Introduction to Logic 7

Last time:

- Meaning and Form
- Formal Systems
- PC as a Formal System
- Proof and Truth
- Decidability

This time:

- PC as an Axiomatic System
- Formal Proofs
- The Deduction Relation
- Deduction and Entailment

Propositional Logic as an Axiomatic System

- *The language of Propositional Logic*
- *The following axiom schemas:*

S1: $(A \to (B \to A))$

S2: $((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$

S3: $(((\neg A) \to (\neg B)) \to (B \to A))$

- *The following rule of inference:*

From A *and* $(A \to B)$ *deduce* B

• B is called a *direct consequence* of A and $(A \to B)$

• The rule is known as **Modus Ponens** (**MP** for short)

- Note once again that the number of axioms is infinite.

- There is an infinite number of *instances* of the axiom schemas S1, S2 and S3.

Intuitively, and instance of an axiom is a sentence of propositional logic formed by *instantiating* the meta-language variables in a schema.

Example:

$(A \to (B \to A))$ – (schema S1)

can be instantiated as:

$((p \to q) \to ((\neg q) \to (p \to q))$

where:

A is instantiated as $(p \to q)$

B is instantiated as $(\neg q)$

• Each instantiation of S1, S2 or S3 is an axiom of the formal system of Propositional Logic.

e.g. The following sentences are all axioms:

Inst S1: $(p \to (q \to p))$

Inst S2: $((p \to (q \to r)) \to ((p \to q) \to (p \to r))$

Inst S3: $((\neg p) \to (\neg q)) \to (q \to p))$

Inst S1: $((p \wedge q) \to (r \to (p \wedge q)))$

Inst S3: $((\neg(p \vee q) \to \neg r) \to (r \to (p \vee q)))$

etc., etc. ...

Formal Proofs and Theorems

• We are interested in formalizing the notion of **proof**

• We can now define a notion of *proof within a formal system* as follows:

Defintion: (Proof) A proof in a formal system is a sequence of sentences

$$A_1, A_2, \ldots, A_n$$

where each A_i $(1 \leq i \leq n)$ is either:

1. an *axiom*; or
2. a *direct consequence* of two earlier sentences A_j and A_k $(j, k < i)$

Definition: (Theorem) If a sequence of sentences $A_1, A_2, \ldots, A_n$ is a proof in a formal system, then the sentence A_n is called a *theorem* of that system.

Example:

- Proof that $(p \to p)$ is a theorem of the formal system of propositional logic.

(1) $((p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p)))$

– Inst S2

(2) $(p \to ((p \to p) \to p))$ – Inst S1

(3) $((p \to (p \to p)) \to (p \to p))$ – MP on (1) & (2)

(4) $(p \to (p \to p)$ – Inst S1

(5) $(p \to p)$ – MP on (3) & (4)

- So $(p \to p)$ is a theorem.

Note:

- Proofs give us a way of generating new theorems from a given stock of 'old' theorems (i.e. the axioms).
- In general, if

  $$A_1, A_2, \dots A_{n-1}, A_n$$

  is a proof, then so is

  $$A_1, A_2, \dots A_{n-1}$$

Question: Why and what does this imply about A_{n-1}?

Deduction

- We may be interested in finding out what follows from an arbitrary stock of sentences (i.e. not just from the axioms).
- We formalize a notion of a **deduction** (in a formal system) as follows:

Definition: (Deduction) Let G be an arbitrary set of sentences. A sequence of sentences

$$A_1, A_2, \ldots, A_n$$

is a **deduction from** G if each sentence A_i $(1 \leq i \leq n)$ is either:

1. an *axiom*; or
2. a *sentence in* G; or
3. a *direct consequence* from two earlier members of the sequence

Note:

1. a deduction from a set G is just like a proof, except that the members of the sequence $A_1, A_2, \ldots, A_n$ can also be drawn from G.
   - The elements of G are like temporary axioms.
2. Also, if a sequence of sentences:

$$A_1, A_2, \ldots, A_n$$

   is a deduction from a set G, then the sentence A_n will *not*, in general, be a theorem.

- We say that A_n is **deducible from** G and this is written:

$$G \vdash A_n$$

Question: What can we say about A_n if G is the empty set?

Example

- We shall show that:

$$\{p, (q \to (p \to r))\} \vdash (q \to r)$$

(1) p – Assumption

(2) $(q \to (p \to r))$ – Assumption

(3) $(p \to (q \to p))$ – Inst S1

(4) $(q \to p)$ MP on (1) & (3)

(5) $((q \to (p \to r)) \to ((q \to p) \to (q \to r)))$ – Inst S2

(6) $((q \to p) \to (q \to r))$ – MP on (2) & (5)

(7) $(q \to r)$ – MP on (4) & (6)

So: $(q \to r)$ is deducible from $\{p, (q \to (p \to r))\}$

Deduction and Entailment

- You may have observed some similarities between notion of the *deduction relation* ($\vdash$) and the notion of *entailment* ($\models$).
  - both relations are defined to hold between a set of sentences G and a sentence A;
  - both attempt to capture a notion of 'consequence'
- We may suspect that the two relations will actually turn out to be identical.

  i.e.

$$G \models A \text{ if and only if } G \vdash A$$

Summary

- Propositional logic can be formalized as an axiomatic system.
- We can define a notion of formal proof within such a system.
- Proofs establish that certain sentences are theorems of the system.
- More generally, we have the notion of a deduction from a set of statements or assumptions.
- Deduction captures the idea of a statement being consequent on some set of assumptions.
- Deduction and entailment have striking similarities, even though they are defined in very different ways.