”About the possible blow up for incompressible 3D Navier-Stokes equation”.

The incompressible Navier-Stokes system in $\mathbb{R}^3$

$$(NS) \quad \begin{cases} \partial_t u + \mathrm{div}(u \otimes u) - \Delta u = -\nabla p \\ \mathrm{div}\, u = 0 \\ u_{|t=0} = u_0 \end{cases}$$

where

$$v = (v^1, \cdots, v^d), \quad v \cdot \nabla = \sum_{j=1}^{d} v^j \frac{\partial}{\partial x_j}, \quad \nabla p = \Big(\frac{\partial p}{\partial x_1}, \cdots, \frac{\partial p}{\partial x_d} \Big),$$

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \quad \text{and} \quad \mathrm{div}\, v = \sum_{j=1}^{d} \frac{\partial v^j}{\partial x_j} \cdot$$

Basic facts

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^3)}\,dt' = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)} \quad \text{and}$$

$\lambda u(\lambda^2 t, \lambda x)$ is solution on $[0, \lambda^{-2}T]$ if u is solution on $[0, T]$.

Global weak solution (J. Leray, 1934) and give the right regularity in Sobolev spaces for the initial data to get uniqueness : $\dot{H}^{\frac{1}{2}}$

The case when the initial data is more regular than the scaling.

Theorem *Let* u_0 *be in* $\dot{H}^s$ *with* s *in* $]1/2, 3/2[$. *Then a unique solution exists in the space in the space*

$$C([0,T]; \dot{H}^s \cap L^2([0,T]; \dot{H}^{s+1}).$$

Moreover a constant c_s *exists such that if* $T^\star(u_0)$ *denotes the maximal time of existence, then*

$$T^\star(u_0)\|u_0\|_{\dot{H}^s}^{\frac{4}{2s-1}} \geq c_s.$$

Corollary *For such solution, if* $T^\star(u_0)$ *is finite, we have*

$$\|u(t)\|_{\dot{H}^s} \geq c_s(T^\star(u_0)-t)^{-\frac{1}{2}\left(s-\frac{1}{2}\right)}.$$

We have (Escauriaza, Segerin and Sverak, 2003)

$$\limsup_{t \to T^\star(u_0)} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} = \infty\,.$$

With the notation $\sigma_s \stackrel{\text{def}}{=} \frac{4}{2s-1}$ do we have

$$\limsup_{t \to T^\star(u_0)} (T^\star(u_0) - t)\|u(t)\|^{\sigma_s}_{\dot{H}^s} = \infty\,?$$

P. Gérard's profiles theory : the concept of scales and cores

Definition *A sequence* $(\lambda_{n,j}, x_{n,j})_{(n,j)\in\mathbb{N}^2}$ *of* $]0,\infty[\times\mathbb{R}^3$ *is a sequence of scales and cores if*

— *for all positive* j *and* j' *such that* $j \neq j'$, *we have*

$$\lim_{n\to\infty} \frac{\lambda_{n,j}}{\lambda_{n,j'}} + \frac{\lambda_{n,j'}}{\lambda_{n,j}} = \infty \quad \text{or} \quad \lambda_{n,j} \equiv \lambda^{j'}$$

— *for all positive* j *and* j' *such that* $j \neq j'$, *we have*

$$\lambda_{n,j} \equiv \lambda^{j'} \Longrightarrow \lim_{n\to\infty} \frac{|x_{n,j} - x_{n,j'}|}{\lambda_{n,j}} = \infty.$$

Notation For a given s in $]0,3/2[$, defining $p \stackrel{\text{def}}{=} \dfrac{6}{3-2s}$,

$$(\Lambda_n^j \varphi)(t,x) \stackrel{\text{def}}{=} \lambda_{n,j}^{-\frac{3}{p}} \varphi\biggl(\frac{t}{\lambda_{n,j}^2}, \frac{x - x_{n,j}}{\lambda_{n,j}}\biggr).$$

P. Gérard's profiles theory : the decomposition theorem

Theorem (P. Gérard, 1996) *Let* s *be in* $]0,3/2[$. *We consider* $(u_n)_{n\in\mathbb{N}}$ *a bounded sequence in* $\dot{H}^s$. *Then there exist*
– *a sequence* $(\varphi^j)_{j\in\mathbb{N}}$ *of functions in* $\dot{H}^s$,
– *a sequence* $(\psi_n^j)_{(n,j)\in\mathbb{N}^2}$ *of functions in* $\dot{H}^s$,
– *a sequence of scales and cores* $(\lambda_{n,j}, x_{n,j})_{(n,j)\in\mathbb{N}^2}$
such that, up to an omitted extraction, we have, for any J *in* $\mathbb{N}$,

$$u_n = \sum_{j=0}^{J} \Lambda_n^j \varphi^j + \psi_n^J \quad \textit{with}$$

$$\|u_n\|_{\dot{H}^s}^2 = \sum_{j=0}^{J} \|\varphi^j\|_{\dot{H}^s}^2 + \|\psi_n^J\|_{\dot{H}^s}^2 + o_J(1) \quad \textit{and}$$

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\psi_n^J\|_{L^p} = 0 \quad \textit{and} \quad p = \frac{6}{3-2s}\cdot$$

Remark We have that $\lambda_{n,j}^{\frac{3}{p}} u_n(\lambda_{n,j}(\cdot + x_{n,j})) \rightharpoonup \varphi^j$.

The non interaction theorem

Theorem [E. Poulon, 2015] *Let* $(u_{0,n})_{n\in\mathbb{N}}$ *be a bounded sequence of* $\dot{H}^s$*. Then, up to an extraction, we have*

$$\lim_{n\to\infty} T^\star(u_{0,n}) \geq \inf\{T^\star(\varphi^j),\ j \in \mathcal{J}_0\}$$

where $\mathcal{J}_0$ *is the set of* j *such that* $\lambda_{n,j} \equiv 1$ *in the profile decomposition.*

Ideas of the proof

— The scaling gives

$$NS(\Lambda_n^j(\varphi^j)) = \Lambda_n^j(NS(\varphi^j))$$

— If $\lambda_{n,j} \not\equiv 1$, then $\lim_{n\to\infty} \|\Lambda_n^j \varphi^j\|_{\dot{H}^{s\pm\varepsilon}} = 0$,

— If $j \neq j'$, $NS(\Lambda_n^j(\varphi^j))$ and $NS(\Lambda_n^{j'}(\varphi^{j'}))$ do not interact and

$$\|NS(u_{0,n})(t)\|^2_{\dot{H}^s} \sim \sum_{j=0}^{J} \|NS(\Lambda_n^j(\varphi^j))(t)\|^2_{\dot{H}^s} + \|e^{t\Delta}\psi_n^J\|^2_{\dot{H}^s}.$$

The concept of critical solution

Definition *Let us define*

$$\rho_s \stackrel{\text{def}}{=} \inf\Big\{\|u_0\|_{\dot H^s} \,/\, T^\star(u_0) = 1\Big\}.$$

The scaling implies that, for any u_0 such that $T^\star(u_0)$ is finite then

$$T^\star(u_0)\|u_0\|_{\dot H^s}^{\sigma_s} \geq \rho_s^{\sigma_s} \quad \text{with} \quad \sigma_s \stackrel{\text{def}}{=} \frac{4}{2s-1}\cdot$$

Theorem *Let us define*

$$\mathcal{M}_s \stackrel{\text{def}}{=} \Big\{u_0 \in \dot H^s \,/\, T^\star(u_0) = 1 \textit{ and } \|u_0\|_{\dot H^s} = \rho_s\Big\}.$$

The set $\mathcal{M}_s$ is non empty. Moreover it is compact up to translation which means that any sequence $(u_{0,n})_{n\in\mathbb{N}}$ of $\mathcal{M}_s$, up to an extraction is such that $u_{0,n}(\cdot - x_n)$ is convergent in $\dot H^s$ for some sequence $(x_n)_{n\in\mathbb{N}}$.

Ideas of the proof

Let us consider a sequence $(u_{0,n})_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|u_{0,n}\|_{\dot{H}^s} = \rho_s \quad \text{and} \quad T^\star(u_{0,n}) = 1\,.$$

Then up to an extraction, we get

$$1 = \limsup_{n\to\infty} T^\star(u_{0,n}) \geq \inf_{j\in\mathcal{J}_0} T^\star(\varphi^j)\,.$$

If there is more than one profils for j in $\mathcal{J}_0$, then their $\dot{H}^s$ norm is less than ρ_s and thus their life span is greater than 1.

Contradiction

The description of the possible blow up

From now on we assume that some initial data u_0 exists in $\dot{H}^s$ with finite blow up time $T^\star(u_0)$ which satisfies

$$(\mathcal{H}) \qquad \sup_{t<T^\star(u_0)} (T^\star(u_0)-t)\|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} \leq M\,.$$

Definition Let $\mathcal{M}_s$ be the infinum of the M such that $(\mathcal{H})$ is satisfied. We say that $u = NS(u_0)$ is a sup-critical solution if it satisfies

$$T^\star(u_0) < \infty \quad \text{and} \quad \limsup_{t\to T(u_0)} (T^\star(u_0)-t)\|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} = \mathcal{M}_s\,.$$

Theorem [E. Poulon, 2015] *An initial data* u_0 *exists in* $\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^s$ *such that* $NS(u_0)$ *is sup-critical and bounded in time with value in* $\dot{B}^{\frac{1}{2}}_{2,\infty}$ *where the Besov norm for regularity* r *in* $]0,1[$ *is defined by*

$$\|a\|_{\dot{B}^r_{2,\infty}} \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^3} \frac{\|a(\cdot - x) - a\|_{L^2}}{|x|^r} \cdot$$

Question

Close to Escauriaza-Segerin-Sverak theorem generalized by I. Gallagher, G. Koch and F. Planchon. . . or (very) far away ?

Ideas of the proof

The method consists in creating solution with additional properties from a solution satisfyng ($\mathcal{H}$).

The main tools are the profile decomposition and the almost orthogonality identity.

The existence of a sup-critical solution

Let us consider a sequence $(u_{0,n})_{n\in\mathbb{N}}$ such that

$$\limsup_{t\to T^\star(u_0)} (T^\star(u_0)-t)\|NS(u_{0,n})(t)\|^{\sigma_s}_{\dot H^s} \leq \mathcal{M}_s + \frac{1}{n}\cdot$$

A sequence $(t_n)_{n\in\mathbb{N}}$ tending to $T^\star(u_0)$ exists such that

$$\sup_{t\geq t_n}(T^\star(u_0)-t)\|NS(u_{0,n})(t)\|^{\sigma_s}_{\dot H^s} \leq \mathcal{M}_s + \frac{2}{n}\cdot$$

Let us define $v_{0,n}(y) \stackrel{\text{def}}{=} (T^\star(u_0)-t_n)^{\frac12} NS(u_{0,n})(t_n,(T^\star(u_0)-t_n)^{\frac12}y)$. We have

$$T(v_{0,n})=1\,,\ \|v_{0,n}\|_{\dot H^s}\leq \mathcal{M}_s \quad \text{and}$$

$$\sup_{\tau<1}(1-\tau)\|NS(v_{0,n})(\tau)\|^{\sigma_s}_{\dot H^s} \leq \mathcal{M}_s + \frac{2}{n}\cdot$$

The profile decomposition of the sequence $(v_{0,n})_{n\in\mathbb{N}}$ is of the form

$$v_{0,n} = \varphi(x-x_n) + \sum_{j\leq J, \lambda_{n,j}\neq 1} \Lambda^j_n \varphi^j + \psi_{n,J}\,.$$

The boundedness in the Besov space $\dot{B}^{\frac{1}{2}}_{2,\infty}$.

Let us write that

$$NS(u_0) = e^{t\Delta}u_0 + F(u_0)\,.$$

Principle The term $F(u_0)$ is better that $NS(u_0)$. More precisely here

$$(T^\star(u_0)-t)\|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} \leq M \Longrightarrow \|F(u_0)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \leq CM^2\,.$$

Then, $NS(u_0)$ being a sup-critical solution, let us write

$$\begin{aligned} v_{0,n}(y) &\stackrel{\text{def}}{=} (T^\star(u_0)-t_n)^{\frac{1}{2}} NS(u_0)(t_n,(T^\star(u_0)-t_n)^{\frac{1}{2}}y) \\ &= (T^\star(u_0)-t_n)^{\frac{1}{2}}(e^{t\Delta}u_0)((T^\star(u_0)-t_n)^{\frac{1}{2}}y) \\ &\qquad + (T^\star(u_0)-t_n)^{\frac{1}{2}}F(u_0)(t_n,(T^\star(u_0)-t_n)^{\frac{1}{2}}y)\,. \end{aligned}$$

THANK YOU VERY MUCH FOR ATTENTION