A Proof of Onsager's Conjecture for the Incompressible Euler Equations

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Outline

- Motivation
  - Weak Solutions to the Euler Equations
  - Onsager's Conjecture and Turbulence
  - Brief Survey of Previous Results
- A Proof of Onsager's Conjecture
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  - ...
  - ...
  - ...
  - ...

Motivation: Weak Solutions to the Euler equations

The incompressible Euler equations for a homogeneous fluid:

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = 0 \tag{1}$$

$$\nabla_j v^j = 0 \tag{2}$$

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make sense in integral form for continuous (v, p):

$$\frac{d}{dt} \int_\Omega v^\ell(t,x) dx = \int_{\partial\Omega} p(t,x) n^\ell d\sigma + \int_{\partial\Omega} v^\ell(t,x)(v \cdot n) d\sigma \tag{3}$$

$$\int_{\partial\Omega} (v \cdot n)(t,x) d\sigma(x) = 0 \tag{4}$$

for all Ω with smooth boundary $\partial\Omega$ and interior unit normal n^ℓ.

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Then, use the divergence free condition div $v = \nabla_\ell v^\ell = 0$, and integrate

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{|v|^2}{2}(t,x) dx = - \int_{\mathbb{R}^n} \text{div} \left[(\frac{|v|^2}{2} + p) v \right] dx = 0$$

Motivation: Onsager's Conjecture (1949)

1. Solutions (v, p) to Euler obeying a Hölder estimate

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = 0 \tag{5}$$

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Kolmogorov (1941): As $\nu \to 0$ for solutions to $3D$ Navier-Stokes:

$$\begin{cases} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nu \Delta v^\ell \\ \nabla_j v^j = 0 \end{cases} \tag{7}$$

the energy dissipation rate remains **strictly positive** as $\nu \to 0$

$$\varepsilon = \lim_{\nu \to 0} \left\langle -\frac{d}{dt} \int \frac{|v_\nu|^2}{2}(t,x)dx \right\rangle > 0.$$

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The velocity fluctuations on average obey a law

$$\langle |v(x+\Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^{1/3} |\Delta x|^{1/3}$$

$$\text{for} \qquad |\Delta x| \geq \left(\nu^3/\varepsilon\right)^{1/4}$$

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Onsager considered the case $\nu = 0$; argued that "frequency cascades" may lead to energy dissipation in the absence of viscosity.

Onsager and Ideal Turbulence

Onsager considered the Euler equations in Fourier series form (which converges for $v \in L^2$)

$$v(x,t) = \sum_k a_k(t) e^{ik\cdot x}$$

$$\frac{da_k}{dt} = i \sum_m a_{k-m} \cdot k \left[-a_m + \frac{(a_m \cdot k)k}{|k|^2} \right]$$

He argued that energy can “cascade” from low wavenumbers to high wavenumbers, and the cascade can happen so rapidly that part of the energy $\sum_k |a_k|^2$ escapes to infinite frequency (i.e. vanishes to small spatial scales) in finite time.

However, only low regularity solutions could behave this way, and he stated that solutions in C^α with $\alpha > 1/3$ must conserve energy.

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By a statistical physics argument, a "typical" turbulent flow should have: $\sum_{\frac{\lambda}{2} \leq |k| \leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ (hence regularity exactly $1/3$).

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Shell models and continuous model equations:
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- Weak solutions in $L^2_{t,x}(\mathbb{R} \times \mathbb{R}^2)$ with compact support in space and time (Scheffer, '93)
- Weak solutions in $L^2_{t,x}(\mathbb{R} \times \mathbb{T}^2)$ (Shnirelman, '97)
- Dissipative solutions in $L^\infty_t L^2_x(\mathbb{R} \times \mathbb{T}^3)$ (Shnirelman, '00)

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- Solutions in $L^\infty_{t,x} \cap C_t L^2_x(\mathbb{R} \times \mathbb{R}^n)$ with any energy density

$$\frac{|v|^2}{2} = e(t, x)$$

(De Lellis, Székelyhidi, '07)

Convex Integration and Isometric Embeddings

- (Nash, '54) Constructs surprising, C^1 isometric embeddings in very low codimension.
  - (Borisov, '65, '04) Irregular $C^{1,\alpha}$ isometric embeddings for analytic metric
- (Gromov, '86) Generalizes Nash's idea to the method of "convex integration" in topology and geometry
- (Müller-Sverak, '04) Elliptic systems with Lipschitz but nowhere C^1 solutions (i.e. $\nabla u \in L^\infty$, but $\nabla u \notin C^0$).

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- (De Lellis-Székelyhidi, '09) Simpler proofs and extensions of Borisov's results on $C^{1,\alpha}$ isometric embeddings

Continuous weak solutions that fail to conserve energy

Theorem (De Lellis, Székelyhidi, '12)

For every $\alpha < 1/10$, $\exists$ *solutions* $(v, p) \in C^{\alpha}_{t,x} \times C^{2\alpha}_{t,x}(\mathbb{R} \times \mathbb{T}^3)$ *that can realize any smooth energy profile*

$$\int \frac{|v|^2}{2}(t, x)dx = e(t) \geq c > 0$$

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Improved regularity of energy non-conserving solutions

Theorem (I., '12)

For every $\alpha < 1/5$ *there exist nontrivial weak solutions to the incompressible Euler equations on* $\mathbb{R} \times \mathbb{T}^3$ *in the class*

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- Shorter proof, solutions with arbitrary smooth $e(t) = \int |v|^2(t,x)dx \geq c > 0$ (Buckmaster-De Lellis-Székelyhidi, '13)
- Solutions with compact support in $\Omega \subseteq \mathbb{R} \times \mathbb{R}^3$ (I.-Oh, '14)

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Onsager's Conjecture in Weaker Topologies

Can the exponent $1/5$ be improved if we weaken the topology?

- (Buckmaster, '13) Solutions in $v \in C^{1/5-\epsilon}_{t,x}$ with $v(t,\cdot) \in C^{1/3-\epsilon}$ for a.e. t
- (Buckmaster-De Lellis-Székelyhidi, '14) C^0 solutions in $v \in L^1_t C^{1/3-\epsilon}_x$
- (Buckmaster-Masmoudi-Vicol, '16) Solutions with $v \in C_t H^{1/3-\epsilon}_x$

Note: The improvement in regularity is in an averaged sense (in L^1 or L^2), but achieves the Onsager critical exponent $1/3-$.
To compare: energy conservation requires $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$.

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such that v *has nonempty, compact support in time.*

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- Gluing technique
  - Hidden special structure in the linearization of the Euler equations to estimate main terms

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- (Buck., De L., Szé., Vicol, '17) Solutions in $v \in C^{1/3-\epsilon}_{t,x}$ with any, smooth energy profile $\int_{\mathbb{T}^3} |v|^2(t,x)dx = e(t) > 0$.

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- (I., '17) Solutions with borderline endpoint regularity

  $$|v(t,x+\Delta x) - v(t,x)| \lesssim |\Delta x|^{\frac{1}{3} - B\sqrt{\frac{\log\log|\Delta x|^{-1}}{\log|\Delta x|^{-1}}}}, \; B = 4/3^{+}.$$

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- Nonstationary Phase Lemma
  - $\approx$ Acceptable errors (high frequency $\cdot$ slowly varying)

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- The Euler Reynolds Equations (= Approximate solutions)
- Nonstationary Phase Lemma
  - $\approx$ Acceptable errors (high frequency $\cdot$ slowly varying)
- Mikado flows (Daneri-Székelyhidi)
  - Convex integration using Mikado flows
  - The difficulty with Mikado flows for Onsager's conjecture
- The Gluing technique
  - Deriving the Gluing equations
  - Dangerous terms
  - Special structure in the equations

Continuous Solutions: The Euler-Reynolds Equations

(De Lellis, Székelyhidi): Consider the **Euler-Reynolds equations**

$$\begin{aligned} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p &= \nabla_j R^{j\ell} \\ \nabla_j v^j &= 0 \end{aligned} \tag{ER}$$

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Examples: If (v, p) solves the Euler equations then

- $(v_\epsilon, p_\epsilon, R_\epsilon^{j\ell})$, $R_\epsilon^{j\ell} = v_\epsilon^j v_\epsilon^\ell - (v^j v^\ell)_\epsilon$, $v_\epsilon^\ell = \eta_\epsilon * v^\ell$
- **Corollary:** Every continuous incompressible Euler flow (v, p) is the uniform limit of a sequence of Euler-Reynolds flows (v_q, p_q, R_q) with $\|R_q\|_{C^0} \to 0$ as $q \to \infty$

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- Any v^ℓ that is incompressible and conveserves momentum

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) = U^\ell$$
$$\int_{\mathbb{T}^3} U^\ell(t,x) dx = 0$$
$$\nabla_j R^{j\ell} = U^\ell$$

Continuous Solutions: Convex Integration for Euler

We construct a sequence (v_q, p_q, R_q) indexed by q solving

$$\partial_t v_q^\ell + \nabla_j (v_q^j v_q^\ell) + \nabla^\ell p_q = \nabla_j R_q^{j\ell} \qquad \text{(ERq)}$$
$$\nabla_j v_q^j = 0$$

where $v_{q+1} = v_q + V_q$, $p_{q+1} = p_q + P_q$ solve (ERq+1) with

much smaller $|R_{q+1}| \ll |R_q|^{1+\delta}$

Continuous Solutions: Convex Integration for Euler

We construct a sequence (v_q, p_q, R_q) indexed by q solving

$$\begin{aligned} \partial_t v_q^\ell + \nabla_j (v_q^j v_q^\ell) + \nabla^\ell p_q &= \nabla_j R_q^{j\ell} \\ \nabla_j v_q^j &= 0 \end{aligned} \qquad \text{(ERq)}$$

where $v_{q+1} = v_q + V_q$, $p_{q+1} = p_q + P_q$ solve (ERq+1) with

$$\textbf{much smaller} \quad |R_{q+1}| \ll |R_q|^{1+\delta}$$

In the limit as $q \to \infty$, we get continuous solutions

$$\|R_q\|_{C^0} \to 0, \quad |V_q| \sim |R_q|^{1/2}, \quad |P_q| \sim |R_q|$$

Continuous Solutions: Convex Integration for Euler

Start with **any** smooth solution to Euler-Reynolds on $\mathbb{R} \times \mathbb{T}^3$

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell}$$
$$\nabla_j v^j = 0$$

and add high-frequency corrections

$$v_1 = v + V, \qquad p_1 = p + P,$$

which are designed to "get rid of" $R^{j\ell}$.

Continuous Solutions: Convex Integration for Euler

Get new solutions $v_1 = v + V$, $p_1 = p + P$ to Euler-Reynolds

$$\begin{aligned} \partial_t v_1^\ell + \nabla_j (v_1^j v_1^\ell) + \nabla^\ell p_1 &= \nabla_j R_1^{j\ell} \\ \nabla_j v_1^j &= 0 \end{aligned}$$

with $||R_1||_{C^0_{t,x}}$ **much smaller than** $||R||_{C^0_{t,x}}$.

Continuous Solutions: Convex Integration for Euler

The corrected $v_1 = v + V$, $p_1 = p + P$ satisfy

$$\begin{aligned}\partial_t v_1^\ell + \nabla_j (v_1^j v_1^\ell) + \nabla^\ell p_1 &= \partial_t V^\ell + \ldots + \nabla_j (V^j V^\ell + P\delta^{j\ell} + R^{j\ell}) \\ &= \textbf{ not} \text{ in the form } \nabla_j R_1^{j\ell} \\ \nabla_j v_1^j &= 0\end{aligned}$$

Continuous Solutions: Convex Integration for Euler

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so we will have to solve a divergence equation:

$$\nabla_j R_1^{j\ell} = \partial_t V^\ell + \nabla_j (v^j V^\ell) + \nabla_j (V^j v^\ell) + \nabla_j (V^j V^\ell + P\delta^{j\ell} + R^{j\ell})$$

to define R_1.

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to define R_1.

The new error $\|R_1\|_{C^0}$ will only be small when V and P are very oscillatory and are designed carefully depending on the given v^ℓ and $R^{j\ell}$.

The Error terms

Let (v, p, R) be a smooth solution to Euler-Reynolds.

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell}$$

Then $v_1 = v + V$ and $p_1 = p + P$ satisfy

$$\begin{aligned} \partial_t v_1^\ell + \nabla_j (v_1^j v_1^\ell) + \nabla^\ell p_1 &= \partial_t V^\ell + \nabla_j (v^j V^\ell) + \nabla_j (V^j v^\ell) \\ &\quad + \nabla_j (V^j V^\ell + P \delta^{j\ell} + R^{j\ell}) \\ \text{want } &= \nabla_j R_1^{j\ell} \\ \text{with } \|R_1\|_{C^0} &\lesssim \lambda^{-1} \end{aligned}$$

where V^ℓ oscillates at large frequency λ.

The Error terms

We name the terms as follows

Transport term:

$$\nabla_j R_T^{j\ell} = \partial_t V^\ell + \nabla_j (v^j V^\ell) + \nabla_j (V^j v^\ell)$$

Stress term:

$$\nabla_j R_S^{j\ell} = \mathsf{LFreq}[\nabla_j (V^j V^\ell + P\delta^{j\ell} + R^{j\ell})]$$

High-Frequency Interference terms:

$$\nabla_j R_H^{j\ell} = \mathsf{HFreq}[\nabla_j (V^j V^\ell + P\delta^{j\ell})]$$

Each one of R_T, R_S and R_H must be $\|R_1\|_{C^0} \lesssim \lambda^{-1}$.

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Each one of R_T, R_S and R_H must be $\|R_1\|_{C^0} \lesssim \lambda^{-1}$.

(There is also another term involving errors from mollifying $v \mapsto v_\epsilon$ and $R \mapsto R_\epsilon$ that we are neglecting here.)

The High-Frequency Correction

The correction V^ℓ is a high-frequency, divergence free wave. For example, in (I., '12), it has the form

$$V^\ell = \sum_I e^{i\lambda\xi_I} v_I^\ell + \delta V^\ell$$

$$\nabla_\ell V^\ell = 0 \quad \text{(by choice of small } \delta V^\ell\text{)}$$

$$(\partial_t + v_\epsilon^j \nabla_j)\xi_I = 0 \quad (\Rightarrow \text{ nonlinear phase functions})$$

$$\nabla \times (e^{i\lambda\xi_I} v_I) \approx \lambda e^{i\lambda\xi_I} v_I \quad \text{(by taking } (i\nabla\xi_I) \times v_I \approx v_I\text{)}$$

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The last condition makes V^ℓ approximate a **Beltrami flow** $(\nabla \times V \approx \lambda V)$, which are special stationary solutions to 3D Euler. It is used to control

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$$\nabla_j R_H^{j\ell} = \mathrm{HFreq}[\nabla_j (V^j V^\ell + P\delta^{j\ell})]$$

The Error terms again

Each one of R_T, R_S and R_H must have size $\|R_1\|_{C^0} \lesssim \lambda^{-1}$, and requires solving a divergence equation:

Transport term:

$$\nabla_j R_T^{j\ell} = \partial_t V^\ell + \nabla_j (v_\epsilon^j V^\ell) + \nabla_j (V^j v_\epsilon^\ell)$$

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Nonstationary phase

Lemma (Nonstationary Phase Lemma)

Suppose $u^\ell(x)$ *and* $\xi(x)$ *are smooth functions on* $\mathbb{T}^3$ *and*

$$U^\ell(x;\lambda) = e^{i\lambda\xi(x)} u^\ell(x)$$

$$\| \, |\nabla\xi|^{-1} \|_{C^0} \le A, \quad \int_{\mathbb{T}^3} U^\ell(x) dx = 0$$

Then U^ℓ *is* **very small in** C^{-1}*. That is, we can solve*

$$\nabla_j R^{j\ell} = e^{i\lambda\xi(x)} u^\ell(x)$$

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The implicit constant depends on A *and* C^k *norms of* $\nabla\xi, u^\ell$

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Nonstationary phase Lemma: Cartoon Proof

In 1D, we want to solve

$$\text{div } R(x) = \frac{dR}{dx} = e^{i\lambda\xi(x)}u(x)$$

$$\Rightarrow R(x) = \int_0^x e^{i\lambda\xi(X)}u(X)dX$$

$$= \int_0^x \frac{1}{i\lambda\xi'(X)} \frac{d}{dX}(e^{i\lambda\xi(X)})u(X)dX$$

$$= \frac{u(X)e^{i\lambda\xi(X)}}{i\lambda\nabla\xi(X)}\Big|_{X=0}^{X=x} - \frac{1}{i\lambda}\int_0^x e^{i\lambda\xi(X)}\frac{d}{dX}\left(\frac{1}{\nabla\xi(X)}u(X)\right)dX$$

Using $\||\nabla\xi|^{-1}\|_{C^0} \leq A$, the solution has size $\|R\|_{C^0} \lesssim \lambda^{-1}$.

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Nonstationary phase: Proof

Proof: To solve $\nabla_j R^{j\ell} = e^{i\lambda\xi(x)} u^\ell(x)$, write

$$e^{i\lambda\xi(x)} u^\ell(x) = \nabla_j \left(\frac{1}{\lambda} e^{i\lambda\xi(x)} q^{j\ell}(x) \right) + \nabla_j \check{R}^{j\ell}$$

$$i\nabla_j \xi q^{j\ell}(x) = u^\ell(x), \quad q^{j\ell} \in C^\infty(\mathbb{T}^3; \mathbb{R}^3 \otimes \mathbb{R}^3) \tag{12}$$

$$\nabla_j \check{R}^{j\ell} = -\frac{1}{\lambda} e^{i\lambda\xi(x)} \nabla_j q^{j\ell}(x) \tag{13}$$

Equation (12) is solved pointwise and leads to a bound

$$\|q^{j\ell}\|_{C^0} \lesssim \| \; |\nabla\xi|^{-1} \; \|_{C^0} \|u^\ell\|_{C^0}$$

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$$\|q^{j\ell}\|_{C^0} \lesssim \| \, |\nabla\xi|^{-1} \, \|_{C^0} \|u^\ell\|_{C^0}$$

We can solve (13) because $\int_{\mathbb{T}^3} e^{i\lambda\xi(x)} u^\ell(x) dx = 0$.
The solution satisfies $\|\check{R}^{j\ell}\|_{C^0} \lesssim \lambda^{-1}$.

Nonstationary phase 2

In order to use Mikado flows:

Lemma (Generalized Nonstationary Phase, Daneri-Székelyhidi)

Suppose $u^\ell(x)$ *and* $\omega(x) \in C^\infty(\mathbb{T}^3)$ *and* $\Gamma \in C^\infty(\mathbb{T}^3; \mathbb{T}^3)$

$$U^\ell(x;\lambda) = \omega(\lambda\Gamma(x))u^\ell(x)$$

$$\|(\nabla\Gamma)^{-1}\|_{C^0} \leq A, \quad \int_{\mathbb{T}^3} U^\ell(x)dx = 0$$

$$\int_{\mathbb{T}^3} \omega(X)dX = 0$$

Then U^ℓ *is* **very small in** C^{-1}. *That is, we can solve*

$$\nabla_j R^{j\ell} = \underbrace{\omega(\lambda\Gamma(x))}_{\textit{fast}}\underbrace{u^\ell(x)}_{\textit{slow}}$$

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Nonstationary phase 2: Proof outline

To solve $\nabla_j R^{j\ell} = \omega(\lambda\Gamma(x))u^\ell(x)$, write (using $\int_{\mathbb{T}^3} \omega(X)dX = 0$)

$$\omega(\lambda\Gamma(x))u^\ell(x) = \sum_{m\neq 0} \hat{\omega}(m)e^{i\lambda m\cdot\Gamma(x)}u^\ell(x) \tag{14}$$

Nonstationary phase 2: Proof outline

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$$\omega(\lambda\Gamma(x))u^\ell(x) = \sum_{m\neq 0} \hat{\omega}(m)e^{i\lambda m\cdot\Gamma(x)}u^\ell(x) \qquad (14)$$

Can apply the previous Lemma if we have *nonstationary phase* functions, which requires

$$\|(\nabla\Gamma)^{-1}\|_{C^0} \leq A \Rightarrow |\nabla(m\cdot\Gamma)|^{-1} \leq A|m|^{-1}$$

Applying the Nonstationary Phase Lemma gives a solution with

$$\|R^{j\ell}\|_{C^0} \lesssim \lambda^{-1}$$

Motivation for Mikado flows

Theorem (Daneri-Székelyhidi, '16)

For every smooth Euler-Reynolds flow $(\bar{v}, p, R)$ *with*

$$-R^{j\ell} \geq c\delta^{j\ell}, \quad c > 0, \tag{15}$$

there exist weak solutions to Euler in $v_{(k)} \in C^{1/5-\epsilon}_{t,x}$ *such that*

$$v^{\ell}_{(k)} \rightharpoonup \bar{v}^{\ell} \quad \text{in } L^{\infty}_{t,x} \tag{16}$$

$$v^{j}_{(k)} v^{\ell}_{(k)} - \bar{v}^{j} \bar{v}^{\ell} \rightharpoonup R^{j\ell} \quad \text{in } L^{\infty}_{t,x} \quad \text{as } k \to \infty \tag{17}$$

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With Beltrami flows, would require $R^{j\ell} = -a(t,x)(\delta^{j\ell} + \text{small})$. To overcome this restriction, they introduce a different family of stationary solutions to Euler ("**Mikado flows**") that provide more algebraic flexibility to achieve an arbitrary stress $R^{j\ell}$.

Elementary Mikado flows on $\mathbb{T}^3$

Fix a constant integer vector $f^\ell \in \mathbb{Z}^3$ and define for $X \in \mathbb{T}^3$

$$W^\ell(X) = f^\ell \psi_f(X), \quad \psi_f \in C^\infty(\mathbb{T}^3)$$

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We choose ψ_f whose level surfaces are concentric cylinders with an axis pointed in the f^ℓ direction. With this choice we have

$$\nabla_\ell \psi_f(X) f^\ell = 0 \qquad \Leftarrow \text{orthogonality}$$

Then $W^\ell(X)$ is a stationary Euler flow with 0 pressure:

$$\begin{aligned} \nabla_\ell W^\ell(X) &= 0 \\ \nabla_j (W^j W^\ell(X)) &= \nabla_j (\psi_f^2(X) f^j f^\ell) \end{aligned}$$

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Then $W^\ell(X)$ is a stationary Euler flow with 0 pressure:

$$\nabla_\ell W^\ell(X) = 0$$
$$\nabla_j (W^j W^\ell(X)) = 2\psi_f(X) \nabla_j \psi_f(X) f^j f^\ell$$

Elementary Mikado flows on $\mathbb{T}^3$

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$$\nabla_\ell W^\ell(X) = 0$$
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Elementary Mikado flows on $\mathbb{T}^3$

Fix a constant integer vector $f^\ell \in \mathbb{Z}^3$ and define for $X \in \mathbb{T}^3$

$$W^\ell(X) = f^\ell \psi_f(X), \quad \psi_f \in C^\infty(\mathbb{T}^3)$$

In addition to $\nabla_j \psi_f(X) f^j = 0$, we require that

$$\int_{\mathbb{T}^3} \psi_f(X) dX = 0, \quad \int_{\mathbb{T}^3} \psi_f^2(X) dX = 1 \tag{18}$$

With these choices, we have:

$$\underbrace{\int_{\mathbb{T}^3} W^\ell(X) dX = 0,}_{\text{Oscillation}} \quad \underbrace{\int_{\mathbb{T}^3} W^j W^\ell(X) dX = f^j f^\ell}_{\text{Nontrivial low-frequency part}} \tag{19}$$

Elementary Mikado flows on $\mathbb{T}^3$

More generally, fix a finite set $\mathbb{F} \subseteq \mathbb{Z}^3$ and coefficients γ_f and set

$$W^\ell(X) = \sum_{f \in \mathbb{F}} \gamma_f \psi_f(X) f^\ell$$

$$\text{supp } \psi_f \cap \text{supp } \psi_{f'} = \emptyset, \quad \text{if } f \neq f' \in \mathbb{F}$$

We still have $\nabla_\ell W^\ell = 0$, $\nabla_j (W^j W^\ell) = 0$ and $\int_{\mathbb{T}^3} W^\ell(X) dX = 0$, but now

$$\int_{\mathbb{T}^3} W^j W^\ell(X) dX = \sum_{f \in \mathbb{F}} \gamma_f^2 f^j f^\ell$$

can be an arbitrary, positive definite tensor.

Designing a wave with Mikado flows

Using these flows, we design our high-frequency wave V^ℓ as follows. At time $t = 0$ it looks like

$$V^\ell(0,x) = \sum_{f \in \mathbb{F}} \underbrace{\gamma_f(0,x)}_{\text{slow}} f^\ell \underbrace{\psi_f(\lambda x)}_{\text{fast}} + \underbrace{\delta V^\ell}_{\text{small}}$$

Designing a wave with Mikado flows

Using these flows, we design our high-frequency wave V^ℓ as follows. At time $t = 0$ it looks like

$$V^\ell(0,x) = \sum_{f \in \mathbb{F}} \underbrace{\gamma_f(0,x)}_{\text{slow}} f^\ell \underbrace{\psi_f(\lambda x)}_{\text{fast}} + \underbrace{\delta V^\ell}_{\text{small}}$$

At nonzero times, it has the form:

$$V^\ell(t,x) = \sum_{f \in \mathbb{F}} \gamma_f(t,x) \underbrace{\tilde{f}^\ell(t,x)}_{\text{slow}} \psi_f(\lambda \underbrace{\Gamma(t,x)}_{\text{slow}}) + \delta V^\ell$$

$$(\partial_t + v_\epsilon \cdot \nabla)\Gamma(t,x) = 0, \qquad \Gamma(0,x) = x$$

Designing a wave with Mikado flows

The vector field $\tilde{f}^\ell$ satisfies

$$V^\ell(t,x) = \sum_{f\in\mathbb{F}} \gamma_f(t,x)\tilde{f}^\ell(t,x)\psi_f(\lambda\Gamma(t,x)) + \delta V^\ell$$

$$\tilde{f}^\ell = (\nabla\Gamma^{-1})^\ell_a f^a$$

$$\Rightarrow \tilde{f}^\ell \nabla_\ell[\psi_f(\lambda\Gamma(t,x))] = 0, \quad \text{since } f^a\nabla_a\psi_f = 0$$

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We can then make V^ℓ divergence free by solving

$$\nabla_\ell V^\ell = 0 = \sum_{f\in\mathbb{F}} \underbrace{\nabla_\ell[\gamma_f(t,x)\tilde{f}^\ell(t,x)]}_{\text{slow}} \underbrace{\psi_f(\lambda\Gamma(t,x))}_{\text{fast}} + \nabla_\ell \delta V^\ell$$

$\Rightarrow \|\delta V^\ell\|_{C^0} \lesssim \lambda^{-1}$ (starting now we will neglect this term...)

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Recalling the Error terms again

Each one of R_T, R_S and R_H must have size $\|R_1\|_{C^0} \lesssim \lambda^{-1}$, and requires solving a divergence equation:

Transport term:

$$\nabla_j R_T^{j\ell} = \partial_t V^\ell + \nabla_j(v_\epsilon^j V^\ell) + \nabla_j(V^j v_\epsilon^\ell)$$

Stress term:

$$\nabla_j R_S^{j\ell} = \mathsf{LFreq}[\nabla_j(V^j V^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell})]$$

High-Frequency Interference terms:

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The Main Error Terms

With this Ansatz the Transport term is under control:
Letting $D_t := (\partial_t + v_\epsilon^j \nabla_j)$ be the "advective derivative" we have

$$\partial_t V^\ell + \nabla_j (v_\epsilon^j V^\ell) + \nabla_j (V^j v_\epsilon^\ell) = (\partial_t + v_\epsilon^j \nabla_j) V^\ell + V^j \nabla_j v_\epsilon^\ell$$

$$= \sum_{f \in \mathbb{F}} D_t [\gamma_f \tilde{f}^\ell \psi_f(\lambda \Gamma(t,x))] + \gamma_f \tilde{f}^j \psi_f(\lambda \Gamma(t,x)) \nabla_j v_\epsilon^\ell$$

$$\nabla_j R_T^{j\ell} = \sum_{f \in \mathbb{F}} \underbrace{(D_t[\gamma_f \tilde{f}^\ell] + \gamma_f \tilde{f}^j \nabla_j v_\epsilon^\ell)}_{\text{slow}} \underbrace{\psi_f(\lambda \Gamma(t,x))}_{\text{fast}}$$

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Letting $D_t := (\partial_t + v_\epsilon^j \nabla_j)$ be the "advective derivative" we have

$$\begin{aligned}
\partial_t V^\ell + \nabla_j(v_\epsilon^j V^\ell) + \nabla_j(V^j v_\epsilon^\ell) &= (\partial_t + v_\epsilon^j \nabla_j) V^\ell + V^j \nabla_j v_\epsilon^\ell \\
&= \sum_{f \in \mathbb{F}} D_t[\gamma_f \tilde{f}^\ell \psi_f(\lambda \Gamma(t,x))] + \gamma_f \tilde{f}^j \psi_f(\lambda \Gamma(t,x)) \nabla_j v_\epsilon^\ell \\
\nabla_j R_T^{j\ell} &= \sum_{f \in \mathbb{F}} \underbrace{(D_t[\gamma_f \tilde{f}^\ell] + \gamma_f \tilde{f}^j \nabla_j v_\epsilon^\ell)}_{\text{slow}} \underbrace{\psi_f(\lambda \Gamma(t,x))}_{\text{fast}}
\end{aligned}$$

(Used $\nabla_j V^j = 0$.)

The Main Error Terms

The Stress term is controlled as follows:

$$\mathsf{LFreq}[\nabla_j(V^j V^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell})]$$

$$= \mathsf{LFreq}\Big[\nabla_j\Big(\sum_{f_1,f_2\in\mathbb{F}} \gamma_{f_1}\gamma_{f_2}\overbrace{\psi_{f_1}\psi_{f_2}(\lambda\Gamma)}^{\text{disjoint if }\neq}\tilde{f}_1^j\tilde{f}_2^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell}\Big)\Big]$$

$$= \mathsf{LFreq}\Big[\nabla_j\Big(\sum_{f\in\mathbb{F}}\gamma_f^2\psi_f^2(\lambda\Gamma)\tilde{f}^j\tilde{f}^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell}\Big)\Big]$$

$$:= \nabla_j\Big[\sum_{f\in\mathbb{F}}\gamma_f^2(t,x)\tilde{f}^j\tilde{f}^\ell + P(t,x)\delta^{j\ell} + R_\epsilon^{j\ell}\Big]$$

$$= \nabla_j[0] = 0$$

Here we solve for the $\gamma_f^2(t,x)$ at each point using that the $(\tilde{f}^j\tilde{f}^\ell)_{f\in\mathbb{F}}$ span the space of symmetric tensors.

The Main Error Terms

The remaining High-Frequency Interference term is controlled as follows using the orthogonality $\tilde{f}^j \nabla_j [\psi_f^2(\lambda\Gamma)] = 0$

$$\text{HFreq}[\nabla_j(V^j V^\ell)] = \nabla_j \left[\sum_{f \in \mathbb{F}} \gamma_f^2 \tilde{f}^j \tilde{f}^\ell (\psi_f^2(\lambda\Gamma(t,x)) - 1) \right]$$

$$\nabla_j R_H^{j\ell} = \sum_{f \in \mathbb{F}} \nabla_j [\gamma_f^2 \tilde{f}^j \tilde{f}^\ell](\psi_f^2(\lambda\Gamma(t,x)) - 1)$$

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$$\nabla_j R_H^{j\ell} = \sum_{f \in \mathbb{F}} \underbrace{\nabla_j [\gamma_f^2 \tilde{f}^j \tilde{f}^\ell]}_{\text{slow}} \underbrace{(\psi_f^2(\lambda\Gamma(t,x)) - 1)}_{\text{fast } := \omega(\lambda\Gamma(t,x))}$$

The last term is "fast-oscillating" since $\int_{\mathbb{T}^3} (\psi_f^2(X) - 1) dX = 0$. (Using Beltrami flows, the corresponding term is under control only for a very short period of time.)

Can we use Mikado flows for Onsager's conjecture?

All the error terms discussed above appear sufficiently small for the method of convex integration to yield regularity $1/3 - \epsilon$.

However, there is a substantial difficulty standing in the way of using Mikado flows to prove Onsager's conjecture, namely:

Can we use Mikado flows for Onsager's conjecture?

All the error terms discussed above appear sufficiently small for the method of convex integration to yield regularity $1/3 - \epsilon$.

However, there is a substantial difficulty standing in the way of using Mikado flows to prove Onsager's conjecture, namely:

Problem: To iterate the previous construction again and again (i.e. perform convex integration) we need to use multiple waves (see next slide). The difficulty comes in dealing with the interactions of **distinct** Mikado flows that start from different times.

Why we Need Multiple Waves

A crucial assumption we are using is the bound $\|(\nabla\Gamma^{-1})\|_{C^0} \leq A$ for the solution to

$$(\partial_t + v_\epsilon^j \nabla_j)\Gamma(t,x) = 0, \qquad \Gamma(0,x) = x$$

We can see that this assumption holds only for times of the order $|t| \lesssim \|\nabla v\|_{C^0}^{-1}$ from the PDE:

$$\begin{aligned}(\partial_t + v_\epsilon^j \nabla_j)(\nabla\Gamma^{-1})_b^a &= \nabla_j v_\epsilon^a (\nabla\Gamma^{-1})_b^j \\ (\nabla\Gamma^{-1})_b^a &= \mathrm{Id}_b^a \qquad \text{at } t = 0\end{aligned}$$

Since $\|\nabla v\|_{C^0} \to \infty$ as v converges to a $C^{1/3-\epsilon}$ vector field, we need to use more and more waves starting at different times!

Difficulty with Mikado Flows

It seems very difficult to control the interactions between two Mikado flow based waves. Suppose we have two such waves

$$V_1^\ell = \sum_{f \in \mathbb{F}_1} \gamma_{f,1} f^j f^\ell \psi_f(\lambda \Gamma_1), \qquad V_0^\ell = \sum_{f \in \mathbb{F}_0} \gamma_{f,0} f^j f^\ell \psi_f(\lambda \Gamma_0)$$

where Γ_1 and Γ_0 both solve $(\partial_t + v_\epsilon \cdot \nabla)\Gamma_I = 0$, but start as the identity at different times

$$|t_1 - t_0| \sim \|\nabla v\|_{C^0}^{-1}.$$

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$$|t_1 - t_0| \sim \|\nabla v\|_{C^0}^{-1}.$$

Then the supports of the $\psi_f(\lambda\Gamma_I)$ (which are unions of long, λ^{-1}-thin, λ^{-1}-separated cylinders deformed by the flow) will in general overlap and we will lose control over the interference term

$$\nabla_j [V_1^j V_0^\ell + V_0^j V_1^\ell]$$

Strategy to Fix the Problem

Idea: Find a new stress error $\widetilde{R}$ that is supported in disjoint time intervals of width $\theta \sim |\nabla v|^{-1}$

$$\mathrm{supp}_t \widetilde{R} \subseteq \bigcup_I [t(I) - \theta, t(I) + \theta]$$

so that the new velocity field is a perturbation of the old one $v \mapsto \tilde{v} = v + y$ and $\widetilde{R}$ obeys the same estimates as the original R.

Strategy to Fix the Problem

Idea: More precisely, starting with (v, p, R), find a new Euler-Reynolds flow $(\tilde{v}, \tilde{p}, \widetilde{R})$ with $\tilde{v}$ close to v such that

$$\partial_t \tilde{v}^\ell + \nabla_j(\tilde{v}^j \tilde{v}^\ell) + \nabla^\ell \tilde{p} = \nabla_j \widetilde{R}^{j\ell}, \quad \widetilde{R} = \sum_{I \in \mathbb{Z}} R_I$$

$$\text{supp } R_I \subseteq [t(I) - \theta, t(I) + \theta], \quad \theta \sim |\nabla v|^{-1}$$

$$|t(I) - t(I')| \geq 4\theta, \quad I \neq I'$$

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Rules: $(\tilde{v}, \tilde{p}, \tilde{R})$ must obey the same C^k estimates as (v, p, R). In particular, the new error $\widetilde{R}$ cannot be much larger than the previous error R! ($\|\widetilde{R}\|_{C^0} \lesssim \|R\|_{C^0}$ is OK.) Also, we require $\tilde{v}$ to be close to v: $\|v - \tilde{v}\|_{C^0} \lesssim \|R\|_{C^0}^{1/2}$

Constructing the new $(\tilde{v}, \tilde{p}, \widetilde{R})$

We introduce the velocity increment y^ℓ and pressure increment $\bar{p}$, which satisfy $\tilde{v}^\ell = v^\ell + y^\ell, \qquad \tilde{p} = p + \bar{p}$ and

$$\partial_t y^\ell + v^j \nabla_j y^\ell + y^j \nabla_j v^\ell + \nabla_j (y^j y^\ell) + \nabla^\ell \bar{p} = \nabla_j \widetilde{R}^{j\ell} - \nabla_j R^{j\ell}$$
$$\nabla_j y^j = 0$$

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$$\nabla_j y^j = 0$$

Need $\widetilde{R} = \sum_I R_I$ where $\operatorname{supp}_t R_I \subseteq [t(I) - \theta, t(I) + \theta]$, $\theta \sim \|\nabla v\|_{C^0}^{-1}$. Also need

$$\|y\|_{C^0} \lesssim e_R^{1/2} \sim \|R\|_{C^0}^{1/2}$$

and $\|\widetilde{R}\|_{C^0} \lesssim e_R \sim \|R\|_{C^0}$

The Gluing Technique

Want the new error $\widetilde{R} = \sum_I R_I$ supported in disjoint intervals:

$$\operatorname{supp}_t R_I \subseteq [t(I) - \theta, t(I) + \theta]$$
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So the new $\tilde{v}^\ell$ should solve the Euler equations *exactly* in the gaps between the intervals

$$[t(I) - \theta, t(I) + \theta] \text{ and } [t(I+1) - \theta, t(I+1) + \theta]$$

Also, $\tilde{v}^\ell$ needs to be a close approximation to v^ℓ.

The Gluing Technique

Let $u_I^\ell = v^\ell + y_I^\ell$ be the unique, smooth solution to Euler starting at the middle of the Ith gap $t_0(I)$ with initial data

$$u_I^\ell(t_0(I), x) = v^\ell(t_0(I), x), \qquad y_I^\ell(t_0(I), x) = 0$$

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Theorem (Classical Existence Result)

There exists a unique open interval $\widetilde{J}_I$ containing $t_0(I)$ such that u_I is smooth on $\widetilde{J}_I \times \mathbb{T}^3$ and for all $T^* \in \partial \widetilde{J}_I$ endpoints of $\widetilde{J}_I$,

$$\limsup_{t \to T^*} \|\nabla u_I(t)\|_{C^0} = \infty$$

(We will have to prove that $\mathrm{supp}_t\, \eta_I \subseteq \widetilde{J}_I$ to know the formula is well-defined).

The New Stress

With y_I^ℓ and $y^\ell = \sum_I \eta_I y_I^\ell$ as above, the new $\widetilde{R}^{j\ell}$ is a solution to

$$\nabla_j \widetilde{R}^{j\ell} = \sum_I \eta_I'(t) y_I^\ell + \sum_I \eta_I \eta_{I+1} \nabla_j (y_I^j y_{I+1}^\ell + y_{I+1}^j y_I^\ell) + \sum_I (\eta_I^2 - \eta_I) \nabla_j (y_I^j y_I^\ell),$$

where each $y_I^\ell = u_I^\ell - v^\ell$ solves

$$\partial_t y_I^\ell + v^j \nabla_j y_I^\ell + y_I^j \nabla_j v^\ell + \nabla_j (y_I^j y_I^\ell) + \nabla^\ell \bar{p}_I = -\nabla_j R^{j\ell}$$
$$\nabla_j y_I^j = 0$$
$$y_I^\ell (t_0(I), x) = 0$$

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Choosing $r_I^{j\ell}$ such that $\nabla_j r_I^{j\ell} = y_I^\ell$, the new stress will be

$$\widetilde{R}^{j\ell} = \sum_I \eta_I'(t) r_I^{j\ell} + \sum_I \eta_I \eta_{I+1} (y_I^j y_{I+1}^\ell + y_{I+1}^j y_I^\ell) + \sum_I (\eta_I^2 - \eta_I) y_I^j y_I^\ell$$

Note that $\operatorname{supp}_t \widetilde{R} \subseteq \bigcup_I \operatorname{supp}_t \eta_I' \subseteq \bigcup_I [t(I) - \theta, t(I) + \theta]$.

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Note that $\mathrm{supp}_t\, \widetilde{R} \subseteq \bigcup_I \mathrm{supp}_t\, \eta_I' \subseteq \bigcup_I [t(I) - \theta, t(I) + \theta]$.

Finding a good Anti-Divergence: Attempt 1

Problem: we get bad estimates from solving

$$\nabla_j r_I^{j\ell} = y_I^\ell. \tag{20}$$

Suppose that e_R is the size of the error ($\|R\|_{C^0} \le e_R$) and suppose (optimistically) that $\|y_I^\ell\|_{C^0} \sim e_R^{1/2}$ obeys the bound we desire for $y^\ell = \tilde{v}^\ell - v^\ell$. Then our new error has size

$$\begin{aligned} \|\widetilde{R}\|_{C^0} =& \|\eta_I'(t) r_I + \ldots\|_{C^0} \\ &\lesssim \theta^{-1}\|r_I\|_{C^0} \lesssim \theta^{-1}\|y_I\|_{C^0} \\ \|\widetilde{R}\|_{C^0} \lesssim& \theta^{-1} e_R^{1/2} + \ldots \end{aligned}$$

Our goal was e_R. Having $e_R^{1/2}$ is already too big, and having θ^{-1} makes this bound diverge to ∞!

Finding a good Anti-Divergence: Attempt 2

We can find a better solution to $\nabla_j r_I^{j\ell} = y_I^\ell$ using the equation

$$\partial_t y_I^\ell = -v^j \nabla_j y_I^\ell - y_I^j \nabla_j v^\ell - \nabla_j (y_I^j y_I^\ell) - \nabla^\ell \bar{p}_I - \nabla_j R^{j\ell}$$

Finding a good Anti-Divergence: Attempt 2

We can find a better solution to $\nabla_j r_I^{j\ell} = y_I^\ell$ using the equation

$$\partial_t y_I^\ell = -\nabla_j (v^j y_I^\ell + y_I^j v^\ell + y_I^j y_I^\ell + \bar{p}_I \delta^{j\ell} + R^{j\ell})$$

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$$y_I^\ell(t,\cdot) = -\nabla_j \overbrace{\int_0^t (v^j y_I^\ell(\tau,\cdot) + y_I^j v^\ell(\tau,\cdot) + \ldots + R^{j\ell}(\tau,\cdot))d\tau}^{=-r_I^{j\ell}(t,\cdot)}$$

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$$\|\widetilde{R}\|_{C^0} \lesssim \theta^{-1}\|r_I\|_{C^0} + \ldots \lesssim \|v\|_{C^0}\|y_I\|_{C^0} + \ldots$$

$$\|\widetilde{R}\|_{C^0} \lesssim e_R^{1/2} + \ldots$$

Still not the desired $\|\widetilde{R}\|_{C^0} \lesssim e_R$.

Finding a good Anti-Divergence: Attempt 3

Idea: Set $r_I^{j\ell}(t_0(I), x) = 0$ and solve a transport equation

$$(\partial_t + v^i \nabla_i)[\nabla_j r_I^{j\ell}] = (\partial_t + v^i \nabla_i) y_I^\ell,$$
$$(\partial_t + v^i \nabla_i)[\nabla_j r_I^{j\ell}] = -y_I^j \nabla_j v^\ell - \nabla_j (y_I^j y_I^\ell) - \nabla^\ell \bar{p}_I - \nabla_j R^{j\ell}$$

(Motivation: "integration" over trajectories is more natural than integrating in time at fixed x.)

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Idea: Set $r_I^{j\ell}(t_0(I), x) = 0$ and solve a transport equation

$$(\partial_t + v^i \nabla_i)[\nabla_j r_I^{j\ell}] = (\partial_t + v^i \nabla_i) y_I^\ell,$$
$$(\partial_t + v^i \nabla_i)[\nabla_j r_I^{j\ell}] = -y_I^j \nabla_j v^\ell - \nabla_j (y_I^j y_I^\ell) - \nabla^\ell \bar{p}_I - \nabla_j R^{j\ell}$$

Setting $r_I^{j\ell} = \rho_I^{j\ell} + z_I^{j\ell}$, we can solve away the last few terms:

$$(\partial_t + v^j \nabla_j) z_I^{j\ell} = -y_I^j y_I^\ell - \bar{p}_I \delta^{j\ell} - R^{j\ell} \qquad (21)$$

Then $\|z_I\|_{C^0}$ looks good if we have

$$\|y_I\|_{C^0} \lesssim e_R^{1/2}, \qquad \|\bar{p}_I\|_{C^0} \lesssim e_R$$

Finding a good Anti-Divergence: Attempt 3

Idea: Set $r_I^{j\ell}(t_0(I), x) = 0$ and solve a transport equation

$$(\partial_t + v^i \nabla_i)[\nabla_j r_I^{j\ell}] = (\partial_t + v^i \nabla_i) y_I^\ell,$$
$$(\partial_t + v^i \nabla_i)[\nabla_j r_I^{j\ell}] = -y_I^j \nabla_j v^\ell - \nabla_j (y_I^j y_I^\ell) - \nabla^\ell \bar{p}_I - \nabla_j R^{j\ell}$$

To handle the linear term, let $r_I^{j\ell} = \rho_I^{j\ell} + z_I^{j\ell}$ where

$$\nabla_j [(\partial_t + v^i \nabla_i) \rho_I^{j\ell}] = \nabla_j v^i \nabla_i r_I^{j\ell} - y_I^j \nabla_j v^\ell$$

(Obtained by commuting ∇_j and $(\partial_t + v^i \nabla_i)$.)

Finding a good Anti-Divergence: Attempt 3

To handle the linear term, let $r_I^{j\ell} = \rho_I^{j\ell} + z_I^{j\ell}$ where

$$\nabla_j[(\partial_t + v^i\nabla_i)\rho_I^{j\ell}] = \nabla_j v^i \nabla_i r_I^{j\ell} - y_I^i \nabla_i v^\ell \tag{22}$$

Equation (22) can only be solved if we can invert the divergence on both sides. We need to know the right hand side has integral 0:

$$\nabla_j v^i \nabla_i r_I^{j\ell} - y_I^j \nabla_j v^\ell = \nabla_i[\nabla_j v^i r_I^{j\ell} - y_I^i v^\ell]$$

Here we use that $\nabla_i v^i = \nabla_i y_I^i = 0$.

We now invert the divergence to obtain an equation for ρ_I.

Finding a good Anti-Divergence: Attempt 3

We let $\rho_I^{j\ell}$ solve a "transport-elliptic" equation:

$$(\partial_t + v^i \nabla_i)\rho_I^{j\ell} = \mathcal{R}^{j\ell}[\nabla_a v^i \nabla_i(\rho_I^{ab} + z_I^{ab}) - y_I^i \nabla_i v^b]$$

where $\mathcal{R}^{j\ell} = \text{div}^{-1}$ is an order -1 operator that inverts divergence. This type of equation can be solved as in (I. '12) as long as y_I and z_I are smooth.

Question: Are the estimates good enough?
(e.g. Do we have $\|\widetilde{R}\|_{C^0} \lesssim e_R$?)

Finding a good Anti-Divergence: Attempt 3

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The corresponding estimate for $\widetilde{R}^{j\ell} = \eta_I'(t) r_I + \ldots$ is:

$$\begin{aligned}
\|\widetilde{R}\|_{C^0} &\lesssim \theta^{-1}\|\rho_I\|_{C^0} + \ldots \lesssim \|(\partial_t + v\cdot\nabla)\rho_I\|_{C^0} + \ldots \\
&\lesssim \|\mathcal{R}^{j\ell}[y_I^i \nabla_i v^b]\|_{C^0} + \text{other terms} \\
&\lesssim e_R^{1/2} \cdot \theta^{-1} + \ldots
\end{aligned}$$

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\|\widetilde{R}\|_{C^0} &\lesssim \theta^{-1}\|\rho_I\|_{C^0} + \dots \lesssim \|(\partial_t + v\cdot\nabla)\rho_I\|_{C^0} + \dots \\
&\lesssim \|\mathcal{R}^{j\ell}\nabla_i[y_I^i v^b]\|_{C^0} + \text{other terms} \\
&\lesssim e_R^{1/2}\cdot 1 + \dots \\
&\text{(if we pretend } \mathcal{R}\nabla = \text{div}^{-1}\nabla \text{ is bounded on } C^0)
\end{aligned}$$

But that is still not good enough for $\|\widetilde{R}\|_{C^0} \lesssim e_R$...

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But that is still not good enough for $\|\widetilde{R}\|_{C^0} \lesssim e_R$...
Key point: We can actually prove $\|\mathcal{R}^{j\ell}[y_I^i \nabla_i v^b]\|_{C^0} \lesssim e_R$! (almost)

The Pressure Has a Similar Bad Term

The pressure increment has a similar bad term

$$\bar{p}_I = -2\Delta^{-1}\nabla_\ell[y_I^j \nabla_j v^\ell] - \Delta^{-1}\nabla_\ell \nabla_j [y_I^j y_I^\ell + R^{j\ell}]$$

Note that the highlighted operator is of order -1, similar to $\mathcal{R}^{j\ell}$. Let us show how to (almost) estimate this term by

$$\|\Delta^{-1}\nabla_\ell[y_I^j \nabla_j v^\ell]\|_{C^0} \lesssim e_R$$

The Pressure Has a Similar Bad Term

Notation: We define the Littlewood-Paley projections

$$P_q u^\ell(x) = \int_{\mathbb{R}^3} u^\ell(x-h)\eta_q(h)dh$$

$$\text{supp } \hat{\eta}_q(\xi) \subseteq \{2^{q-2} \leq |\xi| \leq 2^{q+2}\}$$

$$\eta_q(h) = 2^{3q}\eta_0(2^q h)$$

$$u^\ell(x) = \Pi_0 u^\ell + \sum_{q=0}^{\infty} P_q u^\ell(x), \qquad x \in \mathbb{T}^3$$

The Pressure Has a Similar Bad Term

Choose $\widehat{\Xi}$ such that $\theta^{-1} \sim \|\nabla v\|_{C^0} \lesssim \widehat{\Xi} e_R^{1/2}$ and choose $\hat{q} \in \mathbb{Z}$ such that $2^{\hat{q}-1} \leq \widehat{\Xi} < 2^{\hat{q}}$. Then

$$\begin{aligned} \Delta^{-1}\nabla_\ell[y_I^j \nabla_j v^\ell] &= \Delta^{-1}\nabla_\ell P_{\leq \hat{q}}[y_I^j \nabla_j v^\ell] + \sum_{q > \hat{q}} \Delta^{-1}\nabla_\ell P_q[y_I^j \nabla_j v^\ell] \\ &= \qquad \bar{p}_{I,L} \qquad + \qquad \bar{p}_{I,H} \end{aligned}$$

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The high frequency term is bounded by

$$\begin{aligned}\|\bar{p}_{I,H}\|_{C^0} &\leq \sum_{q > \hat{q}} \|\Delta^{-1}\nabla_\ell P_q [y_I^j \nabla_j v^\ell]\|_{C^0} \\ &\leq \sum_{q > \hat{q}} \underbrace{\|\Delta^{-1}\nabla_\ell P_q\|}_{(C^0 \mapsto C^0)\text{norm}} \; \|y_I^j \nabla_j v^\ell\|_{C^0}\end{aligned}$$

(Note the operator convolves with an L^1 Schwartz kernel.)

The Pressure Has a Similar Bad Term

Choose $\widehat{\Xi}$ such that $\theta^{-1} \sim \|\nabla v\|_{C^0} \lesssim \widehat{\Xi} e_R^{1/2}$ and choose $\hat{q} \in \mathbb{Z}$ such that $2^{\hat{q}-1} \leq \widehat{\Xi} < 2^{\hat{q}}$. Then

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The high frequency term is bounded by

$$\begin{aligned} \|\bar{p}_{I,H}\|_{C^0} &\leq \sum_{q > \hat{q}} \|\Delta^{-1}\nabla_\ell P_q\| \; \|y_I^j \nabla_j v^\ell\|_{C^0} \\ &\lesssim \sum_{q > \hat{q}} 2^{-q} \|y_I^j \nabla_j v^\ell\|_{C^0} \\ &\lesssim \widehat{\Xi}^{-1} e_R^{1/2} (\theta^{-1}) = e_R \end{aligned}$$

It now remains to bound the low frequency term.

The Low Frequency Term

The low frequency term has the form

$$\bar{p}_{I,L} = \Delta^{-1}\nabla_\ell P_{\leq \hat{q}}[y_I^j \nabla_j v^\ell] = \sum_{q=0}^{\hat{q}} \Delta^{-1}\nabla_\ell P_q[y_I^j \nabla_j v^\ell]$$

In this case, we do not gain smallness from bounding

$$\|\Delta^{-1}\nabla_\ell P_q\| \lesssim 2^{-q} \lesssim 1$$

The Low Frequency Term

Step 2: Decompose v into high and low frequencies

$$\begin{aligned} \bar{p}_{I,L} &= \bar{p}_{I,LL} + \bar{p}_{I,LH} \\ \bar{p}_{I,LL} &= \Delta^{-1}\nabla_\ell P_{\leq \hat{q}}[y_I^j \nabla_j P_{\leq \hat{q}} v^\ell] \\ \bar{p}_{I,LH} &= \sum_{q > \hat{q}} \Delta^{-1}\nabla_\ell P_{\leq \hat{q}}[y_I^j \nabla_j P_q v^\ell] \end{aligned}$$

The Low Frequency Term

Step 2: Decompose v into high and low frequencies

$$\bar{p}_{I,L} = \bar{p}_{I,LL} + \bar{p}_{I,LH}$$
$$\bar{p}_{I,LL} = \Delta^{-1}\nabla_\ell P_{\leq \hat{q}}[y_I^j \nabla_j P_{\leq \hat{q}} v^\ell]$$
$$\bar{p}_{I,LH} = \sum_{q > \hat{q}} \Delta^{-1}\nabla_\ell P_{\leq \hat{q}}[y_I^j \nabla_j P_q v^\ell]$$

And bound the LH term using $\nabla_j y_I^j = 0$:

$$\|\bar{p}_{I,H}\|_{C^0} \leq \sum_{q > \hat{q}} \|\Delta^{-1}\nabla_\ell \nabla_j P_{\leq \hat{q}}\| \|y_I\|_{C^0} \|P_q v\|_{C^0}$$
$$\lesssim \sum_{q > \hat{q}} \log \widehat{\Xi} \qquad e_R^{1/2}(2^{-q}\|\nabla v\|_{C^0})$$
$$\lesssim \log \widehat{\Xi}\, e_R^{1/2} \widehat{\Xi}^{-1}(\widehat{\Xi} e_R^{1/2}) \lesssim \log \widehat{\Xi}\, e_R$$

Remaining Problematic Term

The remaining problematic term is

$$\bar{p}_{I,LL} = \Delta^{-1} \nabla_\ell P_{\leq \hat{q}} [y_I^j \nabla_j P_{\leq \hat{q}} v^\ell]$$

or

$$\bar{p}_{I,LL} = \Delta^{-1} \nabla_\ell P_{\leq \hat{q}} [P_{\leq \hat{q}+3} y_I^j \nabla_j P_{\leq \hat{q}} v^\ell]$$

using that high frequencies of y_I do not contribute.

Remaining Problematic Term

The remaining problematic term is

$$\bar{p}_{I,LL} = \Delta^{-1}\nabla_\ell P_{\leq \hat{q}}[y_I^j \nabla_j P_{\leq \hat{q}} v^\ell]$$

We treat this term by decomposing into frequency increments

$$\bar{p}_{I,LL} = \sum_{q=-1}^{\hat{q}} \delta_q \bar{p}_{I,LL}$$

$$\delta_q \bar{p}_{I,LL} = \Delta^{-1}\nabla_\ell P_{\leq q+1}[y_I^j \nabla_j P_{\leq q+1} v^\ell] - \Delta^{-1}\nabla_\ell P_{\leq q}[y_I^j \nabla_j P_{\leq q} v^\ell]$$

Note: Starting now, 2^q is in the low to medium range of frequencies.

Frequency Increments

The frequency increment can either fall on the operator or on v:

$$\delta_q \bar{p}_{I,LL} = \Delta^{-1}\nabla_\ell P_{q+1}[y_I^j \nabla_j P_{\leq q+1} v^\ell] + \Delta^{-1}\nabla_\ell P_{\leq q}[y_I^j \nabla_j P_{q+1} v^\ell]$$

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Consider the second term. Using $\nabla_j y_I^j = 0$, we have

$$\begin{aligned} \Delta^{-1}\nabla_\ell P_{\leq q}[y_I^j \nabla_j P_{q+1} v^\ell] &= \Delta^{-1}\nabla_\ell \nabla_j P_{\leq q}[y_I^j P_{q+1} v^\ell] \\ &= \Delta^{-1}\nabla_\ell \nabla_j P_{\leq q}[P_{\leq q+6} y_I^j P_{q+1} v^\ell] \end{aligned}$$

Frequency Increments

The frequency increment can either fall on the operator or on v:

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In the last line, we observe that frequencies of y_I^ℓ above 2^{q+4} do not contribute to the product by the frequency localization.

Frequency Increments

Now use that we can solve $y_I^j = \nabla_i r_I^{ij}$ to write

$$\begin{aligned}
&\Delta^{-1}\nabla_\ell\nabla_j P_{\leq q}[P_{\leq q+6} y_I^j P_{q+1} v^\ell] \\
&= \Delta^{-1}\nabla_\ell\nabla_j P_{\leq q}[P_{\leq q+6}\nabla_i r_I^{ij} P_{q+1} v^\ell] \\
\|\cdot\|_{C^0} &\lesssim \|\Delta^{-1}\nabla_\ell\nabla_j P_{\leq q}\| \; \|P_{\leq q+6}\nabla_i\| \; \|r_I\|_{C^0}[2^{-q}\|\nabla v\|_{C^0}] \\
&\lesssim (2+q)2^q\|r_I\|_{C^0} 2^{-q}\widehat{\Xi} e_R^{1/2}
\end{aligned}$$

Note how the 2^q and 2^{-q} cancel out.

Frequency Increments

Now use that we can solve $y_I^j = \nabla_i r_I^{ij}$ to write

$$\begin{aligned}
&\Delta^{-1}\nabla_\ell \nabla_j P_{\leq q}[P_{\leq q+6} y_I^j P_{q+1} v^\ell] \\
&= \Delta^{-1}\nabla_\ell \nabla_j P_{\leq q}[P_{\leq q+6} \nabla_i r_I^{ij} P_{q+1} v^\ell] \\
\| \cdot \|_{C^0} &\lesssim \|\Delta^{-1}\nabla_\ell \nabla_j P_{\leq q}\| \, \|P_{\leq q+6}\nabla_i\| \, \|r_I\|_{C^0}[2^{-q}\|\nabla v\|_{C^0}] \\
&\lesssim (2+q)\|r_I\|_{C^0}(\widehat{\Xi} e_R^{1/2})
\end{aligned}$$

Almost closes if there exists r_I such that $\|r_I\|_{C^0}\widehat{\Xi} \lesssim e_R^{1/2}$

Frequency Increments

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\| \cdot \|_{C^0} &\lesssim \|\Delta^{-1}\nabla_\ell \nabla_j P_{\leq q}\| \; \|P_{\leq q+6}\nabla_i\| \; \|r_I\|_{C^0}[2^{-q}\|\nabla v\|_{C^0}] \\
&\lesssim (2+q)\|r_I\|_{C^0}(\widehat{\Xi} e_R^{1/2})
\end{aligned}$$

Idea: impose a bootstrap assumption on ρ_I and z_I that implies

$$\widehat{\Xi}\|r_I\|_{C^0} \lesssim e_R^{1/2}$$

Then summing over $q \leq \hat{q} \sim \log \widehat{\Xi}$ leads to $\|\widetilde{R}_I\|_{C^0} \lesssim (\log \widehat{\Xi})^2 e_R$, which is the correct estimate (except for the $(\log \widehat{\Xi})^2$)!

Loss of Derivatives

It turns out that (if one furthermore shrinks the time scale θ by a logarithmic factor) it is possible to close the argument implying the above estimates by using certain weighted $C^{3,\alpha}$ norms.

But there is a catch...

Loss of Derivatives

It turns out that (if one furthermore shrinks the time scale θ by a logarithmic factor) it is possible to close the argument implying the above estimates by using certain weighted $C^{3,\alpha}$ norms.

But there is a catch, namely this gluing construction loses derivatives. E.g., ∇v and ∇R both enter in the equation for y_I

$$\partial_t y_I^\ell + v^j \nabla_j y_I^\ell + y_I^j \nabla_j v^\ell + \nabla_j (y_I^j y_I^\ell) + \nabla^\ell \bar{p}_I = -\nabla_j R^{j\ell}$$

Similarly, bounds on $\nabla^2 v$ and $\nabla^2 R$ are required to estimate ∇y_I and so on...

Loss of Derivatives

To fully close the argument, we first regularize the Euler-Reynolds flow $(v, p, R) \mapsto (v_\epsilon, p_\epsilon, R_\epsilon)$ using a mollifier $\eta_\epsilon *$

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell}$$
$$\Rightarrow \partial_t v^\ell_\epsilon + \nabla_j (v^j_\epsilon v^\ell_\epsilon) + \nabla^\ell p_\epsilon = \nabla_j [v^j_\epsilon v^\ell_\epsilon - (v^j v^\ell)_\epsilon + \eta_\epsilon * R^{j\ell}]$$

We apply the Constantin-E-Titi commutator estimate to bound the resulting Stress for $\epsilon \sim \widehat{\Xi}^{-1}$ not too small.

This regularization gains derivatives (with acceptable bounds on higher, "borrowed" derivatives), and allows the whole scheme (i.e. Regularize $\mapsto$ Gluing $\mapsto$ Convex integration with Mikado flows $\mapsto$ repeat) to close.

Thank you!