Introduction to Logic 6

Last time:

- Valuations
- Consistency/Inconsistency
- The Entailment relation
- Some Facts about entailment

This time:

- Meaning and form
- Formal systems
- PC as a formal system
- Proof and Theorems
- Soundness and Completeness
- Decidability

Meaning and Form

- We have introduced a simple language for expressing propositions and sets of propositions.
- We have studied this language from the point of view of its meaning

  i.e.

  - Sentences are taken to denote truth-values
  - The connectives are truth-functions
  - We have looked at how the truth-value of a compound sentence is calculated from the meaning of its parts

- By investigating the meaning of our language we have found ways to:
  - classify sentences as tautologous, contingent or inconsistent
  - decide whether simple arguments are valid or not
  - decide whether two sentences are logically equivalent
  - determine the consistency/inconsistency of sets of sentences
  - formalize a notion of logical consequence between sentences and sets of sentences (entailment)
  - etc. etc. ...

- Studying a language from the point of view of its meaning seems natural.
- It is not the only way to proceed however.
  - We can examine the **form** of the sentences in our language rather than the **content**
  - We can provide rules for manipulating sentences in a purely formal (i.e. symbolic, syntactical) way.
  - We can devise techniques for determining consistency, inconsistency, equivalence, validity, etc., etc., that *do not depend on meaning or truth*.

- This all raises a couple of questions:

Question 1: Why study logic in this purely formal way?

Answer: Formal techniques are often more convenient, both for people and computers

  - recall the limitations of the method of truth tables that we uncovered

Question 2: If it's all a matter of symbol manipulation, without regard to meaning at all, how do we know that it makes any sense?

Answer: Good question!

  - Ultimately we have to *demonstrate* that the formal rules are sensible (and this *does* require reference to meaning and truth).

Logic as a Formal System

In general, a formal system is made up of

1. A language of some kind for making statements (expressing propositions)
2. A designated set of sentences called **axioms**
3. A set of rules for generating new sentences from old — the **rules of inference**.

In studying logic as a formal system we are interested in the notion of formal deduction or **proof**

- The **axioms** are sentences that we hold to be true in virtue of their form.
- The rules of inference allow us to prove **theorems**
- idea is that the formal notion of a **theorem** should coincide exactly with our previous semantic notion of a **tautology**.

Axiomatic Propositional Logic

- We can now view propositional logic as a formal system.
- One way is the following:

1. *The language of propositional logic*
2. *The following* **axiom schemas**:

   A1 $(A \to (B \to A))$

   A2 $((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$

   A3 $(((\neg A) \to (\neg B)) \to (B \to A))$
3. **Modus Ponens**:

   from A *and* $(A \to B)$ *infer* B

Note:

- We have specified the axioms of the system using three **schemas**
  - Each schema must be *instantiated* to provide an axiom
  - There is an infinite number of instantiations of each schema!
- The axioms (axiom schemas) do not appear very natural.
  - it is hard to see where they come from
  - it is also hard to see how they might be used
- On the other hand, there is a single, and reasonably intuitive rule of inference

Proofs and Theorems

- Our intention is to provide a formal definition of our informal notion of **proof**.
- What is a proof?
  - Informally, we might say that a proof is a demonstration that some statement follows from some set of statements
  - A connected sequence of statements that go together to establish a conclusion
- Informal forms of proof often leave much of the structure or working *implicit*.

  Note that this includes mathematical proofs. While these are precise, many obvious steps (obvious to mathematicians!) are typically left out.
- To provide a *formal* notion of proof, we must make everything *explicit*.

Soundness and Completeness

- Of course, in the end we must show that our formal notion of proof makes sense.
- The formal notion of proof must be related back to our notion of logical consequence (the semantic relation $\models$)

Soundness: If there is a proof of a statement A (i.e. A is a theorem), then $\models A$ (i.e. A is a tautology).

Completeness: If $\models A$ (A is a tautology), then it must be possible to prove that A (i.e. A is a theorem).

- Only if our formal system is both **sound** and **complete** can we regard it as adequate.

Note:

- Soundness and completeness is something that we must *prove* about a formal system of logic. We cannot just take it for granted.
  - Proving that a formal system is sound is generally quite straightforward.
  - Proving completeness can be very tricky.
- Having said this, we will not actually attempt to prove soundness and completeness for axiomatic propositional logic.
- In fact, the system is both sound and complete (see e.g. Kelly chapter 4, section 5 for proofs).

Decidability

- A further property of a formal system of logic of interest to us is **decidability**.
- A formal system of logic is **decidable** if there exists an **effective procedure** for determining whether or not an arbitrary statement A is a theorem; i.e.:
  - If A is a theorem the procedure should halt and answer **yes**
  - If A is not a theorem the procedure should halt and answer **no**

Proposition: *Axiomatic propositional logic is decidable.*

Proof: *(Sketch) A is a theorem if and only if it is a tautology (soundness and completeness).*

We can check whether A is a tautology in a purely mechanical way (e.g. by constructing its truth table). $\square$

Summary

- We can study logic according to the meaning or content of statement.
- An alternative is to examine the form of statements.
- A formal system of logic has axioms and inference rules.
- The aim is to formalize a notion of *proof*.
- To be adequate, a formal system of logic must be both *sound* and *complete* – axiomatic propositional logic is adequate in this sense.
- A useful property of a formal system of logic is *decidability* — axiomatic propositional logic is decidable.