Introduction to Logic 16

Last time:

- Interpretations and Assignments
- The Meaning of Terms
- The Meaning of Formulas
  - Atomic Formulas
  - Compound Formulas

This time:

- Interpretations
- Models
- First Order Theories

Interpretations and Models

- We give meaning to the sentences of predicate logic by providing an interpretation

$$\mathcal{I} = (D, I)$$

- For a given interpretation we can judge whether a sentence is **true** or **false** .

  e.g.

$$\forall x.\forall y.\forall z.((R(x,y) \land R(y,z)) \rightarrow R(x,z))$$

  Considerand interpretation with:

  $D = \mathit{Integers}$

  $I(R) = \mathit{less\text{-}than}$ (i.e. $<$)

- An interpretation $\mathcal{I}$ that makes a formula A **true** is said to *satsify* A.
- We may write

$$\mathcal{I} \models A$$

  to mean "*interpretation* $\mathcal{I}$ *satisfies* A"
- Also, if $\mathcal{I} \models A$ we may refer to $\mathcal{I}$ as " a *model for* A".
- More generally, this applies to *sets* of formulas.

Definition *Let G be a set of formulas and $\mathcal{I}$ and interpretation. We write*

$$\mathcal{I} \models G$$

and say $\mathcal{I}$ is a model for *G (I* satisfies *G) if and only if $\mathcal{I}$ is a model for every formula in G.*

First Order Theories

- A **first order theory** is a (possibly infinite) set of formulas of first order logic.
- A theory may be defined by a set of **axioms**: e.g.

  $$\forall x.\forall y.\forall z.((R(x, y) \land R(y, z)) \to R(x, z))$$
  $$\forall x.\neg R(x, x)$$

- A theory provides a characterisation of the salient properties of some domain of interest.
  1. R is the *less-than* relation in the *Integer* domain.
  2. R is the *ancestor* relation in the *People* domain.
- There may be many models satisfying a given theory.

- In general, we have an intended interpretation in mind as our model.
- aim to **axiomatize** the domain of interest;
- axiomatic theories provide a basis for:
  - *formalizing aspects of mathematics and computer science*

    characterizing data-structures or the behviour of differents kinds of mathematical spaces/structures
  - *studying properties of programs*

    proofs of correctness, program transformations
  - *automated reasoning*

    building 'intelligent' systems capable of reasoning about particular domains
  - *computer programming*

    logic programming (e.g. Prolog)

- A set of axioms G determines a set of **theorems**.
- If G is a set of axioms and A is a statement, then as usual we may write:

  $$G \models A$$

  to mean that A is entailed by G.
- And we may write:

  $$G \vdash A$$

  to mean that A can be deduced from G.
- We need techniques for determining what follows from a set of axioms (*theorem-hood*).
  - examine the models for G
  - develop viable proof methods.

Summary

- Interpretations determine the truth-value of statements of the predicate calculus
- An interpretation that satisfies a set of statements is called a model.
- A first order theory is a set of statements.
- We may define a theory axiomatically
- A first order theory characterizes a set of models with common structure.
- We need proof-theoretic techniques for determining theorem-hood.