Introduction to Logic 8

Last time:

- PC as an Axiomatic System
- Formal Proofs
- The Deduction Relation
- Deduction and Entailment

This time:

- Consistency and Inconsistency
- Semantic Tableau
- The Tableaux Technique
- Tableaux Derivation Rules

Consistency and Inconsistency

- Recall that a set of sentences G is *consistent* if there is at least one valuation that makes every sentence in G true (and otherwise G is *inconsistent*).
- We can test consistency/inconsistency using the method of truth tables:

  e.g.

$$G = \{(p \wedge q), (p \rightarrow \neg q)\}$$

p	q	$(p \wedge q)$	$\neg q$	$(p \rightarrow \neg q)$
t	t	t	f	f
t	f	f	t	t
f	t	f	f	t
f	f	f	t	t

Thus G is inconsistent!

Semantic Tableaux

- There are more effective ways of testing for consistency/inconsistency.
- The method of semantic tableaux provides a means of testing *inconsistency* of sets of sentences.
- Semantic tableaux are more expressive and in some ways easier to use than truth tables
- Can also be used to test *entailement* :

  Is it the case that $G \models A$?

- Based on the idea of generating *descriptions* of situations.

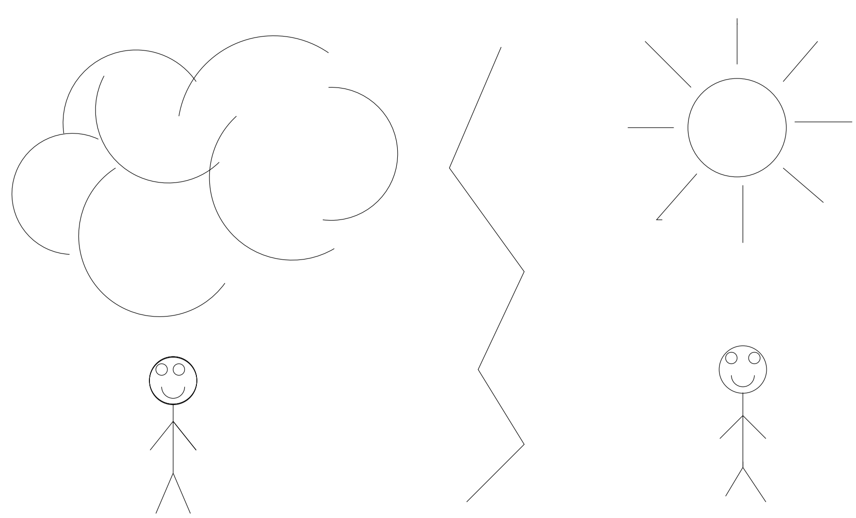
The Tableaux Technique

- Consider a set of sentences G.
  - We can think of G as describing different possible situations....
  - ....those situations which make every sentence of G true.

i.e.

$$\{\textit{(it is cloudy} \vee \textit{it is sunny)}, \textit{Bill is happy}\}$$

Situation 1   Situation 2



- Semantic tableaux provide a systematic method for finding what possible situations are described by a set of sentences G.
  - We use G to produce new descriptions of the situations....
  - ....the new descriptions are obtained by *simplifying* the complex sentences in G.

e.g.

$$\{\textit{(it is cloudy} \vee \textit{it is sunny)}, \textit{Bill is happy}\}$$

Description 1	Description 2
$\{\textit{it is cloudy}, \textit{Bill is happy}\}$	$\{\textit{it is sunny}, \textit{Bill is happy}\}$

Tableaux and Inconsistency

- The tableaux method has various **tableaux derivation rules** that allow us to construct a ‘picture’ of all the different possible descriptions.
- This picture is a tree diagram (the **tableau**). e.g.

(it is cloudy $\vee$ *it is sunny)*
Bill is happy

it is cloudy | *it is sunny*

- This tableau has two branches, where each branch represents a situation.

- Sometimes, branches *fail* to represent a possible situation.

  e.g.

$$(\textit{it is cloudy} \lor \textit{it is sunny})$$
$$\neg \textit{it is sunny}$$

$$\textit{it is cloudy} \qquad\qquad \textit{it is sunny}$$

- This tableau has two branches.
  - One branch fails to describe a situation – it contains inconsistent information.
  - The branch is said to be **closed**
- We write a line under the branch to show that it is closed.

- In general, whenever a branch contains a statement A and a statement $\neg A$, then it contains inconsistent information, and is said to be *closed*.
- Closed branches are not extended further.
- If *all* the branches of a tableau are closed, then we have shown that the set of statements we started from is inconsistent.

  e.g.

$$(\textit{it is cloudy} \vee \textit{it is sunny})$$
$$(\neg \textit{it is cloudy} \wedge \neg \textit{it is sunny})$$

$$\neg \textit{it is cloudy}$$
$$\neg \textit{it is sunny}$$

$$\underline{\textit{it is cloudy}} \qquad \underline{\textit{it is sunny}}$$

Tableaux Rules

$$\begin{array}{c} \neg\neg A \\ | \\ A \end{array}$$

$$\begin{array}{c} (A \land B) \\ | \\ A \\ B \end{array} \qquad\qquad \begin{array}{cc} \multicolumn{2}{c}{\neg(A \land B)} \\ / & \backslash \\ \neg A & \neg B \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{(A \lor B)} \\ / & \backslash \\ A & B \end{array} \qquad\qquad \begin{array}{c} \neg(A \lor B) \\ | \\ \neg A \\ \neg B \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{(A \to B)} \\ / & \backslash \\ \neg A & B \end{array} \qquad\qquad \begin{array}{c} \neg(A \to B) \\ | \\ A \\ \neg B \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{(A \leftrightarrow B)} \\ / & \backslash \\ A & \neg A \\ B & \neg B \end{array} \qquad\qquad \begin{array}{cc} \multicolumn{2}{c}{\neg(A \leftrightarrow B)} \\ / & \backslash \\ A & \neg A \\ \neg B & B \end{array}$$

Example

- Is the set

$$\{\neg(p \wedge \neg q), (q \rightarrow r), (p \wedge \neg r)\}$$

consistent or inconsistent?

1.

$$\neg(p \wedge \neg q)$$
$$(q \rightarrow r)$$
$$(p \wedge \neg r)$$

2.

$$\neg(p \wedge \neg q)$$
$$(q \rightarrow r)$$
$$(p \wedge \neg r)$$

$$\begin{array}{ccc} & \diagup \quad \diagdown & \\ \neg p & & \neg\neg q \end{array}$$

3.

$$\neg(p \land \neg q)$$
$$(q \to r)$$
$$(p \land \neg r)$$
$$\begin{array}{cc} \neg p & \neg\neg q \\ p & p \\ \underline{\neg r} & \neg r \end{array}$$

4.

$$\neg(p \land \neg q)$$
$$(q \to r)$$
$$(p \land \neg r)$$
$$\begin{array}{cc} \neg p & \neg\neg q \\ p & p \\ \underline{\neg r} & \neg r \\ & \begin{array}{cc} \underline{\neg q} & \underline{r} \end{array} \end{array}$$

- Each branch or the tableau is closed.

- Because each branch of the tableau closed, we say that *the tableau is closed*.
- This means that every branch of the tableau contains contradictory information.
  - we cannot find a valuation that will make every sentence on a given branch true.
  - there is no valuation that makes every sentence in the original set true.
- It follows that the set

$$\{\neg(p \wedge \neg q), (q \to r), (p \wedge \neg r)\}$$

  is inconsistent.

Summary

- Semantic tableaux provide a technique for testing consistency/inconsistency of sets of sentences
- Tableaux are more expressive, and easier to use than truth tables
- The method is based on the idea of simplifying descriptions/sentences and looking for contradictions.
- The tableaux derivation rules allow us to grow a tree diagram representing possible situations.
- In contrast to the axiomatic system of propositional logic, the tableaux proof method is simple and straightforward to use.