_1$, then the solution is global and satisfies*

$$\|u\|_{L^\infty(\mathbb{R}^+;L^3)} + \|u\|_{L^5(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim \|u_0\|_{L^3}.$$

Proof. It relies on the following lemma.

Lemma 3.4.1. *For j in $\{1,2,3\}$, we have*

$$L_j f(x) = \sum_{k=1}^3 \int_0^t \int_{\mathbb{R}^3} \Gamma^j_k(t-t', x-y) f(t',y)dydt'.$$

where the functions $\Gamma^j_{k,\ell}$ belongs to the space $L^{\frac54}_w$ and satisfies

$$|\Gamma^j_{k,\ell}(\tau,z)| \lesssim \frac{1}{(\sqrt{\tau}+|z|)^4}\,\cdot$$

Proof. In Fourier space, we have

$$\mathcal{F}L_j f(t,\xi) = i\int_0^t e^{-(t-t')|\xi|^2} \sum_{k,\ell} \alpha_{j,k\ell}\xi_j\xi_k\xi_\ell|\xi|^{-2}\widehat{f}(t',\xi)dt'.$$

In order to write this operator as a convolution operator, it is enough to compute the inverse Fourier transform of $\xi_j\xi_k\xi_\ell|\xi|^{-2}e^{-t|\xi|^2}$. Using the fact that

$$e^{-t|\xi|^2}|\xi|^{-2} = \int_t^\infty e^{-t'|\xi|^2}dt',$$

we get that

$$\begin{aligned}\Gamma^j_{k,\ell}(t,x) &= i\int_t^\infty \int_{\mathbb{R}^3} \xi_j\xi_k\xi_\ell e^{i(x|\xi)-t'|\xi|^2}dt'd\xi\\ &= \partial_j\partial_k\partial_\ell \int_t^\infty \int_{\mathbb{R}^3} e^{i(x|\xi)-t'|\xi|^2}dt'd\xi.\end{aligned}$$

Using the formula about the Fourier transform of the Gaussian functions, we get

$$\begin{aligned}\Gamma^j_{k,\ell}(t,x) &= \partial_j\partial_k\partial_\ell \int_t^\infty \frac{1}{(4\pi t')^{\frac32}} e^{-\frac{|x|^2}{4t'}}dt'\\ &= \frac{1}{\pi^{\frac32}}\int_t^\infty \frac{1}{(4t')^3}\Psi^j_{k,\ell}\Big(\frac{x}{\sqrt{4t'}}\Big)dt' \quad \text{with} \quad \Psi^j_{k,\ell}(z) \stackrel{\text{def}}{=} \partial_j\partial_k\partial_\ell e^{-|z|^2}.\end{aligned}$$

Changing variable $r=(4t')^{-1}|x|^2$ gives

$$|\Gamma^j_{k,\ell}(t,x)| \leq \frac{1}{\pi^{\frac32}}\frac{1}{|x|^4}\int_0^{\frac{|x|^2}{4t}} r\Psi^j_{k,\ell}\Big(\frac{x}{|x|}r\Big)dr.$$

This implies that

$$|\Gamma^j_{k,\ell}(t,x)| \lesssim \min\Big\{\frac{1}{t^2},\frac{1}{|x|^4}\Big\} \lesssim \frac{1}{(\sqrt{t}+|x|)^4}\,\cdot$$

The fact that $(\sqrt{t}+|x|)^{-4}$ belongs to $L^{\frac54}_w$ from the fact that the function is homogenenous of order -4 in the sapce of dimension 5 because the homogeneity is with respect of the dilation $(t,x)\mapsto(\lambda^2 t,\lambda x)$. This proves the lemma. □

Corollary 3.4.1. *The operators L_k maps continuously from the space $L^{\frac52}([0,T]\times\mathbb{R}^3)$ into the space $C([0,T];L^3)\cap L^5([0,T];L^5)$.*

Proof. Using Lemma 3.4.1, we immediatly infer that Γ^j_k belongs to $L^{\frac54}_w$. As we have

$$1+\frac15 = \frac45+\frac25$$

then using Hardy-Littlewood-Sobolev inequality (see for instance Theorem 1.7 on page 10 of [2]) we infer that

$$\|L_k f\|_{L^5([0,T]\times\mathbb{R}^3)} \lesssim \|f\|_{L^{\frac52}([0,T]\times\mathbb{R}^3)}. \tag{3.11}$$

Now let us observe that Lemma 3.4.1 implies that

$$\|\Gamma^j_k(\cdot,z)\|_{L^{\frac53}(\mathbb{R}^+)} \lesssim \frac{1}{|z|^{\frac{14}{5}}}\,\cdot$$

Hölder inequality in time implies that

$$\begin{aligned}\|L_k f(\cdot,x)\|_{L^\infty([0,T])} &\lesssim \int_{R^3}\|\Gamma^j_k(\cdot,x-y)\|_{L^{\frac53}(\mathbb{R}^+)}\|f(\cdot,y)\|_{L^{\frac52}([0,T])}dy\\ &\lesssim \int_{\mathbb{R}^3}\frac{1}{|x-y|^{\frac{14}{5}}}\|f(\cdot,y)\|_{L^{\frac52}([0,T])}dy.\end{aligned}$$

As

$$1+\frac{1}{3}=\frac{14}{15}+\frac{2}{5}\,,$$

Hardy-Littlewood-Sobolev inequality implies that

$$\|L_k f\|_{L^3(\mathbb{R}^3;L^\infty([0,T])}\lesssim \|f\|_{L^{\frac{5}{2}}([0,T]\times\mathbb{R}^3)}.$$

As $\|g\|_{L^\infty([0,T];L^3(\mathbb{R}^3)}\leq \|g\|_{L^3(\mathbb{R}^3;L^\infty([0,T]))}$ this gives the result. In order to prove the continuity, we use (3.8) and then proceed exactly as in the proof of Lemma 3.2.1. □

Conclusion of the proof of theorem 3.4.1 The fact that the bilinear operator B maps continuously $L^5([0,T]\times\mathbb{R}^3)\times L^5([0,T]\times\mathbb{R}^3)$ into $L^\infty(\mathbb{R}_+;L^3)\cap L^5([0,T]\times\mathbb{R}^3)$ is an obvious consequence of Corollary 3.4.1. Then we simply have to prove that

$$\|e^{t\Delta}u_0\|_{L^5(\mathbb{R}^+\times\mathbb{R}^3)}\lesssim \|u_0\|_{L^3}. \tag{3.12}$$

As we have

$$|e^{t\Delta}u_0(x)|\leq \frac{1}{(4\pi t)^{\frac{3}{2}}}\int_{\mathbb{R}^3}e^{-\frac{|x-y|^2}{4t}}|u_0(y)|dy,$$

we get

$$\begin{aligned}\|e^{\cdot\Delta}u_0(x)\|_{L^5([0,T])} &\leq \int_{\mathbb{R}^3}\Big\|\frac{1}{(4\pi\cdot)^{\frac{3}{2}}}e^{-\frac{|x-y|^2}{4\cdot}}\Big\|_{L^5([0,T])}|u_0(y)|dy\\ &\lesssim \int_{R^3}\frac{1}{|x-y|^{\frac{13}{5}}}|u_0(y)|dy.\end{aligned}$$

As

$$1+\frac{1}{5}=\frac{13}{15}+\frac{1}{3}\,,$$

Hardy-Littlewood-Sobolev inequality give the result in the case when u_0 is small. For the local version, let us use that the space $L^3\cap L^5$ is dense in L^3. Thus for any positive ε, a function $u_{0,\varepsilon}$ exists in $L^3\cap L^5$ such that $\|u_0-u_{0,\varepsilon}\|_{L^3}\leq\varepsilon$. Using (3.12) gives that

$$\|e^{t\Delta}u_0\|_{L^5([0,T]\times\mathbb{R}^3)}\lesssim \varepsilon+T^{\frac{1}{5}}\|u_{0,\varepsilon}\|_{L^5}.$$

This concludes the proof of the theorem. □

3.5 A stability result in the L^p framework

The following theorem refines Theorem 3.3.2. It will be useful in the proof of forthcoming Theorem 4.1.1.

Theorem 3.5.1. *A constant C_0 exists such that for any couple (u_0,v_0) of divergence free vector fields in $\dot H^{\frac{1}{2}}$ and an external force f in $L^2(\mathbb{R}^+;\dot H^{\frac{1}{2}})\cap L^{\frac{5}{2}}(\mathbb{R}^+\times\mathbb{R}^3)$. Let us assume that u belongs to $L^\infty(\mathbb{R}^+;\dot H^{\frac{1}{2}})\cap L^4(\mathbb{R}^+;\dot H^1)$ and that*

$$\|v_0\|_{L^3}+\|f\|_{L^{\frac{5}{2}}(\mathbb{R}^+\times\mathbb{R}^3)}\leq\frac{1}{C_0}\exp\Big(-C_0\|u\|^5_{L^5(\mathbb{R}^+\times\mathbb{R}^3)}\Big).$$

Then the system

$$(NS_u)\left\{\begin{aligned}&\partial_t v-\Delta v+v\cdot\nabla v+\operatorname{div}\big(u\otimes v+v\otimes u\big)=-\nabla p+\sum_{\ell=1}^3\partial_\ell f^\ell\\ &\qquad\qquad\operatorname{div} v=0\quad\textit{and}\quad v_{|t=0}=v_0.\end{aligned}\right.$$

has a global solution which satisfies

$$\|v\|_{L^\infty(\mathbb{R}^+;L^3)} + \|v\|_{L^5(\mathbb{R}^+\times\mathbb{R}^3)} \leq \frac{1}{C_0}\big(\|v_0\|_{L^3} + \|f\|_{L^{\frac52}(\mathbb{R}^+\times\mathbb{R}^3)}\big)\exp\big(C_0\|u\|^5_{L^5(\mathbb{R}^+\times\mathbb{R}^3)}\big).$$

Proof. Let us first notice that for any subinterval $I=[a,b)$, the bilinear operator B_I defined by

$$\begin{cases} \partial_t B_I - \Delta B_I = -\dfrac{1}{2}\operatorname{div}\big(u\otimes v + v\otimes u\big) - \nabla p \\ \operatorname{div} B_I = 0 \quad \text{and} \quad B_{I|t=a} = 0 \end{cases}$$

maps continuously $L^5(I\times\mathbb{R}^3)\times L^5(I\times\mathbb{R}^3)$ into $L^\infty(I;L^3(\mathbb{R}^3))\cap L^5(I\times\mathbb{R}^3)$ with a constant of continuity which does not depend on I; let us denote by $\|B\|$ this constant. Let us also denote by C_1 the best constant such that

$$\|e^{t\Delta}a\|_{L^5(\mathbb{R}^+\times\mathbb{R}^3)} \leq C_1\|a\|_{L^3}.$$

Let us remark that Theorem 3.3.1 implies that u belongs to $L^\infty(\mathbb{R}^+;\dot H^{\frac12})\cap L^2(\mathbb{R}^+;\dot H^{\frac32})$. Thus Sobolev embedding and interpolation inequalities between Sobolev spaces implies that u belongs to the space $L^5(\mathbb{R}^+\times\mathbb{R}^3)$. The local theory of wellposedness for initial data in $\dot H^{\frac12}$ claims in particular that a unique maximal solution w exists in $C[0,T^\star[;\dot H^{\frac12})\cap L^2_{loc}([0,T^\star[;\dot H^{\frac32})$ and if $T^\star$ is finite, then

$$\int_0^{T^\star}\|w(t,\cdot)\|^5_{L^5}dt = \infty.$$

We want to control the $\|w\|_{L^5([0,T]\times\mathbb{R}^3)}$ to prevent blow up. In order to do so, let us write the solution $w=u+v$. Then v is the solution of

$$(NS_u)\quad \begin{cases} \partial_t v - \Delta v + v\cdot\nabla v + \operatorname{div}\big(u\otimes v + v\otimes u\big) = -\nabla p + \displaystyle\sum_{\ell=1}^3 \partial_\ell f^\ell \\ \operatorname{div} v = 0 \quad \text{and} \quad v_{|t=0} = v_0. \end{cases}$$

Let us decompose $\mathbb{R}^+$ as a disjoint union of intervals $(I_j)_{1\leq j\leq N}$ such that if $I_j=[I_j^-,I_j^+]$, then $I_j^-=0$, $I_j^+=I_{j+1}^-$ and $I_N^+=\infty$ and

$$\forall j\in\{1,\cdots,N\}\,,\ \int_{I_j}\|u(t,\cdot)\|^5_{L^5}dt = \Big(\frac{1}{8\|B\|}\Big)^5 \quad\text{and}\quad \int_{I_N}\|u(t,\cdot)\|^5_{L^5}dt \leq \Big(\frac{1}{8\|B\|}\Big)^5.$$

Let us notice that, by addition, we get

$$(N-1)\Big(\frac{1}{8\|B\|}\Big)^5 \leq \int_{\mathbb{R}^+}\|u(t,\cdot)\|^5_{L^5}dt \leq N\Big(\frac{1}{8\|B\|}\Big)^5$$

We shall prove by induction that if

$$\|v_0\|_{L^3} + \|f\|_{L^{\frac52}(\mathbb{R}^+\times\mathbb{R}^3)} \leq \frac{1}{8C_1\|B\|}(2C_1)^{-N-1}, \tag{3.13}$$

then, for any j, we have $\widehat I_j \overset{\text{déf}}{=} \displaystyle\bigcup_{j'=1}^{j} I'_j \subset [0,T^\star[$ and

$$T^\star > I_j^+ \quad\text{and}\quad \|v(I_j^+)\|_{L^3} \leq \big(\|v_0\|_{L^3} + \|f\|_{L^{\frac52}(I_1\times\mathbb{R}^3)}\big)(2C_1)^j. \tag{3.14}$$

Let us point out that it prevents blow up and thus $u_0 + v_0$ belongs to $\mathcal{G}$. Let us prove the induction hypothesis for $j = 1$. Let us define

$$\overline{T} \stackrel{\text{def}}{=} \sup\Big\{T < \min\{T^\star, I_1^+\} \,/\, \max\{\|v\|_{L^\infty([0,T];L^3(\mathbb{R}^3))}, \|v\|_{L^5([0,T]\times\mathbb{R}^3)}\} \leq \big(\|v_0\|_{L^3} + \|f\|_{L^5(I_1\times\mathbb{R}^3)}\big)(2C_1)\Big\}.$$

The system (NS_u) writes

$$v = e^{t\Delta}v_0 + 2B(u,v) + B(v,v) + \sum_{k=1}^3 L_k f^k$$

For any $T < \overline{T}$, we have

$$\max\{\|v\|_{L^\infty([0,T];L^3(\mathbb{R}^3))}, \|v\|_{L^5([0,T]\times\mathbb{R}^3)}\} \leq C_1\big(\|v_0\|_{L^3} + \|f\|_{L^{\frac52}(I_1\times\mathbb{R}^3)}\big) + 2\|B\|\|u\|_{L^5(I_1\times\mathbb{R}^3)}\|v\|_{L^5([0,T]\times\mathbb{R}^3)} + \|B\|\|v\|^2_{L^5([0,T]\times\mathbb{R}^3)}.$$

By definition of T and the intervals I_j, we get, thanks to Hypothesis (3.13)

$$\max\{\|v\|_{L^\infty([0,T];L^3(\mathbb{R}^3))}, \|v\|_{L^5([0,T]\times\mathbb{R}^3)}\} \leq C_1\big(\|v_0\|_{L^3} + \|f\|_{L^{\frac52}(I_1\times\mathbb{R}^3)}\big) + \frac12\|v\|_{L^5([0,T]\times\mathbb{R}^3)}$$

which gives

$$\forall T < \overline{T}\,,\ \max\{\|v\|_{L^\infty([0,T];L^3(\mathbb{R}^3))}, \|v\|_{L^5([0,T]\times\mathbb{R}^3)}\} \leq 2C_1\big(\|v_0\|_{L^3} + \|f\|_{L^{\frac52}(I_1\times\mathbb{R}^3)}\big).$$

This implies that $T^\star > I_1^+$ and $\overline{T} = I_1^+$ which is exactly the induction hypothesis for $j = 1$.

Let us assume the induction hypothesis for $j < N$. The system (NS_u) can be written

$$v = e^{(t-I_{j+1}^-)\Delta}v(I_{j+1}^-) + 2B_{I_{j+1}}(u,v) + B(v,v) + \sum_{k=1}^3 L_k f^k.$$

Let us define

$$\overline{T}_{j+1} \stackrel{\text{def}}{=} \sup\Big\{T < \min\{T^\star, I_{j+1}^+\} \,/\, \max\{\|v\|_{L^\infty([I_{j+1}^-,T];L^3(\mathbb{R}^3))}, \|v\|_{L^5(I_{j+1}^-,T]\times\mathbb{R}^3)}\} \leq (2C_1)^{j+1}\big(\|v_0\|_{L^3} + \|f\|_{L^{\frac52}(I_{j+1}\times\mathbb{R}^3)}\big)\Big\}.$$

For any $I_{j+1}^- \leq T < \overline{T}_{j+1}$, we have

$$\max\{\|v\|_{L^\infty([I_{j+1}^-,T];L^3(\mathbb{R}^3))}, \|v\|_{L^5(I_{j+1}^-,T]\times\mathbb{R}^3)}\} \leq C_1\big(\|v(I_{j+1}^-)\|_{L^3} + \|f\|_{L^{\frac52}(I_{j+1}\times\mathbb{R}^3)}\big) + 2\|B\|\|u\|_{L^5(I_1\times\mathbb{R}^3)}\|v\|_{L^5(I_{j+1}^-,T]\times\mathbb{R}^3)} + \|B\|\|v\|^2_{L^5(I_{j+1}^-,T]\times\mathbb{R}^3)}.$$

By definition of T and of the intervals I_j, we get that

$$\max\{\|v\|_{L^\infty([I_{j+1}^-,T];L^3(\mathbb{R}^3))}, \|v\|_{L^5(I_{j+1}^-,T]\times\mathbb{R}^3)}\} \leq C_1\big(\|v(I_{j+1}^-)\|_{L^3} + \|f\|_{L^{\frac52}(I_{j+1}\times\mathbb{R}^3)}\big) + \frac12\|v\|_{L^5(I_{j+1}^-,T]\times\mathbb{R}^3)}$$

which gives

$$\forall T < \overline{T}\,,\ \max\{\|v\|_{L^\infty([I_{j+1}^-,T];L^3(\mathbb{R}^3))}, \|v\|_{L^5(I_{j+1}^-,T]\times\mathbb{R}^3)}\} \leq 2C_1\big(\|v(I_{j+1}^-)\|_{L^3} + \|f\|_{L^{\frac52}(I_{j+1}\times\mathbb{R}^3)}\big).$$

This implies that $T^\star > I_{j+1}^+$ and $\overline{T} = I_{j+1}^+$ which is exactly the induction hypothesis for $j+1$ and the theorem is proved. □

Chapter 4

Applications of profile decomposition to the Navier-Stokes equations

The following theorem is the basis of application of profiles theory to incompressible Navier-Stokes system.

4.1 Bounded sequences of initial data

Theorem 4.1.1. *Let $(u_{0,n})_{n\in\mathbb{N}}$ be a sequence of initial data which is bounded in $\dot H^{\frac12}$. Let us consider an extraction ϕ, a sequence of profiles $(\varphi^j)_{j\in\mathbb{N}}$, a sequence of scales and cores $(\lambda_n^j, x_n^j)_{(j,n)\in\mathbb{N}^2}$ in the sense of Definition 2.1.1, a sequence $(r_n^j)_{(j,n)\in\mathbb{N}^2}$ of functions given by Theorem 2.1.1 on page 13.*

Let us define $\mathcal{J}_f$ as the set of indices j such that φ^j does not belongs to $\mathcal{G}$ i.e. φ^j does not give birth to a global solution in $L^4(\mathbb{R}^+;\dot H^1)$. This set is finite.

If $\mathcal{J}_f$ is empty, then, for n large enough, $u_{\phi(n)}$ belongs to $\mathcal{G}$.

If $\mathcal{J}_f$ is non empty, an index j_0 exists in $\mathcal{J}_f$ such that

$$\forall j\in\mathcal{J}_f\,,\ \exists n_j \,/\, n\geq n_j \Longrightarrow (\lambda_n^{j_0})^2 T^\star(\varphi^{j_0}) \leq (\lambda_n^j)^2 T^\star(\varphi^j).$$

For any positive real number ε, let us define $\tau_n^\varepsilon \stackrel{\text{def}}{=} (\lambda_n^{j_0})^2(T^\star(\varphi^{j_0})-\varepsilon)$. Then

$$\exists n_\varepsilon \,/\, n\geq n_\varepsilon \Longrightarrow T^\star(u_{0,\phi(n)}) \geq \tau_n^\varepsilon.$$

Proof. Let us search the solution u_n^J associated with $u_{0,\phi(n)}$ of the form

$$\begin{aligned} u_n^J = u_{n,\text{app}}^J + R_n^J \quad \text{with} \quad u_{n,\text{app}}^J \stackrel{\text{def}}{=} \sum_{j=0}^J \Phi_n^j + e^{t\Delta} r_n^J \\ \text{and} \quad \Phi_n^j(t,x) = \frac{1}{\lambda_n^j}\Phi^j\left(\frac{t}{(\lambda_n^j)^2}, \frac{x-x_n^j}{\lambda_n^j}\right) \end{aligned} \tag{4.1}$$

where Φ^j denotes the solution of (NS) associated with φ^j. The vector field R_n^J satisfies

$$
\begin{aligned}
&\partial_t R_n^J - \Delta R_n^J + R_n^J \cdot \nabla R_n^J + u_{n,\mathrm{app}}^J \cdot \nabla R_n^J + R_n^J \cdot \nabla u_{n,\mathrm{app}}^J = -\nabla p - \operatorname{div} \sum_{\ell=1}^3 F_n^{J,\ell} \quad \text{with} \\
&F_n^{J,1} \stackrel{\text{def}}{=} \sum_{\substack{0\leq j,k\leq J \\ j\neq k}} F_n^{j,k} \quad \text{with} \quad F_n^{j,k} \stackrel{\text{def}}{=} \Phi_n^j \otimes \Phi_n^k\,, \\
&F_n^{J,2} \stackrel{\text{def}}{=} \Big(\sum_{j=0}^J \Phi_n^j\Big) \otimes e^{t\Delta} r_n^J + e^{t\Delta} r_n^J \otimes \Big(\sum_{j=0}^J \Phi_n^j\Big) \quad \text{and} \\
&F_n^{J,3} \stackrel{\text{def}}{=} e^{t\Delta} r_n^J \otimes e^{t\Delta} r_n^J.
\end{aligned}
\tag{4.2}
$$

Now let us prove the following lemma.

Lemma 4.1.1. *A real number M exists such that, for J,*

$$
\limsup_{n\to\infty} \Big\| \sum_{j=0}^J \Phi_n^j \Big\|_{L^5([0,\tau_n^\varepsilon]\times\mathbb{R}^3)} \leq M
$$

with the agreement that τ_n^ε equal to infinity if $\mathcal{J}_f$ is empty.

Proof. Using the scaling invariance of the space-time L^5 norm, let us write that

$$
\begin{aligned}
\bigg| \Big\| \sum_{j=0}^J \Phi_n^j \Big\|^5_{L^5([0,\tau_n^\varepsilon]\times\mathbb{R}^3)} &- \sum_{j=0}^J \|\Phi^j\|^5_{L^5([0,\tau_n^\varepsilon]\times\mathbb{R}^3)} \bigg| \\
&\leq \sum_{(j_1,j_2,j_3,j_4,j_5)\in\{0,\cdot,J\}^5\setminus\Delta} \int_{[0,\tau_n^\varepsilon]\times\mathbb{R}^3} \prod_{\ell=1}^5 |\Phi_n^{j_\ell}(t,x)| dxdt\,.
\end{aligned}
$$

where $\Delta \stackrel{\text{def}}{=} \big\{(j,j,j,j,j)\,/\, j\in\{0,\cdots,J\}^5\big\}$. Up to a permutation of indices, we can assume that j_1 is different from j_2. Using Hölder inequality and the scaling invariance of the space-time L^5 norm we get

$$
\begin{aligned}
\int_{[0,\tau_n^\varepsilon]\times\mathbb{R}^3} \prod_{\ell=1}^5 |\Phi_n^{j_\ell}(t,x)| dxdt \leq &\Big(\prod_{\ell=3}^5 \|\Phi^{j_\ell}\|_{L^5([0,T^\star(\varphi^j)-\varepsilon]\times\mathbb{R}^3)} \Big) \\
&\times \Big(\int_{[0,\tau_n^\varepsilon]\times\mathbb{R}^3} |\Phi_n^{j_1}(t,x)|^{\frac52} \, |\Phi_n^{j_2}(t,x)|^{\frac52} dxdt \Big)^{\frac25}.
\end{aligned}
$$

Let us assume that $\displaystyle\lim_{n\to\infty} \Big| \log\Big(\frac{\lambda_n^{j_1}}{\lambda_n^{j_2}}\Big)\Big| = +\infty$. Using the scaling, we get

$$
\begin{aligned}
\mathcal{J}_n^{j_1,j_2} &\stackrel{\text{def}}{=} \int_{[0,\tau_n^\varepsilon]\times\mathbb{R}^3} |\Phi_n^{j_1}(t,x)|^{\frac52} \, |\Phi_n^{j_2}(t,x)|^{\frac52} dxdt \\
&\leq \int_0^{\frac{\tau_n^\varepsilon}{(\lambda_n^{j_2})^2}} \frac{\lambda_n^{j_2}}{\lambda_n^{j_1}} \Big\| \Phi^{j_1}\Big(\Big(\frac{\lambda_n^{j_2}}{\lambda_n^{j_1}}\Big)^2 t,\cdot\Big)\Big\|^{\frac52}_{L^5(\mathbb{R}^3)} \|\Phi^{j_2}(t,\cdot)\|^{\frac52}_{L^5(\mathbb{R}^3)} dt.
\end{aligned}
$$

Using that the weak converge in L^p namely the fact that, in p is in $]1,\infty[$, for any (f,g) in $L^p(\mathbb{R}^d)\times L^{p'}(\mathbb{R}^d)$, we have

$$\lim_{\Lambda\to\infty}\int_{\mathbb{R}^d}\Lambda^{-\frac{d}{p}}f(\Lambda^{-1}x)g(x)dx=0. \tag{4.3}$$

Thus we have

$$\lim_{n\to\infty}\left|\log\Big(\frac{\lambda_n^{j_1}}{\lambda_n^{j_2}}\Big)\right|=+\infty\Longrightarrow\lim_{n\to\infty}\mathcal{J}_n^{j_1,j_2}=0. \tag{4.4}$$

If the case when $\lambda_n^{j_1}=\lambda_n^{j_2}$, let us write that

$$\mathcal{J}_n^{j_1,j_2}\leq\int_0^{\frac{\tau_n^\varepsilon}{(\lambda_n^{j_2})^2}}\int_{\mathbb{R}^3}\left|\Phi^j\left(t,x-\frac{x_n^j-x_n^k}{\lambda_n^j}\right)\right|^{\frac52}|\Phi^k(t,x)|^{\frac52}dxdt.$$

Using the fact that $\displaystyle\lim_{n\to\infty}\frac{|x_n^j-x_n^k|}{\lambda_n^j}$ tends to infinity this implies that together with (4.4)

$$\forall(j,k)\in\{0,\cdots,J\}^2\,/\,j\neq k\,,\ \lim_{n\to\infty}\mathcal{J}_n^{j_1,j_2}=0. \tag{4.5}$$

Together with (4.4), this implies that

$$\forall J\,,\ \limsup_{n\to\infty}\Big\|\sum_{j=0}^J\Phi_n^j\Big\|^5_{L^5([0,\tau_n^\varepsilon]\times\mathbb{R}^3)}\leq\sum_{j=0}^J\|\Phi^j\|^5_{L^5([0,T^\star(\varphi^j)-\varepsilon]\times\mathbb{R}^3)}. \tag{4.6}$$

Now we have to estimate the righthandside term of the above inequality independently of J. Let us denote by $\mathcal{J}_{\rho_0}$ the set of indices such that $\|\varphi^j\|_{\dot H^{\frac12}}$ is greater than or equal to than $\rho_0/2$ (where ρ_0 is the radius given by Theorem 3.2.1 on page 29). As the series $(\|\varphi^j\|^2_{\dot H^{\frac12}})_{j\in\mathbb{N}}$ is summable, this set is finite. If j does not belongs to $\mathcal{J}_{\rho_0}$, Theorem 3.2.1 claims that

$$\|\Phi^j\|^2_{L^\infty(\mathbb{R}^+;\dot H^{\frac12})}+\|\Phi^j\|^2_{L^2(\mathbb{R}^+;\dot H^{\frac32})}\leq\|\varphi^j\|^2_{\dot H^{\frac12}}.$$

Sobolev embeddings and interpolation inequality between Sobolev spaces imply that

$$\begin{aligned}\|\Phi^j\|^2_{L^5(\mathbb{R}^+;L^5(\mathbb{R}^3))}&\leq C\|\Phi^j\|^2_{L^5(\mathbb{R}^+;\dot H^{\frac12+\frac25}(\mathbb{R}^3))}\\&\leq C\|\varphi^j\|^2_{\dot H^{\frac12}}.\end{aligned}$$

Thus, as the series $(\|\varphi^j\|^2_{\dot H^{\frac12}})_{j\in\mathbb{N}}$ is summable and as $\ell^2(\mathbb{Z})$ is continuously included in $\ell^5(\mathbb{Z})$, we get that

$$\sum_{j=0}^J\|\Phi^j\|^5_{L^5([0,T^\star(\varphi^j)-\varepsilon]\times\mathbb{R}^3)}\leq C\sup_n\|u_{0,n}\|^5_{\dot H^{\frac12}}+\sum_{j\in\mathcal{J}_{\rho_0}}\|\Phi^j\|^5_{L^5([0,T^\star(\varphi^j)-\varepsilon]\times\mathbb{R}^3)}$$

Using Theorem 3.3.1 on page 32 and Inequality (4.6), we get the lemma. □

Continuation of the proof of Theorem 4.1.1 Let η be a positive real number which will be choosen later on. Two integers J_η and n_η such that

$$n\geq n_\eta\Longrightarrow\|r_n^{J_\eta}\|_{L^3}\leq\eta. \tag{4.7}$$

Lemma 4.1.1 and Inequality (3.12) on page 36 imply that, if n large enough (depending on η)

$$\begin{aligned}\|u_{n,\mathrm{app}}^{J_\eta}\|_{L^5([0,T^\star(\varphi^j)-\varepsilon]\times\mathbb{R}^3)} &\leq M+\|e^{t\Delta}r_n^{J_\eta}\|_{L^5(\mathbb{R}^+\times\mathbb{R}^3)}\\ &\leq M+C\eta. \end{aligned}\tag{4.8}$$

Moreover, Inequality (3.12) on page 36 implies that, if n large enough,

$$\|F_n^{J_\eta,3}\|_{L^{\frac{5}{2}}([0,\tau_n^\varepsilon]\times\mathbb{R}^3)}\leq C\eta^2. \tag{4.9}$$

Hölder inequality and Lemma 4.1.1 imply that, if n is large enough ,

$$\begin{aligned}\|F_n^{J\eta,2}\|_{L^{\frac{5}{2}}([0,\tau_n^\varepsilon]\times\mathbb{R}^3)} &\leq \Big\|\sum_{j=0}^{J_\eta}\Phi_n^j\Big\|_{L^5([0,\tau_n^\varepsilon]\times\mathbb{R}^3)}\|e^{t\Delta}r_{n,J}\|_{L^5(\mathbb{R}^+\times\mathbb{R}^3)}\\ &\leq CM\eta. \end{aligned}\tag{4.10}$$

Inequality (4.5) on page 41 implies that

$$\lim_{n\to\infty}\|F_n^{J\eta,1}\|_{L^{\frac{5}{2}}([0,\tau_n^\varepsilon]\times\mathbb{R}^3)}=0.$$

Together with Inequalities (4.8)–(4.10), this implies that, choosing η small enough, we have, for n large enough,

$$\|F_n^{J_\eta}\|_{L^{\frac{5}{2}}([0,\tau_n^\varepsilon]\times\mathbb{R}^3)}\leq C_0^{-1}\exp\big(-C_0\|u_{n,\mathrm{app}}^{J_\eta}\|^5_{L^5([0,\tau_n^\varepsilon]\times\mathbb{R}^3)}\big).$$

In the case when τ_n^ε equals to infinity, then the theorem is proved. In the case when τ_n^ε in finite, the proof of Theorem 3.5.1 on page 36 can be repeated words for words to ensures that $T^\star(u_{\phi(n)})$ is greater than or equal to τ_n^ε.

□

4.2 A first application to the structure of the set $\mathcal{G}$

Theorem 4.2.1. *Let us assume that $\mathcal{G}\neq\dot H^{\frac12}$. Let us define*

$$\rho_c\overset{def}{=}\sup\big\{\rho\in]0,\infty[\ /\ B(0,\rho)\subset\mathcal{G}\big\}.$$

Then if $S(0,\rho)$ denotes the sphere on radius ρ and center 0 in $\dot H^{\frac12}$, we have that $S(0,\rho_c)\cap\mathcal{G}^c$ is non empty. It is compact up to translation and dilation in the following sense. If $(u_n)_{n\in\mathbb{N}}$ is a sequence of $S(0,\rho_c)\cap\mathcal{G}^c$, then a sequence $(\lambda_n,x_n)_{n\in\mathbb{N}}$ of $(]0,\infty[\times\mathbb{R}^3)^{\mathbb{N}}$ and a function v in $S(0,\rho_c)\cap\mathcal{G}^c$ exists such that, up to an extraction, we have

$$\lim_{n\to\infty}\|\lambda_n u_n(\lambda_n(\cdot+x_n))-v\|_{\dot H^{\frac12}}=0.$$

Proof. By definition of ρ_c, a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of $\dot H^{\frac12}$ exists such that

$$T^\star(u_n)<\infty\quad\text{and}\quad\lim_{n\to\infty}\|u_n\|_{\dot H^{\frac12}}=\rho_c. \tag{4.11}$$

Theorem 2.1.1 claims that a sequence of scales and cores $(\lambda_n^j, x_n^j)_{(j,n)\in\mathbb{N}^2}$, a sequence $(\varphi^j)_{j\in\mathbb{N}}$ in $\dot H^{\frac12}$, a sequence $(r_n^j)_{(j,n)\in\mathbb{N}^2}$ which satisfies the following, up to an extraction on $(u_n)_{n\in\mathbb{N}}$,

$$\forall J\in\mathbb{N}\,,\ u_n(x)=\sum_{j=0}^J \frac{1}{\lambda_n^j}\varphi^j\Big(\frac{x-x_n^j}{\lambda_n^j}\Big)+r_n^J(x)\quad\text{with}$$

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|r_n^J\|_{L^3}=0\quad\text{and}$$

$$\forall J\in\mathbb{N}\,,\ \lim_{n\to\infty}\Big(\|u_n\|^2_{\dot H^{\frac12}}-\|r_n^J\|^2_{\dot H^{\frac12}}\Big)=\sum_{j=0}^J\|\varphi^j\|^2_{\dot H^{\frac12}}.$$

The last relation implies in particular that

$$\sum_{j=0}^\infty\|\varphi^j\|^2_{\dot H^{\frac12}}\leq\rho_c^2. \tag{4.12}$$

Theorem 4.1.1 implies that an integer j_0 exists such that φ^{j_0} does not belongs to $\mathcal{G}$. As the (open) ball of center 0 and radius ρ_c is included in $\mathcal{G}$, then an integer j_0 exists such that $\|\varphi^{j_0}\|_{\dot H^{\frac12}}\geq\rho_c$. Inequality (4.12) implies that

$$\|\varphi^{j_0}\|_{\dot H^{\frac12}}=\rho_c\quad\text{and}\qquad j\neq j_0\Longrightarrow\varphi^j=0.$$

Thus φ^{j_0} is an element of $S(0,\rho_c)\cap\mathcal{G}^c$ which proves the first part of the theorem. For the second part, let us consider a sequence $(u_n)_{n\in\mathbb{N}}$ of $S(0,\rho_c)\cap\mathcal{G}$. Arguing exactly as above, we deduce that, up to an extraction

$$u_n=\frac{1}{\lambda_n^{j_0}}\varphi^{j_0}\Big(\frac{\cdot-x_n^{j_0}}{\lambda_n^{j_0}}\Big)+r_n\quad\text{with}\quad\lim_{n\to\infty}\|r_n\|_{\dot H^{\frac12}}=0.$$

As the $\dot H^{\frac12}$ norm is scaling invariant, this gives the whole theorem. □

4.3 Describtion of bounded sequences of solutions

The purpose of this section is to prove the following theorem.

Theorem 4.3.1. *Let $(u_{0,n})_{n\in\mathbb{N}}$ be a sequence of initial data bounded in $\dot H^{\frac12}$. With notations of Theorem 4.1.1, we have*

$$\lim_{J\to\infty}\limsup_{n\to\infty}\big(\|R_n^J\|_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})}+\|R_n^J\|_{L^2([0,\tau_n^\varepsilon];\dot H^{\frac32})}\big)=0. \tag{4.13}$$

Moreover, for any J, we have

$$\lim_{n\to\infty}\Big\|\,\|u_{\phi(n)}(t)\|^2_{\dot H^{\frac12}}-\sum_{j=0}^J\|\Phi_n^j(t)\|^2_{\dot H^{\frac12}}-\|e^{t\Delta}r_n^J\|^2_{\dot H^{\frac12}}\Big\|_{L^\infty([0,\tau_n^\varepsilon])}=0. \tag{4.14}$$

Proof. The first step is the proof of the following lemma which is the analog of Lemma 4.1.1.

Lemma 4.3.1. *A real number M exists such that, for J,*

$$\limsup_{n\to\infty}\Big\|\sum_{j=0}^{J}\Phi_n^j\Big\|_{L^4([0,\tau_n^\varepsilon];\dot H^1)}\leq M$$

with the agreement that τ_n^ε equal to infinity if $\mathcal{J}_f$ is empty.

Proof. Let us write that for any t in $[0,\tau_n^\varepsilon]$, we have

$$\Big\|\sum_{j=0}^{J}\Phi_n^j(t)\Big\|_{\dot H^1}^2=\sum_{j=0}^{J}\|\Phi_n^j(t)\|_{\dot H^1}^2+\sum_{\substack{0\leq j,k\leq J\\ j\neq k}}\big(\Phi_n^j(t)|\Phi_n^k(t)\big)_{\dot H^1}. \tag{4.15}$$

Let us prove that for any (j,k) in $\{0,\cdots,J\}^2$ such that j and k are different, we have

$$\lim_{n\to\infty}\big\|\big(\Phi_n^j(t)|\Phi_n^k(t)\big)_{\dot H^1}\big\|_{L^2([0,\tau_n^\varepsilon[)}=0. \tag{4.16}$$

Let us first consider the case when the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^k)_{n\in\mathbb{N}}$ are different.

Using the scaling invariance of the space $L_t^4(\dot H^1)$ norm, let us write that

$$\begin{aligned}\big\|\big(\Phi_n^j(t)|\Phi_n^k(t)\big)_{\dot H^1}\big\|_{L^2([0,\tau_n^\varepsilon[)}^2 &\leq \int_0^\infty \frac{1}{\lambda_n^j\lambda_n^k}\mathbb{1}_{[0,T^\star(\varphi^j)-\varepsilon]}\Big(\frac{t}{(\lambda_n^j)^2}\Big)\Big\|\Phi^j\Big(\frac{t}{(\lambda_n^j)^2}\Big)\Big\|_{\dot H^1}^2\\ &\qquad\times\mathbb{1}_{[0,T^\star(\varphi^k)-\varepsilon]}\Big(\frac{t}{(\lambda_n^k)^2}\Big)\Big\|\Phi^k\Big(\frac{t}{(\lambda_n^k)^2}\Big)\Big\|_{\dot H^1}^2 dt\\ &\leq \int_0^\infty\frac{\lambda_n^k}{\lambda_n^j}\mathbb{1}_{[0,T^\star(\varphi^j)-\varepsilon]}\Big(\Big(\frac{\lambda_n^k}{\lambda_n^j}\Big)^2 t\Big)\Big\|\Phi^j\Big(\Big(\frac{\lambda_n^k}{\lambda_n^j}\Big)^2 t\Big)\Big\|_{\dot H^1}^2\\ &\qquad\times\mathbb{1}_{[0,T^\star(\varphi^k)-\varepsilon]}(t)\|\Phi^k(t)\|_{\dot H^1}^2 dt.\end{aligned}$$

Because of the hypothesis of orthogonality on the scales, this implies (4.16) in the case when the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^k)_{n\in\mathbb{N}}$ are different.

In the case when the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^k)_{n\in\mathbb{N}}$ are equal, let us observe

$$\big\|\big(\Phi_n^j(t)|\Phi_n^k(t)\big)_{\dot H^1}\big\|_{L^2([0,\tau_n^\varepsilon[)}^2\leq\int_0^{T_{j,k}^\varepsilon}\Big|\Big(\Phi^j(t,\cdot)\Big|\Phi^k\Big(t,\cdot-\frac{x_n^j-x_n^k}{\lambda_n^j}\Big)\Big)_{\dot H^1}\Big|^2 dt$$

with $T_{j,k}^\varepsilon\stackrel{\text{def}}{=}\min\{T^\star(\varphi^j),T^\star(\varphi^k)\}-\varepsilon$. Lebesgue's theorem implies that for any positive η, a positive real number α_J exists such that, for any j in $\{0,\cdots,J\}$, we have

$$\begin{aligned}&\|\Phi^j-\Phi_{\alpha_J}^j\|_{L^4([0,T^\star(\varphi^j)-\varepsilon];\dot H^1)}\leq\frac{\eta}{\sup_{j\in J}\|\Phi^j\|_{L^4([0,T^\star(\varphi^j)-\varepsilon];\dot H^1)}}\quad\text{with}\\ &\qquad\Phi_{\alpha_J}^j\stackrel{\text{def}}{=}\Big(1-\chi\Big(\frac{\cdot}{\alpha_J}\Big)+\chi(\cdot\,\alpha_J)\Big)\Phi^j\end{aligned}\tag{4.17}$$

where χ if a compactly supported function in $\mathbb{R}^3$ with value 1 near the origin. Thus we have

$$\big\|\big(\Phi_n^j(t)|\Phi_n^k(t)\big)_{\dot H^1}\big\|_{L^2([0,\tau_n^\varepsilon[)}^2\leq 2\eta+\int_0^{T_{j,k}^\varepsilon}\Big|\Big(\Phi_{\alpha_J}^j(t,\cdot)\Big|\Phi_{\alpha_J}^k\Big(t,\cdot-\frac{x_n^j-x_n^k}{\lambda_n^j}\Big)\Big)_{\dot H^1}\Big|^2 dt.$$

As $\lim_{n\to\infty} \frac{|x_n^j - x_n^k|}{\lambda_n^j} = \infty$, for n large enough, the support of the two functions

$$\Phi^j_{\alpha_J}(t,\cdot) \quad \text{and} \quad \Phi^k_{\alpha_J}\Big(t, \cdot - \frac{x_n^j - x_n^k}{\lambda_n^j}\Big)$$

are disjoint. Thus we get (4.16) in all cases. Taking the L^2 norm and passing to the limit when n tends to infinity in Relation (4.15), we get

$$\limsup_{n\to\infty} \Big\|\sum_{j=0}^J \Phi_n^j(t)\Big\|^2_{L^4([0,\tau_n^\varepsilon];\dot H^1)} \leq \sum_{j=0}^J \|\Phi^j\|^2_{L^4([0,T^\star(\varphi^j)-\varepsilon];\dot H^1)}.$$

With the notations of Theorem 4.1.1, let us write that

$$\sum_{j=0}^J \|\Phi^j\|^2_{L^4([0,T^\star(\varphi^j)-\varepsilon];\dot H^1)} \leq \sum_{j\in\mathcal{J}_{\rho_0}} \|\Phi^j\|^2_{L^4([0,T^\star(\varphi^j)-\varepsilon];\dot H^1)} + \sum_{\substack{j\notin\mathcal{J}_{\rho_0}\\ j\leq J}} \|\Phi^j\|^2_{L^4\,\mathbb{R}^+;\dot H^1)}$$

Using Theorem 3.2.1, we get that

$$\sum_{j=0}^J \|\Phi^j\|^2_{L^4([0,T^\star(\varphi^j)-\varepsilon];\dot H^1)} \leq \sum_{j\in\mathcal{J}_{\rho_0}} \|\Phi^j\|^2_{L^4([0,T^\star(\varphi^j)-\varepsilon];\dot H^1)} + \limsup_{n\in\mathbb{N}} \|u_n\|^2_{\dot H^{\frac12}}$$

and the lemma is proved. □

Continuation of the proof of Theorem 4.3.1 Thanks to Lemma 4.3.1, it is enough to prove that

$$\lim_{J\to\infty}\lim_{n\to\infty} \|F_n^J\|_{L^2([0,\tau_n^\varepsilon];\dot H^{\frac12})} = 0. \tag{4.18}$$

We use the notations of Definition (4.2) of R_n^J and F_n^J. Let us first study $F_n^{J,1}$. If the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^k)_{n\in\mathbb{N}}$ are different, we have, because of law of product in Sobolev spaces

$$\begin{aligned}
\|F_n^{j,k}\|^2_{L^2([0,\tau_n^\varepsilon];\dot H^{\frac12})} &\stackrel{\text{def}}{=} \|\Phi^j\otimes\Phi^k\|^2_{L^2([0,\tau_n^\varepsilon];\dot H^{\frac12})}\\
&\leq \int_0^\infty \frac{1}{\lambda_n^j\lambda_n^k} \mathbb{1}_{[0,T^\star(\varphi^j)-\varepsilon]}\Big(\frac{t}{(\lambda_n^j)^2}\Big)\Big\|\Phi^j\Big(\frac{t}{(\lambda_n^j)^2}\Big)\Big\|^2_{\dot H^1}\\
&\qquad\qquad \times \mathbb{1}_{[0,T^\star(\varphi^k)-\varepsilon]}\Big(\frac{t}{(\lambda_n^k)^2}\Big)\Big\|\Phi^k\Big(\frac{t}{(\lambda_n^k)^2}\Big)\Big\|^2_{\dot H^1} dt\\
&\leq \int_0^\infty \frac{\lambda_n^k}{\lambda_n^j} \mathbb{1}_{[0,T^\star(\varphi^j)-\varepsilon]}\Big(\frac{(\lambda_n^k)^2 t}{(\lambda_n^j)^2}\Big)\Big\|\Phi^j\Big(\frac{(\lambda_n^k)^2 t}{(\lambda_n^j)^2}\Big)\Big\|^2_{\dot H^1}\\
&\qquad\qquad \times \mathbb{1}_{[0,T^\star(\varphi^k)-\varepsilon]}(t)\|\Phi^k(t)\|^2_{\dot H^1} dt.
\end{aligned}$$

Thus we infer that if the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^k)_{n\in\mathbb{N}}$ are different, then

$$\lim_{n\to\infty} \|F_n^{j,k}\|_{L^2([0,\tau_n^\varepsilon];\dot H^{\frac12})} = 0. \tag{4.19}$$

In the case when the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^k)_{n\in\mathbb{N}}$ are equal, let us observe, using (4.17), we have, for any appropriated choice of α_J,

$$\begin{aligned}\|F_n^{j,k}\|^2_{L^2([0,\tau_n^\varepsilon];\dot H^{\frac12})} &\leq \int_0^{T^\varepsilon_{j,k}} \Big\|\Phi^j(t,\cdot)\otimes\Phi^k\Big(t,\cdot-\frac{x_n^j-x_n^k}{\lambda_n^j}\Big)\Big\|^2_{\dot H^1}dt\\ &\leq 2\eta+\int_0^{T^\varepsilon_{j,k}} \Big\|\Phi^j_{\alpha_J}(t,\cdot)\otimes\Phi^k_{\alpha_J}\Big(t,\cdot-\frac{x_n^j-x_n^k}{\lambda_n^j}\Big)\Big\|^2_{\dot H^1}dt\end{aligned}$$

with $T^\varepsilon_{j,k}\stackrel{\rm def}{=}\min\{T^\star(\varphi^j),T^\star(\varphi^k)\}-\varepsilon$. As $\displaystyle\lim_{n\to\infty}\frac{|x_n^j-x_n^k|}{\lambda_n^j}=\infty$, for n large enough, the support of the two functions

$$\Phi^j_{\alpha_J}(t,\cdot)\quad\text{and}\quad\Phi^k_{\alpha_J}\Big(t,\cdot-\frac{x_n^j-x_n^k}{\lambda_n^j}\Big)$$

are disjoint and thus, thanks to (4.19), we have

$$\forall J\,,\ \lim_{n\to\infty}\|F_n^{J,1}\|_{L^2([0,\tau_n^\varepsilon];\dot H^1)}=0. \tag{4.20}$$

In order to treat the term $F_n^{J,2}$, let us observe that interpolation inequalities imply that

$$\|a\|_{\dot B^{\frac32\theta-1}_{\frac2\theta,\frac2\theta}}\leq\|a\|^\theta_{\dot H^{\frac12}}\|a\|^{1-\theta}_{\dot B^{-1}_{\infty,\infty}}.$$

Using Lemma 2.4 of [2] and the Minkowski inequality, we have

$$\|e^{t\Delta}r_n^J\|_{L^4(\mathbb{R}^+;\dot B^{\frac32\theta-\frac12}_{\frac2\theta,\frac2\theta})}\leq\|r_n^J\|^\theta_{\dot H^{\frac12}}\|r_n^J\|^{1-\theta}_{\dot B^{-1}_{\infty,\infty}}.$$

Properties of the sequence $(r_n^j)_{(j,n)\in\mathbb{N}^2}$ implies that

$$\forall\theta\in[0,1[\,,\ \lim_{J\to\infty}\limsup_{n\to\infty}\|e^{t\Delta}r_n^J\|_{L^4(\mathbb{R}^+;\dot B^{\frac32\theta-\frac12}_{\frac2\theta,\frac2\theta})}=0.$$

Law of product in Besov spaces (see for instance Chapter 2 of [2]) say that, for $\theta>2/3$, we have

$$\dot B^{\frac32\theta-\frac12}_{\frac2\theta,\frac2\theta}\cdot\dot B^{\frac32\theta-\frac12}_{\frac2\theta,\frac2\theta}\subset\dot H^{\frac12}.$$

It immediatly ensures that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|F_n^{J,3}\|_{L^4(\mathbb{R}^+;\dot H^{\frac12})}=0. \tag{4.21}$$

Moreover, as $\dot H^1$ is continuously embedded in $\dot B^{\frac32\theta-\frac12}_{\frac2\theta,\frac2\theta}$, then Lemma 4.3.1 ensures that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|F_n^{J,2}\|_{L^4(\mathbb{R}^+;\dot H^{\frac12})}=0.$$

Then Inequalitites (4.20) and (4.21) imply (4.18) and the first part of Theorem 4.3.1 is proved.

In order to prove the second part of Theorem 4.3.1, let us first write that

$$
\begin{aligned}
\|u_{\phi(n)}(t)\|^2_{\dot H^{\frac12}} &= \sum_{j=0}^J \|\Phi^j_n(t)\|^2_{\dot H^{\frac12}} + \|e^{t\Delta}r^J_n\|^2_{\dot H^{\frac12}} + \|R^J_n(t)\|^2_{\dot H^{\frac12}} + 2\sum_{\ell=1}^4 E^{J,\ell}_n(t) \quad \text{with}\\
E^{J,1}_n(t) &\stackrel{\text{def}}{=} \Big(\sum_{j=0}^J \Phi^j_n(t)\Big|R^J_n(t)\Big)_{\dot H^{\frac12}}\,,\\
E^{J,2}_n(t) &\stackrel{\text{def}}{=} \big(e^{t\Delta}r^J_n|R^J_n(t)\big)_{\dot H^{\frac12}}\,,\\
E^{J,3}_n(t) &\stackrel{\text{def}}{=} \Big(\sum_{j=0}^J \Phi^j_n(t)\Big|e^{t\Delta}r^J_n\Big)_{\dot H^{\frac12}}\,,\\
E^{J,4}_n(t) &\stackrel{\text{def}}{=} \sum_{\substack{0\leq j,k\leq J\\ j\neq k}} \big(\Phi^j_n(t)|\Phi^k_n(t)\big)_{\dot H^{\frac12}}\,.
\end{aligned}
\tag{4.22}
$$

We shall make a very frequent use of the fact that

$$
\forall (j,n)\in\mathbb{N}^2\,,\ t\in[0,\tau^\varepsilon_n] \Longrightarrow \frac{t}{(\lambda^j_n)^2}\in[0,T^\star(\varphi^j)-\varepsilon].
$$

The first step is the proof of the following lemma

Lemma 4.3.2. *For any (j,k) in $\mathbb{N}^2$ such that j and k are different, we have*

$$
\lim_{n\to\infty}\big\|\big(\Phi^j_n(t)|\Phi^k_n(t)\big)_{\dot H^{\frac12}}\big\|_{L^\infty([0,\tau^\varepsilon_n])} = 0.
$$

Proof. Let us consider a positive real number η. We first study the case when $(\lambda^j_n)_{n\in\mathbb{N}}$ and $(\lambda^k_n)_{n\in\mathbb{N}}$ are different. For any j, the set $\big\{\Phi^j(t)\,,t\in[0,T^\star(\varphi^j)-\varepsilon]\big\}$ is a compact subset of $\dot H^{\frac12}$ as the range of the compact interval $[0,T^\star(\varphi^j)-\varepsilon]$ by the continuous ap Φ^j. Thus it can be covered by a finite number of balls centered at functions the Fourier transform of which is included in a ring of $\mathbb{R}^3$ and of radius

$$
\frac{\eta}{2}\max\{\|\Phi^j\|^{-1}_{L^\infty([0,T^\star(\varphi^j)-\varepsilon];\dot H^{\frac12})}, \|\Phi^k\|^{-1}_{L^\infty([0,T^\star(\varphi^k)-\varepsilon];\dot H^{\frac12})}\}.
$$

Then we deduce that for any t in $[0,\tau^\varepsilon_n]$, it exists two functions f_j and g_k (which are choosen a finite family of functions depending on η, on (j,k) and on ε such that

$$
\begin{aligned}
\Big|\big(\Phi^j_n(t)|\Phi^k_n(t)\big)_{\dot H^{\frac12}} - \frac{1}{\lambda^j_n\lambda^k_n}\Big(f_j\Big(\frac{\cdot-x^j_n}{\lambda^j_n}\Big)\Big|g_k\Big(\frac{\cdot-x^k_n}{\lambda^k_n}\Big)\Big)_{\dot H^{\frac12}}\Big|&\\
= \Big|\big(\Phi^j((\lambda^j_n)^2t)|\Phi^k((\lambda^k_n)^2t)\big)_{\dot H^{\frac12}} - (f_j|g_k)_{\dot H^{\frac12}}\Big| &\leq \eta.
\end{aligned}
$$

When $\Big|\log\dfrac{\lambda^j_n}{\lambda^k_n}\Big|$ large enough, the support of the Fourier transform of two functions

$$
f_j\Big(\frac{\cdot-x^j_n}{\lambda^j_n}\Big) \quad\text{and}\quad g_k\Big(\frac{\cdot-x^k_n}{\lambda^k_n}\Big)
$$

are disjoint; thus the two functions are orthogonal in $\dot H^{\frac12}$ and we have, for $\Big|\log\dfrac{\lambda^j_n}{\lambda^k_n}\Big|$ large enough

$$
\big\|\big(\Phi^j_n(t)|\Phi^k_n(t)\big)_{\dot H^{\frac12}}\big\|_{L^\infty([0,\tau^\varepsilon_n])}\leq\eta. \tag{4.23}
$$

In the case when the two scales $(\lambda_n^j)_{n\in\mathbb{N}}$ and $(\lambda_n^k)_{n\in\mathbb{N}}$ are equal, let us write that because of the scaling invariance of the space $\dot H^{\frac12}$ we have

$$(\Phi_n^j(t)|\Phi_n^k(t))_{\dot H^{\frac12}} = \biggl(\Phi^j\Bigl(\frac{t}{(\lambda_n^j)^2},\cdot\Bigr)|\Phi^k\Bigl(\frac{t}{(\lambda_n^j)^2},\cdot-y_n^{j,k}\Bigr)\biggr)_{\dot H^{\frac12}} \quad\text{with}\quad y_n^{j,k}\stackrel{\text{def}}{=}\frac{x_n^k-x_n^j}{\lambda_n^k}\,\cdot$$

By definition of the $\dot H^{\frac12}$ scalar product, we get

$$(\Phi_n^j(t)|\Phi_n^k(t))_{\dot H^{\frac12}} = \Bigl\langle(|D|\Phi^j\Bigl(\frac{t}{(\lambda_n^j)^2},\cdot\Bigr),\Phi^k\Bigl(\frac{t}{(\lambda_n^j)^2},\cdot-y_n^{j,k}\Bigr)\Bigr\rangle.$$

Using that the set

$$\bigl\{|D|\Phi^j(t)\,,t\in[0,T^\star(\varphi^j)-\varepsilon]\bigr\}$$

is a compact subset of $\dot H^{-\frac12}$, it can be covered with a finite number of ball of radius less than

$$\frac{\eta}{2}\bigl(\|\Phi^j\|^{-1}_{L^\infty([0,T^\star(\varphi^j)-\varepsilon]}+1\bigr)^{-1}$$

and centered on functions of $\mathcal{D}(\mathbb{R}^3\setminus\{0\})$ for the $\dot H^{-\frac12}$ topology. The same is true for the set

$$\bigl\{\Phi^k(t)\,,t\in[0,T^\star(\varphi^j)-\varepsilon]\bigr\}$$

in the $\dot H^{\frac12}$ topology. Thus two functions f_j and g_k exists in $\mathcal{D}(\mathbb{R}^3\setminus\{0\})$ such that

$$\bigl|(\Phi_n^j(t)|\Phi_n^k(t))_{\dot H^{\frac12}}\bigr|\leq\eta+\Bigl|\int_{\mathbb{R}^3}f_j(x)g_k(x-y_n^{j,k})dx\Bigr|.$$

As $\lim\limits_{n\to\infty}|y_n^{j,k}|=\infty$, the above integral is 0. With (4.23), the lemma is proved. □

Continuation of the proof of Theorem 4.3.1. Lemma 4.3.2 implies that, for any J

$$\lim_{n\to\infty}\biggl\|\Bigl\|\sum_{j=0}^J\Phi_n^j(t)\Bigr\|^2_{\dot H^{\frac12}}-\sum_{j=0}^J\|\Phi_n^j(t)\|^2_{\dot H^{\frac12}}\biggr\|_{L^\infty([0,\tau_n^\varepsilon])}=0.$$

Let us introduce $\mathcal{J}_{\rho_0}$ the set of indices such that the associated profile φ^j does not belong to the open ball contered at the origin and of radius ρ_0 where ρ_0 is given by Theorem 3.2.1 on page 29. It is a finite set. Moreover thanks to Theorem 3.2.1, we have

$$\forall j\notin\mathcal{J}_{\rho_0}\,,\ \|\Phi_n^j\|_{L^\infty(\mathbb{R}^+;\dot H^{\frac12})}=\|\Phi^j\|_{L^\infty(\mathbb{R}^+;\dot H^{\frac12})}\leq\|\varphi^j\|_{\dot H^{\frac12}}\,. \tag{4.24}$$

Thus we infer that

$$\limsup_{n\to\infty}\Bigl\|\sum_{j=0}^J\Phi^j(t)\Bigr\|^2_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})}\leq M_\varepsilon\stackrel{\text{def}}{=}\sum_{j\in\mathcal{J}_{\rho_0}}\|\Phi^J\|^2_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})}+\sum_j\|\varphi^j\|^2_{\dot H^{\frac12}}\,.$$

Using Cauchy-Schwarz inequality, we get, with the notations of (4.22),

$$\begin{aligned}\limsup_{n\to\infty}\|E_n^{J,1}\|_{L^\infty([0,\tau_n^\varepsilon])} &\leq \limsup_{n\to\infty}\Bigl\|\sum_{j=0}^J\Phi^j(t)\Bigr\|_{L^\infty([0,\tau_n^\varepsilon])}\limsup_{n\to\infty}\|R_n^J\|_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})}\\ &\leq M_\varepsilon\|R_n^J\|_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})}\,.\end{aligned}$$

Thus, using Inequality (4.13) (which is now proven), we get

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|E_n^{J,1}\|_{L^\infty([0,\tau_n^\varepsilon])}=0. \tag{4.25}$$

The estimate of $E_n^{J,2}$ is easy. Indeed, Cauchy-Schwarz inequality implies that

$$\big\|\big(e^{t\Delta}r_n^J|R_n^J(t)\big)_{\dot H^{\frac12}}\big\|_{L^\infty([0,\tau_n^\varepsilon])}\leq\|r_n^J\|_{\dot H^{\frac12}}\|R_n^J\|_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})}.$$

Because $\limsup_{n\to\infty}\|r_n^J\|_{\dot H^{\frac12}}$ is less than or equal to $M\stackrel{\rm def}{=}\sup_n\|u_{0,n}\|_{\dot H^{\frac12}}$, we get

$$\limsup_{n\to\infty}\big\|\big(e^{t\Delta}r_n^J|R_n^J(t)\big)_{\dot H^{\frac12}}\big\|_{L^\infty([0,\tau_n^\varepsilon])}\leq M\limsup_{n\to\infty}\|R_n^J\|_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})}.$$

Using Lemma 4.13, we get

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|E_n^{J,2}\|_{L^\infty([0,\tau_n^\varepsilon])}=0. \tag{4.26}$$

The term $E_n^{J,3}$ requires more care. Let us consider an integer J' greater than the maximum of $\mathcal{J}_{\rho_0}$. For any J greater than J', we have, because of Lemma 4.3.2 and Assertion (4.24)

$$\begin{aligned}\limsup_{n\to\infty}\Big\|\sum_{j=J'}^J\Phi_n^j(t)\Big\|^2_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})} &\leq \limsup_{n\to\infty}\sum_{j=J'}^J\|\Phi_n^j\|^2_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})}\\ &\leq \sum_{j=J'}^J\|\Phi^j\|^2_{L^\infty(\mathbb{R}^+;\dot H^{\frac12})}\\ &\leq \sum_{j=J'}^J\|\varphi^j\|^2_{\dot H^{\frac12}}.\end{aligned}$$

Let us define J_η such that

$$\sum_{j\geq J_\eta}\|\varphi^j\|^2_{\dot H^{\frac12}}\leq\frac{\eta}{2\sup_{n,J}\|r_n^J\|_{\dot H^{\frac12}}}\,\cdot$$

Now let us consider J such that J_η. By definition of $E_n^{J,3}(t)$, we have

$$\limsup_{n\to\infty}\|E_n^{J,3}(t)\|_{L^\infty([0,\tau_n^\varepsilon])}\leq\frac{\eta}{2}+\sum_{j=0}^{J_\eta}\limsup_{n\to\infty}\big\|\big(\Phi_n^j(t)|e^{t\Delta}r_n^J\big)_{\dot H^{\frac12}}\big\|_{L^\infty([0,\tau_n^\varepsilon])}.$$

Using the scaling properties of the space $\dot H^{\frac12}$, let us write that, by definition of the scalar product of $\dot H^{\frac12}$,

$$\big(\Phi_n^j(t)|e^{t\Delta}r_n^J\big)_{\dot H^{\frac12}}=\Big\langle|D|\Phi^j\Big(\frac{t}{(\lambda_n^j)^2}\,,\cdot\Big),\lambda_n^j(e^{(\lambda_n^j)^2t\Delta})r_n^j(\lambda_n^j\cdot+x_n)\Big\rangle.$$

Using the compactness of $\big\{\Phi^j(t)\,,t\in[0,T^\star(\varphi^j)-\varepsilon]\big\}$, we can recover the set

$$\big\{|D|\Phi^j(t)\,,t\in[0,T^\star(\varphi^j)-\varepsilon]\big\}$$

by a finite number of balls (for the $\dot H^{-\frac12}$-topology) of radius

$$\frac{\eta}{4J_\eta \sup_{n,J} \|r_n^J\|_{\dot H^{\frac12}}}$$

and centered at function of $\mathcal{D}(\mathbb{R}^3)$. Thus we get, for any t in $[0,\tau_n^\varepsilon]$,

$$\begin{aligned}\big|\big(\Phi_n^j(t)|e^{t\Delta}r_n^J\big)_{\dot H^{\frac12}}\big| &\leq \frac{\eta}{4J_\eta} + \big\langle f_\ell, \lambda_n^j(e^{(\lambda_n^j)^2t\Delta})r_n^j(\lambda_n^j\cdot+x_n)\big\rangle\\ &\leq \frac{\eta}{4J_\eta} + \|f_\ell\|_{\dot B^1_{1,1}}\|r_n^J\|_{\dot B^{-1}_{\infty,\infty}}.\end{aligned}$$

Thus we get

$$\|E_n^{J,3}\|_{L^\infty([0,\tau_n^\varepsilon]} \leq \frac{3}{4\eta} + C\eta\|r_n^J\|_{\dot B^{-1}_{\infty,\infty}}.$$

Using (4.25), (4.26) and Lemma 4.3.2, we conclude the proof of the Theorem. $\square$

4.4 A blow up theorem

The purpose of this section is to prove the following theorem.

Theorem 4.4.1. *Let us consider a maximal solution u to (NS) which belongs to the space*

$$C([0,T^\star[;\dot H^{\frac12})\cap L^2_{\rm loc}([0,T^\star[;\dot H^{\frac32}).$$

We have

$$T^\star<\infty \Longrightarrow \limsup_{t\to T^\star}\|u(t)\|_{\dot H^{\frac12}} = \infty.$$

Proof. We argue by contradiction. Let us denote by $I_{\mathcal B}$ the interval of real numbers A such that it exists a solution u to (NS) which blows up for some finite $T^\star$ and is such that

$$\limsup_{t\to T^\star}\|u(t)\|_{\dot H^{\frac12}} \leq A.$$

The theorem claims that $I_{\mathcal B}$ is empty. Let us assume that $I_{\mathcal B}$ is not empty; this will lead to some contradiction.

Because of Theorem 3.2.1 on page 29, $I_{\mathcal B}$ does not intersect the ball centered at the origin and of radius ρ_0 defined in this theorem. Thus if we denote by $\mathcal{A}_c$ the infimum of $I_{\mathcal B}$, we have

$$\mathcal{A}_c \geq \rho_0. \tag{4.27}$$

Now let us consider a sequence $(u_n)_{n\in\mathbb N}$ of solutions to (NS) which blow up for finite time $(T_n^\star)_{n\in\mathbb N}$ such that

$$\mathcal{A}_c \leq \limsup_{t\to T_n^\star}\|u_n(t)\|_{\dot H^{\frac12}} \leq \mathcal{A}_c + \frac{1}{2n}\cdot \tag{4.28}$$

By definition of the upper limit, a sequence $(t_n)_{n\in\mathbb N}$ exists such that

$$\|u_n(t_n)\|_{\dot H^{\frac12}} \geq \mathcal{A}_c - \frac1n \quad\text{and}\quad \|u_n\|_{L^\infty([t_n,T_n^\star[;\dot H^{\frac12})} \leq \mathcal{A}_c + \frac1n\cdot \tag{4.29}$$

Let us consider the sequence

$$v_{0,n} \stackrel{\rm def}{=} \sqrt{T_n^\star - t_n}\, u_n\Big(t_n, \sqrt{T_n^\star-t_n}\, x\Big)\cdot$$

By definition of $(u_n)_{n\in\mathbb{N}}$, the sequence $(v_{0,n})_{n\in\mathbb{N}}$ gives birth to a family of solutions $(v_n)_{n\in\mathbb{N}}$ to (NS) the life span of which is 1; because of (4.29), we have

$$\forall n \in \mathbb{N}\,,\ \|v_{0,n}\|_{\dot H^{\frac12}} \geq \mathcal{A}_c - \frac{1}{n} \quad \text{and} \quad \|v_n\|_{L^\infty([0,1[;\dot H^{\frac12})} \leq \mathcal{A}_c + \frac{1}{n}\,\cdot \tag{4.30}$$

Let us prove the following proposition, which is the one of the main step of the proof.

Proposition 4.4.1. *Let $(v_{0,n})_{n\in\mathbb{N}}$ be a family of initial data such that for any n, the life span of the solution associated with $v_{0,n}$ is 1 and which satisfies (4.30). Let us consider an extraction, a sequence of scales and cores, and a sequence of profiles given by Theorem 2.1.1 on page 13. For any j, we have that λ_n^j is less than or equal to* 1.

Proof. Let us apply Theorem 2.1.1. We have

$$v_{0,\phi(n)} = \sum_{j=0}^{J} \frac{1}{\lambda_n^j}\varphi^j\left(\frac{\cdot - x_n^j}{\lambda_n^j}\right) + r_n^J. \tag{4.31}$$

A consequence of (4.30) is that

$$\lim_{n\to\infty} \|v_n\|_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})} = \mathcal{A}_c.$$

Inequality (4.14) of Theorem 4.3.1 allows us to write that, for any t in $[0,\tau_n^\varepsilon]$,

$$\|v_n(t)\|^2_{\dot H^{\frac12}} \geq \left\|\frac{1}{\lambda_n^0}\Phi^0\Big(\frac{t}{(\lambda_n^0)^2},\frac{x-x_n^0}{\lambda_n^0}\Big)\right\|^2_{\dot H^{\frac12}} + \sum_{j=1}^{J}\left\|\frac{1}{\lambda_n^j}\Phi^j\Big(\frac{t}{(\lambda_n^j)^2},\frac{x-x_n^j}{\lambda_n^j}\Big)\right\|^2_{\dot H^{\frac12}} - o_n^J$$

where o_n^J is non negative and $\lim\limits_{J\to\infty}\limsup\limits_{n\to\infty} o_n^J = 0$. We deduce that, for any t in $[0,\tau_n^\varepsilon]$ we get

$$\begin{aligned}\|v_n(t)\|^2_{\dot H^{\frac12}} \geq{}& \left\|\frac{1}{\lambda_n^0}\Phi^0\Big(\frac{t}{(\lambda_n^0)^2},\frac{x-x_n^0}{\lambda_n^0}\Big)\right\|^2_{\dot H^{\frac12}}\\ &+\sum_{j=1}^{J}\inf\bigg\{\|\Phi^j(t)\|^2_{\dot H^{\frac12}}\,,t\in\Big[0,\Big(\frac{\lambda_n^0}{\lambda_n^j}\Big)^2(T^\star(\varphi_0)-\varepsilon)\Big]\bigg\} - o_n^J.\end{aligned}$$

Passing to the supremum in time gives

$$\begin{aligned}\|v_n\|^2_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})} \geq{}& \|\Phi^0\|^2_{L^\infty([0,T^\star(\varphi^0)-\varepsilon];\dot H^{\frac12})}\\ &+\sum_{j=1}^{J}\inf\bigg\{\|\Phi^j(t)\|^2_{\dot H^{\frac12}}\,,t\in\Big[0,\Big(\frac{\lambda_n^0}{\lambda_n^j}\Big)^2(T^\star(\varphi_0)-\varepsilon)\Big]\bigg\} - o_n^J.\end{aligned} \tag{4.32}$$

Let us analyze this inequality. First of all, we get,

$$\|v_n\|^2_{L^\infty([0,\tau_n^\varepsilon];\dot H^{\frac12})} + o_n^J \geq \|\Phi^0\|^2_{L^\infty([0,T^\star(\varphi^0)-\varepsilon];\dot H^{\frac12})}.$$

Passing to the limit in n and J gives that

$$\|\Phi^0\|^2_{L^\infty([0,T^\star(\varphi^0)-\varepsilon];\dot H^{\frac12})} \leq \mathcal{A}_c^2.$$

By definition of $\mathcal{A}_c$, this implies the following proposition.

Proposition 4.4.2. *A solution Φ to (NS) exists such that $\|\Phi\|_{L^\infty([0,T^\star[;\dot H^{\frac12})}$ is equal to $\mathcal{A}_c$ and $\limsup_{t\to T^\star} \|\Phi(t)\|_{\dot H^{\frac12}} = \mathcal{A}_c$.*

Let us continue the proof of Proposition 4.4.1. Let us consider in Inequality (4.32) the indices j such that j does not belongs to $\mathcal{G}$. Because of Theorem 3.2.1 on page 29, we get

$$\inf\big\{\|\Phi^j(t)\|_{\dot H^{\frac12}}\,,\ t\in[0,T^\star(\varphi^j)[\big\}\geq \rho_0.$$

Inequality (4.32) implies that this set is empty. Let us consider that the set of j such that

$$\varphi^j\in\mathcal{G}\quad\text{and}\quad \lim_{n\to\infty}\frac{\lambda_n^0}{\lambda_n^j}=0.$$

As the solution is continuous in time with value in $\dot H^{\frac12}$, we have

$$\lim_{n\to\infty}\inf\left\{\|\Phi^j(t)\|^2_{\dot H^{\frac12}}\,, t\in\left[0,\Big(\frac{\lambda_n^0}{\lambda_n^j}\Big)^2(T^\star(\varphi_0)-\varepsilon)\right]\right\}=\|\varphi^j\|^2_{\dot H^{\frac12}}\,.$$

Thus for such indices j, $\varphi^j\equiv 0$ and the proposition is proved. □

Continuation of the proof of Theorem 4.4.1 It reduces to a backward uniqueness argument once we have proved the following proposition.

Proposition 4.4.3. *Let Φ be a maximal solution of (NS) given by Proposition 4.4.2. Then, for all s in $[-3/2,1/2[$,*

$$\Phi\in C([0,T^\star[;\dot H^s)\quad\textit{and}\quad \lim_{t\to T^\star}\|\Phi(t)\|_{\dot H^s}=0.$$

Proof. Now let us consider a critical blow up solution given by Proposition 4.4.2. Let us consider a sequence $(t_n)_{n\in\mathbb{N}}$ which tends to $T^\star$ such that $\|\Phi(t_n)\|_{\dot H^{\frac12}}$ tends to $\mathcal{A}_c$. Let us apply Proposition 4.4.1 to the sequence

$$v_{0,n}(x)\stackrel{\text{def}}{=}\sqrt{T^\star-t_n}\,\Phi(t_n,\sqrt{T^\star-t_n}\,x).$$

Profiles decomposition of $v_{0,n}$ contains only scales less than or equal to 1. By rescaling, we deduce that

$$\Phi(t_n,x)=\sum_{j=0}^J\frac{1}{\lambda_n^j}\varphi^j\Big(\frac{x-x_n^j}{\lambda_n^j}\Big)+r_n^J\quad\text{with}\quad\forall j\in\mathbb{N}\,,\ \lambda_n^j\leq\sqrt{T^\star-t_n}.$$

This implies that the sequence $(\Phi(t_n))_{n\in\mathbb{N}}$ tends weakly to 0.

Thanks to law of product in Sobolev spaces, the fact that Φ is a solution of Navier-Stokes equation gives that

$$\begin{aligned}\|\partial_t\Phi\|_{L^\infty([0,T^\star[;\dot H^{-\frac32})} &\leq \|\Delta\Phi\|_{L^\infty([0,T^\star[;\dot H^{-\frac32})}+\big\|\mathrm{div}(\Phi\otimes\Phi)\big\|^2_{L^\infty([0,T^\star[;\dot H^{-\frac32})}\\ &\lesssim \|\Phi\|_{L^\infty([0,T^\star[;\dot H^{\frac12})}+\|\Phi\|^2_{L^\infty([0,T^\star[;\dot H^{\frac12})}.\end{aligned}\tag{4.33}$$

This implies that the function $\widetilde\Phi(t)\stackrel{\text{def}}{=}\Phi(t)-\Phi(0)$ in time with value in $\dot H^{-\frac32}$ and satisfies a Cauchy condition when t tends to $T^\star$ which means that

$$\forall\varepsilon>0\,,\ \exists t_\varepsilon>0\,/\ \forall(t,t')\in]T^\star-t_\varepsilon,T^\star[\,,\ \|\widetilde\Phi(t)-\widetilde\Phi(t')\|_{\dot H^{-\frac32}}<\varepsilon.$$

This implies that the function $\widetilde\Phi$ has a limit in the space $\dot H^{-\frac32}$ when t tend to $T^\star$. As the sequence $(\Phi(t_n))_{n\in\mathbb{N}}$ tends weakly to 0 when n tends to infinity, we deduce that $\Phi(0)$ belongs to $\dot H^{-\frac32}$ and then (4.33) implies the result after interpolation. □

Continuation of the proof of Theorem 4.4.1 Now, we are going to explain why such a solution must be 0 which gives a contradiction.

Let us introduce the conpcet of suitable solution of Navier-Stokes equations on a domain of $\mathbb{R}^+ \times \mathbb{R}^3$.

Definition 4.4.1. *Let ω be an open set in $\mathbb{R}^3$. We say that a pair (u,p) is a suitable weak solution of the NavierStokes equations on the set $\omega \times]-T_1,T[$ if the following conditions hold:*

$$u \in L^\infty(]-T_1,T[;L^2(\Omega)) \cap L^2(-T_1,T[;\dot{H}^1(\omega)) \quad \textit{and} \quad p \in L^{\frac{3}{2}}(]-T_1,T[\times\omega)$$

and (u,p) satisfy the Navier-Stokes equations in the distribution sense and in addition the following local energy inequality

$$\int_\omega \varphi(t,x)|u(x,t)|^2 dx + 2\int_{-T_1}^T \int_\omega \varphi(t,x)|\nabla u(t,x)|^2 dxdt$$
$$\leq \int_{-T_1}^T \int_\omega \Big(|u(t,x)|^2 \big(\Delta\varphi(t,x) + \partial_t\varphi(t,x)\big) + u(t,x)\cdot\nabla\varphi(t,x)(|u(t,x)|^2 + 2p(t,x))\Big) dxdt$$

for almost all t in $]-T_1,T[$ and for any non-negative function φ in $\mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^3)$ in a neighbourhood of the parabolic boundary $\partial' Q \stackrel{\text{def}}{=} \omega \times \{t=-T_1\} \cup \partial\omega \times [-T_1,T]$.

The following statement is a local regularity criterion proved in [11]

Lemma 4.4.1. *There exist positive absolute constants ε_0 and $(c_k)_{k\in\mathbb{N}}$ with the following property: If a suitable weak solution (u,p) of (NSE) on Q_1 where $Q_r \stackrel{\text{def}}{=}]-r^2,0[\times B(0,r)$ satisfies the condition*

$$\int_{Q_1} \big(|u(t,x)|^3 + |p(t,x)|^{\frac{3}{2}} dxdt < \varepsilon_0,$$

then u is smooth on $\overline{Q}_{\frac{1}{2}}$ and satisfies the estimate

$$\sup_{(t,x)\in\overline{Q}_{\frac{1}{2}}} |\nabla^{k-1} u(t,x)| \leq c_k.$$

Let us observe that because of Sobolev embeddings, Φ belongs to $L^\infty([0,T^\star];L^3)$. Because Riesz operators maps $L^{\frac{3}{2}}(\mathbb{R}^3)$ into itsself, we have the pressure p belongs to $L^\infty([0,T^\star];L^{\frac{3}{2}})$. Thus

$$\int_0^{T^\star}\int_{\mathbb{R}^3} \big(|\Phi(t,x)|^3 + |p(t,x)|^{\frac{3}{2}} dxdt\big) < \infty.$$

It implies that, for any positive ε, a radius R_ε exists such that, for any x_0 such that $|x_0|$ greater than or equal to R,

$$\int_0^{T^\star}\int_{B(x_0,\sqrt{T^\star})} \big(|\Phi(t,x)|^3 + |p(t,x)|^{\frac{3}{2}}\big) dxdt \leq \varepsilon.$$

Then applying Lemma 4.4.1 after a rescaling and time translation we get that Φ is smooth on $[(3/4)T^\star, T^\star] \times (\mathbb{R}^3 \setminus B(0,R_\varepsilon))$. Let us write the vorticity equation

$$\partial_t\Omega - \Delta\Omega + \Phi\cdot\nabla\Omega + \Omega\cdot\nabla\Phi = 0$$

on the set $[(3/4)T^\star, T^\star] \times (\mathbb{R}^3 \setminus B(0,R_\varepsilon))$. Because Φ is smooth up to $T^\star$, the vorticity $\Omega(T\star)$ is identically equal to 0 on $\mathbb{R}^3 \setminus B(0,R_\varepsilon))$. As Φ in bounded in $[(3/4)T^\star, T^\star] \times (\mathbb{R}^3 \setminus B(0,R_\varepsilon))$, we have the pointwise inequality

$$\big|\partial_t\Omega(t,x) - \Delta\Omega(t,x)\big| \leq C\big(|\Omega(t,x)| + |\nabla\Omega(t,x)|\big).$$

Then we can apply the following backward uniqueness result proved in [12].

Theorem 4.4.2 (Backwards uniqueness). *Let us consider a vector valued distribution* Ω *in* $\dot{H}^2_{\rm loc}((\mathbb{R}^3 \setminus B(0,R_\varepsilon))\times]-\delta,0])$ *and satisfies*

$$\big|\partial_t\Omega(t,x) - \Delta\Omega(t,x)\big| \leq C\big(|\Omega(t,x)| + |\nabla\Omega(t,x)|\big)$$

on $(\mathbb{R}^3 \setminus B(0,R_\varepsilon))\times]-\delta,0]$. *If* $\Omega(0,x) = 0$ *on* $B(\mathbb{R}^3 \setminus B(0,R_\varepsilon))$, *then*

$$\Omega \equiv 0 \text{ on the set } (\mathbb{R}^3 \setminus B(0,R_\varepsilon))\times]-\delta,0].$$

In order to conclude the proof of Theorem 4.4.1, let us notice that for any time t in $]0,T^\star[$, the vector field u has components which are analytic in the variable x on the whole space $\mathbb{R}^3$ (see for instance [6] and [23]) . Thus for any t in $[3/4T^\star, T^\star]$, $\Omega(t) \equiv 0$ and thus $\Phi(t) \equiv 0$ which is obviously in contradiction with the definition of Φ. $\square$

Bibliography

[1] P. Auscher, S. Dubois and P. Tchamitchiann, On the stability of global solutions to NavierStokes equations in the space, *Journal de Mathématiqeus Pures et Appliquées*, **83**, 2004, pages 673–697 .

[2] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, **343**, Springer-Verlag Berlin Heidelberg, 2011.

[3] H. Bahouri and P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, *American Journal of Math*, **121** (1999), pages 131-175.

[4] L. Cafarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weal solutions of Navier-Stokes equations, *Communications in Pure and Applied Mathematics*, **35**, 1982, pages 771–831.

[5] M. Cannone, Y. Meyer and F. Planchon, Solutions autosimilaires des équations de Navier-Stokes, Séminaire "Équations aux Dérivées Partielles de l'École polytechnique, Exposé VIII, 1993–1994.

[6] J.-Y. Chemin, Localization in Fourier space and Navier-Stokes system, *Phase Space Analysis of Partial Differential Equations*, Proceedings 2004, CRM series, pages 53–136.

[7] J.-Y. Chemin and I. Gallagher, Large, global solutions to the Navier-Stokes equations, slowly varying in one direction, *Transactions of the Americal Mathematical Society*, **362**, 2010, pages 2859–2873.

[8] J.-Y. Chemin, I. Gallagher and P. Zhang, On large perturbation of global solutions of the 3D incompressible Navier-Stokes equations, *t̀o appear in Journal für reine und angewandte Mathematik.*

[9] J.-Y. Chemin, I. Gallagher and M. Paicu, Global regularity for some classes of large solutions to the Navier-Stokes equations, *Annals of Mathematics*, **173**, 2011, pages 986–1012.

[10] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non-lipschitziens et équations de Navier-Stokes, *Journal of Differential Equations*, **121**, 1995, pages 314–328.

[11] L. Escauriaza, G. A. Serëgin and V. Sverak, $L^{3,\infty}$ -solutions of Navier-Stokes equations and backward uniqueness, (Russian) *Uspekhi Mat. Nauk*, **58** (2003), no. 2(350), 3-44; translation in *Russian Math. Surveys*, **58** (2003), no. 2, 211-250.

[12] L. Escauriaza, G. Serëgin, V. Sverak, Backward uniqueness for parabolic equations, *Archiv for Rational Mechanic and Analysis*, **169**, 2003, pages 147157.

[13] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archiv for Rational Mechanic Analysis*, **16**, 1964, pages 269–315.

[14] I. Gallagher, Profile decomposition for solutions of the Navier-Stokes equations, *Bulletin de la Société Mathématique de France*, **129**, 2001, pages 285-316.

[15] I. Gallagher, D. Iftimie, and F. Planchon, Non-explosion en temps grand et stabilité de solutions globales des équations de Navier-Stokes, *Notes aux Comptes Rendus de l'Académie des Sciences*, **334** , Série I, 2002, pages 289–292.

[16] I. Gallagher, D. Iftimie, and F. Planchon, Asymptotics and stability for global solutions to the Navier-Stokes equations, *Annales de l' Institut Fourier*, **53**, 2003, pages 1387–1424.

[17] I. Gallagher, G. Koch and F. Planchon, A profile decomposition approach to the $L^\infty_t(L^3_x)$ Navier-Stokes regularity criterion, *Mathematische Annalen*, **355**, 2013, pages 1527–1559.

[18] P. Gérard, Description des défauts de compacité de l'injection de Sobolev, *ESAIM, Control Optimal et Calcul des variations*, **3**, pages 213–233.

[19] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, *Journal of functionnal Analysis*, **9**, 1972, pages 296–305.

[20] C. Kenig and G. Koch, An alternative approach to regularity for the NavierStokes equations in critical spaces *Annales de l'Institut Henri Poincaré Analyse non Linéaire*, **28**, 2011, pages 15–187.

[21] H. Koch and D. Tataru: Well-posedness for the Navier-Stokes equations, *Advances in Mathematics*, **157**, 2001, pages 22–35.

[22] P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, Research Notes in Mathematics, **431**, 2002.

[23] P.-G. Lemarié-Rieusset, Nouvelles remarques sur l'analyticité des solutions milds des équations de Navier-Stokes dans $\mathbb{R}^3$, (French) [Further remarks on the analyticity of mild solutions for the Navier-Stokes equations in R^3] *Compte Rendus de l'Académie des Sciences de Paris, Série I* **338 (6)**, 2004, pages 443446.

[24] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica*, **63**, 1933, pages 193–248.

[25] F. Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem, *Communication in Pure and Applied Mathematics*, **51**, 1998, pages 241257.

[26] Y. Meyer, *Wavelets, Paraproducts and Navier-Stokes*, Current Developments in Mathematics, International Press, Cambridge, Massachussets, 1996.

[27] C. Oseen, Sur les formules de Green généralisées qui se présentent dans l'hydrodynamique et sur quelques unes de leurs applications, (première partie) *Acta Mathematica*, **34**, 1911, pages 205–284.

[28] C. Oseen, Sur les formules de Green généralisées qui se présentent dans l'hydrodynamique et sur quelques unes de leurs applications, (seconde partie) *Acta Mathematica*, **35**, 1912, pages 97–192.

[29] T. Runst and W. Sickel: *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, Nonlinear Analysis and Applications, 3. Walter de Gruyter & Co., Berlin, 1996.

[30] W. Rusin and V. Sverák, Minimal initial data for potential NavierStokes singularities, *Journal of Functional Analysis* **260**, 2011, pages 879–891.

[31] F.Weissler, The Navier-Stokes initial value problem in L^p, *Archiv for Rational Mechanics and Analysis*, **74**, 1980, pages 219–230.