Introduction to Logic 5

Last time:

- Functional completeness
- The Sheffer Stroke
- Logical Equivalences and simplification
- Practical limitations of truth tables.

This time:

- Valuations
- Consistency/Inconsistency
- The Entailment relation
- Some Facts about entailment

Valuations

- A valuation is really just a function that assigns truth values to propositional variables.
  - If we use $\{t, f\}$ to model truth values; and
  - $Prop$ is the set of propositional variables, then
  - $V : Prop \to \{t, f\}$ is a valuation.
- It is useful to extend the notion of a valuation to arbitrary sentences of the PC.
- Given a valuation V, we extend V to a new function V^* that assigns truth-values to all sentences of the PC (not just the propositional variables).
- $V^* : PC \to \{t, f\}$

  Note: The function V^* is also called a valuation (and confusingly, we may sometimes just write it as V).

Consistency and Inconsistency

- The language of PC can be used to represent sets of propositions.
- We may be interested in determining whether it is possible for every proposition in a given set to be true at the same time.

  Consider for example the following set G:

  $$G \quad = \{p, (\neg p \vee \neg q), (q \rightarrow p)\}$$

  Is there a valuation which makes every sentence in G true?

Definition: *A set $G = \{A_1, A_2, \ldots, A_k\}$ of sentences of the PC is said to be* **consistent** *if there exists some valuation V such that $V^*(A_i) = t$ for each sentences $A_i \in G$ ($1 \leq i \leq k$). Otherwise G is said to be* **inconsistent**.

Testing Consistency

- We can use the method of truth tables to test whether a set of sentences is consistent.

  Consider: $\{p, (\neg p \vee \neg q), (q \rightarrow p)\}$

p	q	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$(q \rightarrow p)$	
t	t	f	f	f	t	
t	f	f	t	**t**	**t**	$\Leftarrow$
f	t	t	f	t	f	
f	f	t	t	t	t	

- Note that the second row of the truth table has **t** in each column corresponding to one of the sentences in the set
- The set of sentences is *consistent* for any valuation V such that $V(p) = t$ and $V(q) = f$

- Consider the set of sentences:

$$G = \{p, (p \to q), \neg q\}$$

p	q	$\neg q$	$(p \to q)$
t	t	f	t
t	f	t	f
f	t	f	t
f	f	t	t

- There is no row of the truth-table for which each sentence in G has the value **t**.
- The set of sentences G is *inconsistent*

Entailment

- *Entailment* is a relation that holds between a set of sentences G and a sentence A.
- Entailment is a *semantic* relation:

  i.e. it is defined with reference to the meaning of the sentences involved.
- Entailment captures a notion of *logical consequence*.

Definition: *A set of sentences G* **semantically entails** *a sentence A if and only if there is no valuation that makes all of the sentences in G* **true** *, but makes A* **false**

– i.e. assuming the truth of all the sentences in G has the consequence that A is **true** *as well.*

- We will introduce some special notation to stand for the entailment relation, and write:

$$G \models A$$

to mean "G semantically entails A".

- We can think of $G \models A$ as formalizing the notion that given the **assumptions** in G, then the **conclusion** A is **true** , or A follows from the assumptions.

Note:

- The symbol $\models$ does **not** belong to the language of the PC.
- It belongs to our **meta-language** for talking about a relation between sentences and sets of sentences in our **object language** (the PC).

Example

- Consider the set of sentences:

$$G = \{p, (\neg p \vee \neg q), (q \to p)\}$$

  Then we have:

$$G \models \neg q$$

- To see this, note that (as we showed a little earlier by the method of truth tables) any valuation V which makes each sentence in G true is such that:

$$V(p) = \mathrm{t}$$
$$V(q) = \mathrm{f}$$

- But if $V(q) = \mathrm{f}$, then $V^*(\neg q) = \mathrm{t}$.

- So, assuming the truth of all the sentences in G has the consequence that $\neg q$ is **true** as well.

Some Facts about Entailment

Fact 1:

For any set of sentences G, if $A \in G$, then it must be the case that:

$$G \models A$$

e.g. if $G = \{(p \wedge q), \neg p\}$, then

$$\begin{aligned} G &\models (p \wedge q) \\ G &\models \neg p \end{aligned}$$

But note that $G \models A$ does *not* imply that $A \in G$.

Consider the previous example:

$$G = \{p, (\neg p \vee \neg q), (q \rightarrow p)\}$$

and

$$G \models \neg q$$

Fact 2: An inconsistency entails everything!

Consider a set of sentences G such that G is **inconsistent**. It follows that:

$$G \models A$$

for any sentence A

Proof: *Let G be an inconsistent set of sentences and A an arbitrary sentence. Suppose that A is* **not** *entailed by G. From the definition of entailment, there must exist a valuation that makes every sentence in G* **true** *, but which makes A* **false** *. But G is inconsistent, so no such evaluation can exist. It follows that $G \models A$.* $\square$

Fact 3: Anything entails a tautology

Consider a **tautology** A. From the definition of entailment it follows that

$$G \models A$$

for any set of sentences G.

Proof: *Let* A *be a tautology and* G *an arbitrary set of sentences. Suppose that* G *does not entail* A. *From the definition of entailment, it follows that there must be a valuation which makes every sentence in* G **true** *, but that makes* A **false** *. But* A *is a tautology, so no such valuation can exist. It follows that* $G \models A$. $\square$

Fact 4: Only a tautology follows from the empty set

Consider the case when G is the **empty** set of sentences $\{\}$. From the definition of entailment it must be that:

$$\text{if } \{\} \models A \; \textit{then } A \textit{ is a tautology}$$

Proof: If G is the empty set, then there can be no valuation that makes a sentence in G **false** . In other words, every valuation makes all of the sentences in G **true** . So, if $G \models A$, then from the definition of entailment, every valuation must make A **true** as well. It follows that A is a tautology. $\Box$

We write $\models A$ to mean $\{\} \models A$.

Summary

- A valuation is a function from propositional variables to truth-values.
- A set of sentences is consistent if there exists a valuation which makes each sentence in the set **true**
- We can use the method of truth tables to establish the consistency or inconsistency of sets of sentences.
- Entailment is a semantic relation that holds between sentences and sets of sentences.
- The entailment relation captures a notion of logical consequence