Introduction to Logic 8

Last time:

- PC as an Axiomatic System
- Formal Proofs
- The Deduction Relation
- Deduction and Entailment

This time:

- Consistency and Inconsistency
- Semantic Tableau
- The Tableaux Technique
- Tableaux Derivation Rules

Consistency and Inconsistency

- Recall that a set of sentences G is *consistent* if there is at least one valuation that makes every sentence in G true (and otherwise G is *inconsistent*).
- We can test consistency/inconsistency using the method of truth tables:

  e.g.

$$G = \{(p \wedge q), (p \rightarrow \neg q)\}$$

p	q	$(p \wedge q)$	$\neg q$	$(p \rightarrow \neg q)$
t	t	t	f	f
t	f	f	t	t
f	t	f	f	t
f	f	f	t	t

Thus G is inconsistent!

Semantic Tableaux

- There are more effective ways of testing for consistency/inconsistency.
- The method of semantic tableaux provides a means of testing *inconsistency* of sets of sentences.
- Semantic tableaux are more expressive and in some ways easier to use than truth tables
- Can also be used to test *entailement* :

  Is it the case that $G \models A$?

- Based on the idea of generating *descriptions* of situations.

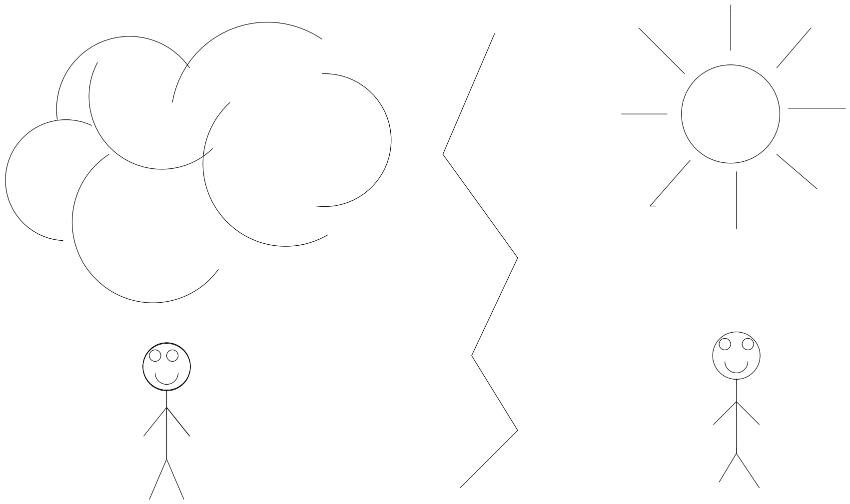
The Tableaux Technique

- Consider a set of sentences G.
  - We can think of G as describing different possible situations....
  - ....those situations which make every sentence of G true.

i.e.

$$\{\textit{(it is cloudy} \vee \textit{it is sunny)}, \textit{Bill is happy}\}$$

Situation 1 Situation 2



- Semantic tableaux provide a systematic method for finding what possible situations are described by a set of sentences G.
  - We use G to produce new descriptions of the situations....
  - ....the new descriptions are obtained by *simplifying* the complex sentences in G.

e.g.

$$\{\textit{(it is cloudy} \vee \textit{it is sunny)}, \textit{Bill is happy}\}$$

Description 1

$$\{\textit{it is cloudy}, \textit{Bill is happy}\}$$

Description 2

$$\{\textit{it is sunny}, \textit{Bill is happy}\}$$

Tableaux and Inconsistency

- The tableaux method has various **tableaux derivation rules** that allow us to construct a ‘picture’ of all the different possible descriptions.
- This picture is a tree diagram (the **tableau**). e.g.

$$\begin{array}{ccc} & \textit{(it is cloudy} \vee \textit{it is sunny)} & \\ & \textit{Bill is happy} & \\ \textit{it is cloudy} & & \textit{it is sunny} \end{array}$$

- This tableau has two branches, where each branch represents a situation.

- Sometimes, branches *fail* to represent a possible situation.

  e.g.

$$(it\ is\ cloudy \lor it\ is\ sunny)$$
$$\neg it\ is\ sunny$$

it is cloudy *it is sunny*

- This tableau has two branches.
  - One branch fails to describe a situation – it contains inconsistent information.
  - The branch is said to be **closed**
- We write a line under the branch to show that it is closed.

- In general, whenever a branch contains a statement A and a statement $\neg A$, then it contains inconsistent information, and is said to be *closed.*
- Closed branches are not extended further.
- If *all* the branches of a tableau are closed, then we have shown that the set of statements we started from is inconsistent.

  e.g.

$$(\textit{it is cloudy} \vee \textit{it is sunny})$$
$$(\neg \textit{it is cloudy} \wedge \neg \textit{it is sunny})$$
$$|$$
$$\neg \textit{it is cloudy}$$
$$\neg \textit{it is sunny}$$
$$\diagup \qquad \diagdown$$
$$\underline{\textit{it is cloudy}} \qquad \underline{\textit{it is sunny}}$$

Tableaux Rules

$$\begin{array}{c} \neg\neg A \\ | \\ A \end{array}$$

$$\begin{array}{c} (A \wedge B) \\ | \\ A \\ B \end{array} \qquad \begin{array}{ccc} & \neg(A \wedge B) & \\ / & & \backslash \\ \neg A & & \neg B \end{array}$$

$$\begin{array}{ccc} & (A \vee B) & \\ / & & \backslash \\ A & & B \end{array} \qquad \begin{array}{c} \neg(A \vee B) \\ | \\ \neg A \\ \neg B \end{array}$$

$$\begin{array}{ccc} & (A \rightarrow B) & \\ / & & \backslash \\ \neg A & & B \end{array} \qquad \begin{array}{c} \neg(A \rightarrow B) \\ | \\ A \\ \neg B \end{array}$$

$$\begin{array}{ccc} & (A \leftrightarrow B) & \\ / & & \backslash \\ A & & \neg A \\ B & & \neg B \end{array} \qquad \begin{array}{ccc} & \neg(A \leftrightarrow B) & \\ / & & \backslash \\ A & & \neg A \\ \neg B & & B \end{array}$$

Example

- Is the set

$$\{\neg(p \wedge \neg q), (q \to r), (p \wedge \neg r)\}$$

consistent or inconsistent?

1.

$$\neg(p \wedge \neg q)$$
$$(q \to r)$$
$$(p \wedge \neg r)$$

2.

$$\neg(p \wedge \neg q)$$
$$(q \to r)$$
$$(p \wedge \neg r)$$

$$\neg p \qquad\qquad \neg\neg q$$

3.

$$\begin{array}{c}
\neg(p \wedge \neg q) \\
(q \to r) \\
(p \wedge \neg r)
\end{array}$$

$$\begin{array}{ccc}
 & \wedge & \\
\neg p & & \neg\neg q \\
p & & p \\
\underline{\neg r} & & \neg r
\end{array}$$

4.

$$\begin{array}{c}
\neg(p \wedge \neg q) \\
(q \to r) \\
(p \wedge \neg r)
\end{array}$$

$$\begin{array}{ccccc}
 & \wedge & & & \\
\neg p & & \neg\neg q & & \\
p & & p & & \\
\underline{\neg r} & & \neg r & & \\
 & & \wedge & & \\
 & \underline{\neg q} & & \underline{r} &
\end{array}$$

- Each branch or the tableau is closed.

- Because each branch of the tableau closed, we say that *the tableau is closed*.
- This means that every branch of the tableau contains contradictory information.
  - we cannot find a valuation that will make every sentence on a given branch true.
  - there is no valuation that makes every sentence in the original set true.
- It follows that the set

$$\{\neg(p \wedge \neg q), (q \to r), (p \wedge \neg r)\}$$

is inconsistent.

Summary

- Semantic tableaux provide a technique for testing consistency/inconsistency of sets of sentences
- Tableaux are more expressive, and easier to use than truth tables
- The method is based on the idea of simplifying descriptions/sentences and looking for contradictions.
- The tableaux derivation rules allow us to grow a tree diagram representing possible situations.
- In contrast to the axiomatic system of propositional logic, the tableaux proof method is simple and straightforward to use.