Introduction to Logic 14

Last time:

- The Language of the Predicate Calculus
- Expressing Propositions
- Semantic Preliminaries

This time:

- Interpretations
  - Examples
- Formalizing Interpretations
  - Examples

Interpretations

- The Predicate Calculus allows us to:

1. make statements about particular individuals:

   e.g.

$$T(b, l) \rightarrow H(b)$$

2. make statements about relationships between individuals:

   e.g.

$$T(b, l)$$

3. make general statements about individuals:

   e.g.

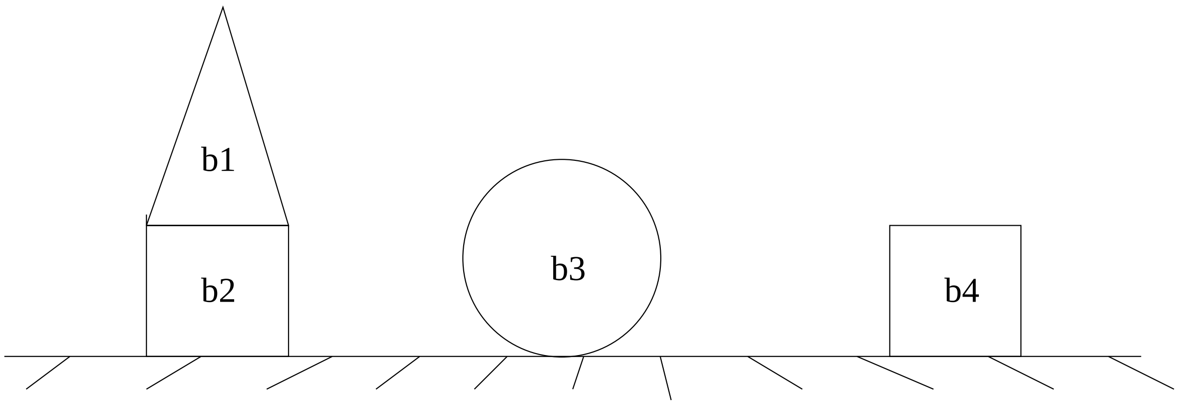
$$\forall x.(T(x, l) \rightarrow \neg H(x))$$

- Just as for Propositional Logic, we can study this new language in two different ways:
  1. in terms of its meaning.
  2. in terms of its form;
- To do the former, we must *interpret* the language.

  i.e. we must:
  1. fix a **domain of interpretation** D
     - the set of things we are interested in talking about
  2. provide an **interpretation function** I
     - relates expressions of the language to our domain D

The 'Blocks World'

Domain:



Interpretation Function:

Notation	*interpreted as*	Denotation
a	$\Longrightarrow$	b_1
b	$\Longrightarrow$	b_2
c	$\Longrightarrow$	b_3
d	$\Longrightarrow$	b_4
C	$\Longrightarrow$	*is a cube*
P	$\Longrightarrow$	*is a pyramid*
O	$\Longrightarrow$	*is on top of*
R	$\Longrightarrow$	*is red*

Question: *Given the Blocks World domain and associated interpretation function, what do the following statements mean?*

$$C(d)$$

$\Longrightarrow$

$$O(a, b)$$

$\Longrightarrow$

$$\exists x.(C(x) \wedge R(x))$$

$\Longrightarrow$

$$\exists x.(R(x) \wedge P(x) \wedge \exists y.(C(y)) \wedge O(x, y)))$$

$\Longrightarrow$

The Integers

Domain: $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

Interpretation Function:

Notation	interpreted as	Denotation
z	$\Longrightarrow$	*integer zero*
p	$\Longrightarrow$	*predecessor function*
s	$\Longrightarrow$	*successor function*
L	$\Longrightarrow$	*is less than*

Question: *What do the following statements mean?*

$$L(z, s(z))$$

$\Longrightarrow$

$$\forall x.\exists y.L(x, y)$$

$\Longrightarrow$

$$\exists x.\forall y.L(x, y)$$

$\Longrightarrow$

Formalizing Intepretations

Definition *An interpretion for a first order language is a pair:*

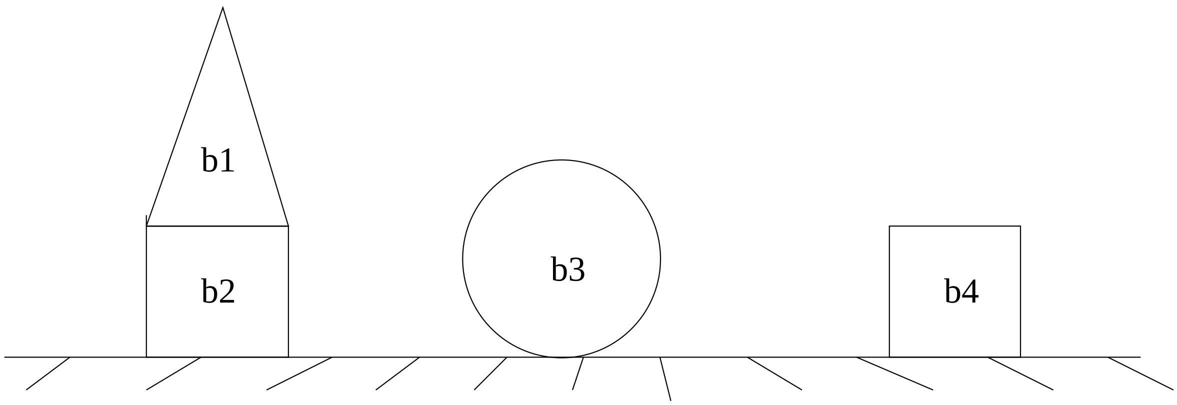
$$\mathcal{I} = (D, I)$$

where:

- *D is a set of individuals (the ”domain interpretation”); and*
- *I is an interpretation function*

- The interpretation function I has to
  - assign a fixed element of D to each individual constant a;
  - assign an n-ary function on D to each function symbol f of arity n;
  - assigns an n-ary relation on D to each predicate symbol P of arity n.

The Blocks World (again)



$$D = \{b_1, b_2, b_3, b_4\}$$

$$\begin{aligned} I(a) &= b_1 \\ I(b) &= b_2 \\ I(c) &= b_3 \\ I(d) &= b_4 \\ I(C) &= \{b_2, b_4\} \\ I(P) &= \{b_1\} \\ I(O) &= \{(b_1, b_2\} \\ I(R) &= \{b_1, b_3\} \end{aligned}$$

The Integers (more formally)

$$D = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

$$\begin{aligned}
I(z) &= 0 \\
I(p) &= \{\ldots \quad (-3,-4) \\
&\qquad\quad (-2,-3) \\
&\qquad\quad (-1,-2) \\
&\qquad\quad (0,-1) \\
&\qquad\quad (1,0) \\
&\qquad\quad (2,1)\ldots\}
\end{aligned}$$

$$\begin{aligned}
I(L) &= \{\ldots \quad (-5,1) \\
&\qquad\quad (-4,1) \\
&\qquad\quad (-3,1) \\
&\qquad\quad (-2,1) \\
&\qquad\quad (-1,1) \\
&\qquad\quad (0,1)\ldots\}
\end{aligned}$$

Notes:

1. So far, we have not mentioned the interpretation of individual variables.
   - We will assume the existence of a separate **variable assignment function** g
   - The assignment function g will map variables onto elements of the domain D
2. Note that a particular interpretation just tells us about the meaning of basic expressions.
   - it does not tell us (directly) about the meaning of compound expressions.
   - .... the meaning of connectives and quantifiers is 'fixed'
   - .... there are general rules for calculating the meaning of compound expressions

Summary

- The language of predicate logic permits us to express propositions about particular individuals, and to make generalizations.
- Like propositional logic, we can study the language from the perspective of its meaning, or its form.
- To study the language in terms of its meaning we must provide an interpretation.
- An interpretation for a first-order language consists of a domain and an intepretation function.
- A particular interpretation fixes the meaning of the basic expressions of a first-order language.
- There are general rules for evaluating the meaning of compound expressions (terms and formulas).