Introduction to Logic 3

Last time:

- Introduction to the PC
- The Language of the PC
- Giving the language meaning
- Arguments

This time:

- Truth-values and the connectives
- Truth tables
- Tautologies and Inconsistencies
- Arguments revisited

Truth Tables and the Connectives

- We said last time that sentences of the PC express propositions and may be either **true** or **false**. What exactly, are we assuming?
  - there are only two truth values: **true** and **false**
  - sentences cannot be *both* **true** and **false** simultaneously.
  - sentences cannot be 'undefined' (i.e. there are not truth-value 'gaps')
- These are fundamental assumptions of 'classical' logic

Questions: *Could you have a non-classical logic? What might that be like?*

- The simplest sentence-types of our language are the propositional variables:

$$p, q, r, s, \ldots$$

- These are combined with the connectives to build 'complex' or 'compound' sentences:

$$((p \to q) \land p) \to q$$

- Clearly, the truth-value of a compound sentence depends on:
  1. the truth-values of the propositional variables that it contains;
  2. the meaning (i.e. truth-functions) of the connectives $\neg$, $\land$, $\lor$, $\to$, $\leftrightarrow$.

- For classical logic, this is *all* we need to know.
  - We don't need to know *how* the variables got their values;
  - We don't need to know anything about the meaning of sentences beyond their truth-values.
- **Analogy:** (arithmetic)
  - Suppose that variable x has value 3, y has value 2 and z has value 5.
  - Given that you know the meaning of $+$ and $-$, you can calculate the value of the expression:

$$(x + y) - z$$

  - Thus: $(3 + 2) - 5 = 5 - 5 = 0$

Truth Tables

- The connectives of our language are truth-functional
- The truth-functions that they correspond to can be expressed conveniently in the form of matrices:

$\neg$	
t	f
f	t

$\wedge$	t	f
t	t	f
f	f	f

$\vee$	t	f
t	t	t
f	t	f

$\rightarrow$	t	f
t	t	f
f	t	t

$\leftrightarrow$	t	f
t	t	f
f	f	t

Example

- Suppose that we want to determine the truth-value of the sentence

$$p \rightarrow q$$

given that $p = \mathrm{t}$ and $q = \mathrm{f}$.

- We know the truth-function for $\rightarrow$:

$\rightarrow$	t	$q = \mathrm{f}$
$p = \mathrm{t}$	t	$p \rightarrow q = \mathrm{f}$
f	t	t

- So, in this case $p \rightarrow q$ is **false** .
- Here's the case when $p = \mathrm{t}$ and $q = \mathrm{t}$:

$\rightarrow$	$q = \mathrm{t}$	f
$p = \mathrm{t}$	$p \rightarrow q = \mathrm{t}$	f
f	t	t

- Note that the truth-value of a compound sentence can vary according to (as a function of!) the truth-values of its parts.
- Given a sentence of the PC, we can display all of the different possible cases in the form of a matrix or **truth table**:

  e.g. $p \rightarrow q$

p	q	$p \rightarrow q$
t	t	t
t	f	f
f	t	t
f	f	t

- Thus we see that $p \rightarrow q$ is alway **true** *except* in case that p is **true** and q is **false** .

- Here's a more complicated example:

$$((p \rightarrow q) \wedge p) \rightarrow q$$

p	q	$(p \rightarrow q)$	$(p \rightarrow q) \wedge p$	$((p \rightarrow q) \wedge p) \rightarrow q$
t	t	t	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

- We calculate the truth-value of the whole expression 'inside-out'.
- Each line of the table corresponds to one way of assigning truth-values to the propositional variables in the sentence
- a *function* from propositional variables to truth-values is called a *valuation*.

Tautologies, Inconsistencies and Equivalences

- A **tautology** is a sentence that is **true** in all possible valuations.
- Consider the sentence $p \vee \neg p$:

p	$\neg p$	$p \vee \neg p$
t	f	t
f	t	t

- We've already seen an example of a sentence that is *not* a tautology:

p	q	$p \rightarrow q$
t	t	t
t	f	f
f	t	t
f	f	t

- An **inconsistency** is a sentence that is **false** in all possible valuations:
- Consider the sentence $p \wedge \neg p$:

p	$\neg p$	$p \wedge \neg p$
t	f	f
f	t	f

  Clearly, $p \wedge \neg p$ is inconsistent.

- A sentence is **contingent** if it is *neither* tautologous *nor* inconsistent.

p	q	$p \rightarrow q$
t	t	t
t	f	f
f	t	t
f	f	t

  So, $p \rightarrow q$ is contingent.

- Two sentences A and B are said to be **equivalent** if, for any given valuation, they have exactly the same truth-value.
- Consider $\neg p \vee q$ and $p \rightarrow q$:

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
t	t	f	t	t
t	f	f	f	f
f	t	t	t	t
f	f	t	t	t

- Note that the last two columns are identical, row-by-row. So, $\neg p \vee q$ is equivalent to $p \rightarrow q$
- **Question** *Suppose that A and B are equivalent. What can you say about $(A \leftrightarrow B)$?*

Arguments Revisited

- We now have a way of distinguishing between 'good' (i.e. valid) and 'bad' (i.e. invalid) arguments.
- Intuitively, an argument is valid if whenever all of its premisses $P_1, P_2, \ldots, P_k$ are **true** , then its conclusion C is also **true** .
- In other words, the sentence:

$$(P_1 \wedge P_2 \wedge \ldots P_k) \rightarrow C$$

  is a tautology.

- So, to check whether an argument is valid we can
  - formalize the argument as a sentence of the PC
  - check whether the resulting sentence is a tautology

Example

Lets return to the argument that we formalized in the last lecture.

$$
\begin{aligned}
P_1 &= \text{If Logic is fun, then Bill is happy} \\
P_2 &= \text{Logic is fun} \\
C &= \text{Therefore, Bill is happy}
\end{aligned}
$$

So:

$$
\begin{array}{ccccc}
P_1 & \wedge & P_2 & \rightarrow & C \\
((p \rightarrow q) & \wedge & p) & \rightarrow & q
\end{array}
$$

p	q	$(p \rightarrow q)$	$(p \rightarrow q) \wedge p$	$((p \rightarrow q) \wedge p) \rightarrow q$
t	t	t	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

Note that the final column contains only t. This means that the sentence is a tautology, and hence the argument is valid.

Summary

- Sentences of the PC can be either **true** or **false** (but not both and they cannot be undefined or have some other value).
- The connectives correspond to *truth-functions*
- Truth tables allow us to set out, systematically, the way the truth-value of a compound sentence varies according to the truth-values of its simpler parts.
- We can distinguish between sentences that are **true** in all valuations (tautologies), and **false** in all valuations (inconsistencies).
- Two sentences are *logically equivalent* if they have the same truth values in all possible valuations
- We can test the validity of arguments by formalizing them as sentences of the PC and then testing to see if they are tautologous.