Introduction to Logic 13

Last time:

- Propositional Logic
- Limitations of Propositional Logic
- The Structure of Propositions
- Individuals, Properties and Quantifiers

This time:

- The Language of the Predicate Calculus
  - Basic Expressions
  - Terms and Formulas
- Expressing Propositions
- Semantic Preliminaries

The Language of FOPC

• The basic expressions of Predicate Logic fall into four separate categories:

1. **Individual names:** a, b, c, ...
   These represent specific objects, persons or events
2. **Individual variables:** x, y, z, ...
   These are variables that range over individuals.
3. **Predicate Symbols:** P, Q, R, ...
   Predicate symbols are used to represent properties or relations over individuals
4. **Function Symbols:** f, g, h, ...
   Function symbols denote functions that map individuals to individuals.

A Note about Predicates and Functions

- **Note:** The predicate and function symbols represent relations over individuals.

  e.g.

Bill **talks**	1 individual
Bill **likes** *Logic*	2 individuals
Bill **teaches** *Moira Logic*	3 individuals

- More generally, we can have relations or functions over an arbitrary number n of individuals

  Terminology: An n-place predicate or function symbol is said to have **arity** n.

- We assume that each predicate or function symbol is associated with a known, fixed arity

- In addition to the four classes of basic expression, the predicate calculus includes:
  - A truth-functionally complete set of connectives: e.g. $\neg, \wedge, \vee, \rightarrow$, and $\leftrightarrow$.
  - Two quantifier symbols:
    The Universal Quantifier: $\forall$
    The Existential Quantifier: $\exists$
  - Brackets '(' and ')', and punctuation symbols ',' and '.'.
- These symbols, taken together with the basic expressions, form the **alphabet** of the language of First Order Predicate Logic.
- The language is defined in two stages: **terms**, and **formulas**

Terms

- Terms are used to pick out individuals:

Definition *A term t is either:*

1. *an individual name; or*
2. *an individual variable; or*
3. *a* functional term $f(t_1, \ldots, t_n)$ *where f is a function symbol of arity n, and $t_1, \ldots, t_n$ are terms*

Question: *Which of the following are terms?*

$$a$$

$$y$$

$$P$$

$$f(a, x)$$

$$Q(x, y)$$

$$(f(x, x))$$

$$f(g(x, a), h(b))$$

Formulas

Definition *A* **well-formed formula** *of Predicate Logic is either:*

1. *an* atomic formula $P(t_1, \ldots, t_n)$ *where* P *is a predicate symbol of arity* n *and* $t_1, \ldots, t_n$ *are terms; or*

2. *a* compound formula *of one of the following forms:*

   (a) $(\neg A)$

   (b) $(A \wedge B)$

   (c) $(A \vee B)$

   (d) $(A \rightarrow B)$

   (e) $(A \leftrightarrow B)$

   (f) $\forall v.A$

   (g) $\exists v.A$

   where A *and* B *are wffs, and* v *is an individual variable.*

Question: *Which of the below are well-formed formulas of Predicate Logic?*

$$P(a)$$

$$(P(a) \to Q(a))$$

$$P(Q(a))$$

$$(\neg P(f(a, x)))$$

$$\forall x.(P(x) \to (\neg Q(x)))$$

$$(P(\forall x.) \wedge Q(a))$$

$$(P(a) \vee \exists x.Q(x))$$

$$\exists x.(P(x) \wedge \forall y.(Q(y) \to R(x, y)))$$

Notes:

- We can drop brackets (as for Propositional Logic) by adopting conventions for operator precedence, etc.
- We may relax conventions for naming predicate symbols, individual names etc.

Representing Propositions

- The FOPC permits finer-grained representation of propositions.

e.g.

Logic is fun

$$F(l)$$

If Logic is fun, then Bill is happy

$$F(l) \rightarrow H(b)$$

Either Logic is fun or Bill is not happy

$$F(l) \vee \neg H(b)$$

All lecturers are happy

$$\forall x.(L(x) \rightarrow H(x))$$

Some lectures are happy

$$\exists x.(L(x) \wedge H(x))$$

Question: *How might you represent the following?*

Some lecturer is not happy

No lecturer is happy

All lecturers teach Logic

All lecturers teach some course

Every course tutor is happy

Semantic Preliminaries

- The semantics of Propositional Logic was particularly simple:
  - **Valuations:** $V : Prop \rightarrow \{t, f\}$
  - Truth tables for connectives:
- For Predicate Logic, the picture is more complicated:
  - four kinds of basic expression;
  - quantification;
  - terms and formulas.
- Meaning is no longer just a matter of **true** and **false** .

- We need a richer semantic domain:
  - individuals – for names
  - functions over individuals – for function symbols
  - relations over individuals - for predicates
- Our semantics should:
  1. map basic expressions onto elements of the semantic domain;
  2. associate individuals with terms; and
  3. give us a way of determining the truth-values of
     - atomic formulas
     - compound formulas (including quantified formulas)

Summary

- The language of First Order Predicate Logic has an alphabet consisting of four basic kind of expression.
- The language is defined in two stages: first order terms, and first order formulas.
- Predicate Logic provides a richer language for representing propositions.
- We can represent propositions concerned with particular individuals, or capture generalisations about groups of classes of individual.
- Our definition of meaning must be correspondingly rich.