Introduction to Logic 9

Last time:

- Consistency and Inconsistency
- Semantic Tableaux
- The Tableaux Technique
- Tableaux Derivation Rules

This time:

- Tableaux Examples
- Satisfying Valuations
- Justification for the Tableaux Rules
- Inconsistency and Entailment
- Bacon and Hamlet (Again)

Semantic Tableaux Examples

- Semantic Tableaux enable us to check consistency/inconsistency of sets of sentences.

  e.g.

  $G = \{(p \land q), (p \to \neg q)\}$

- Construct a tableau as follows:

$$
\begin{array}{ccc}
 & (p \land q) & \\
 & (p \to \neg q) & \\
 \swarrow & & \searrow \\
\neg p & & \neg q \\
\underline{p} & & \underline{q}
\end{array}
$$

- Both branches are closed, so G is inconsistent!

- The method typically requires less effort than the method of truthtables (see start of last lecture for comparison).

- Is the following set of sentences inconsistent?

$$G = \{(p \rightarrow q), (p \vee \neg q)\}$$

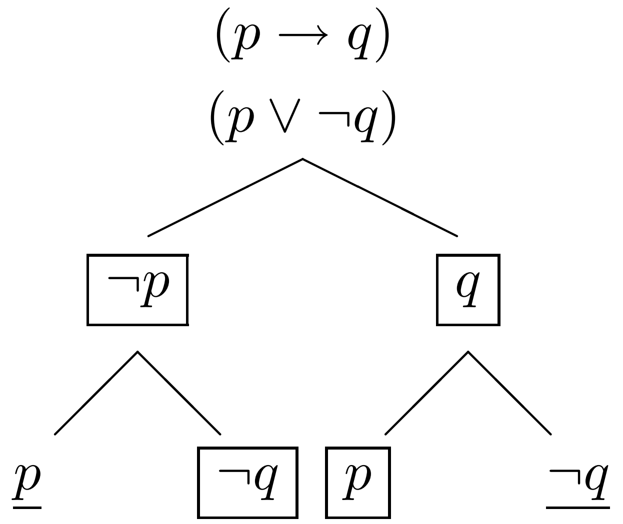
- Construct a tableau as follows:

$$\begin{array}{cccc}
\multicolumn{4}{c}{(p \rightarrow q)} \\
\multicolumn{4}{c}{(p \vee \neg q)} \\
\multicolumn{2}{c}{\neg p} & \multicolumn{2}{c}{q} \\
\underline{p} & \neg q & p & \underline{\neg q}
\end{array}$$

- The tableau is ‘finished’, but it is not closed.
- Two branches remain open: the set G is *consistent*.

Definition: Let G be a set of sentences and V a valuation. We say that V *satisfies* G if and only if V makes every sentence in the set G true.

- We may want to know what valuations satisfy a consistent set G.
- This information can be found from a tableau for G. For example:



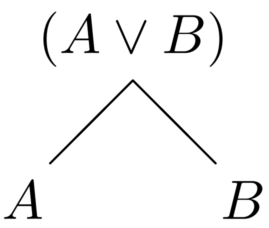
Question: What can we say about valuations that satisfy this set?

Justifying Tableaux Rules

- We can view the tableaux rules *syntactically*.
- We can also view them *semantically*.

  i.e. we can *interpret* the rules and show that they are sensible.
- Tableaux rules can be justified/motivated straightforwardly by considering truth tables.

  e.g.



A	B	$(A \vee B)$
t	t	t
t	f	t
f	t	t
f	f	f

- Note that there are just two sorts of 'situations' in which $(A \vee B)$ is true:
  1. situations where A is true
  2. situations where B is true

- Consider now the tableau rule for $\neg(A \wedge B)$:

$$
\begin{array}{ccc}
 & \neg(A \wedge B) & \\
\swarrow & & \searrow \\
\neg A & & \neg B
\end{array}
$$

- Recall the following equivalence:

$$\neg(A \wedge B) \equiv (\neg A \vee \neg B)$$

  (this is one of De Morgan's equivalences – see lecture 4).

- So, using the tableau rule for disjunction, we can justify the rule by noting that:

$$
\begin{array}{ccc}
 & (\neg A \vee \neg B) & \\
\swarrow & & \searrow \\
\neg A & & \neg B
\end{array}
$$

- Similarly, we can justify the rule for $(A \rightarrow B)$:

$$(A \rightarrow B)$$

$$\neg A \qquad\qquad B$$

- In this case we can make use of the following logical equivalence:

$$(A \rightarrow B) \equiv (\neg A \vee B)$$

(easy to check with truthtables; also given in lecture 4.)

- We can provide a justification for each of the derivation rules of the semantic tableaux method.
- This effectively shows that the method is **sound**

Inconsistency and Entailment

- The tableaux method allows us to test consistency/inconsistency of sets of sentences
- This may seem rather limiting, but it was claimed in the previous lecture that the method can also be used for testing entailment.

Question: *how do we use semantic tableaux to test for entailment?*

The answer to this can be found in the definition of entailment.

- Recall the definition:

  $G \models A$ *if and only if every valuation that makes each sentence in* G **true** *also makes* A **true** .

- or to put it another (and equivalent) way.....

> $G \models A$ *if and only if every valuation that makes each sentence in* G **true** *also makes* $\neg A$ **false** .

- and what this comes down to is...

  > $G \models A$ *if and only if the set of sentences* $G \cup \{\neg A\}$ *is inconsistent.*

- But we can use semantic tableaux to test consistency/inconsistency.
- So we can use semantic tableaux to test entailment.
- To test whether $G \models A$, we:
  1. form the set $G \cup \{\neg A\}$; and
  2. use tableaux to determine if the set is inconsistent (entailment holds) or consistent (entailment does not hold).

Example (Bacon and Hamlet (Again))

- Consider the following argument:

> *If Bacon wrote Hamlet, then Bacon was a great writer. But Bacon did not write Hamlet. So Bacon was not a great writer.*

- We can formalize the premisses and the conclusion of the argument as follows:

$$\begin{array}{ll} \text{Premise 1} & (p \rightarrow q) \\ \text{Premise 2} & \neg p \\ \hline \text{Conclusion} & \neg q \end{array}$$

- Moreover, this argument will be correct (valid, sound) just in case the following entailment holds:

$$\{(p \rightarrow q), \neg p\} \models \neg q$$

- We will test this entailment using the semantic tableaux method.

- To test whether

$$\{(p \to q), \neg p\} \models \neg q$$

we test consistency of the set:

$$\{(p \to q), \neg p, \neg\neg q\}$$

- Applying the tableau method yields:

$$\begin{array}{ccc} & (p \to q) & \\ & \neg p & \\ & \neg\neg q & \\ \neg p & & q \\ q & & q \end{array}$$

- The tableau is ‘finished’, but not closed.
- It follows that the set is *consistent* ... so entailment does *not* hold ... and the argument is *not* valid.

Summary

- Semantic tableaux provide a convenient and systematic technique for testing consistency/inconsistency of sets of sentences
- Tableaux can be used to find the valuations that *satisfy* a set of statements.
- Tableax derivation rules can be given a semantic justification
- There is a close connection between the notions of *inconsistency* and *entailment*.
- This provides the basis for testing entailment using the method of semantic tableaux.