Introduction to Logic 12

Last time:

- Natural Deduction Proof Rules
- Introduction Rules
- Elimination Rules
- Proof by Contradiction

This time:

- Propositional Logic
- Limitations of Propositional Logic
- The Structure of Propositions
- A New Logical Notation

Propositional Logic
(The story so far)

- We have introduced the language of propositional logic as a means of representing propositions and arguments.

  e.g.

  > *If* $x > 3$*, then* $y < 4$*. But* $y \not< 4$*, so* $x \not> 3$*.*

- This can be represented as:

$$((p \to q) \land \neg q) \to \neg p$$

  where:

  > p stands for '$x > 3$'; and
  >
  > q stands for '$y < 4$'

- We have provided a precise notion of meaning for statements of propositional logic:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$(p \rightarrow q) \wedge \neg q$	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$
t	t	f	f	t	f	t
t	f	f	t	f	f	t
f	t	t	f	t	f	t
f	f	t	t	t	t	t

- This allows us to distinguish between statements that are **tautologies**, **contingencies** and **inconsistencies**.

- We can also use truth-tables to determine whether arguments are valid/invalid.

  e.g.

$$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$$

  This is a tautology, so the argument is *valid*.

- The relation of **semantic entailment** ($\models$) captures a notion of logical consequence between propositions.

  e.g.

$$\{(p \to q), \neg q\} \models \neg p$$

> *The statement* $\neg p$ *is a consequence of the set of statements* $\{(p \to q), \neg q\}$

  i.e. Any valuation that makes both $p \to q$ and $\neg q$ **true** , also makes $\neg p$ **true** .

p	q	$\neg p$	$\neg q$	$p \to q$
t	t	f	f	t
t	f	f	t	f
f	t	t	f	t
f	f	**t**	**t**	**t**

- We have looked at purely *formal* techniques for ‘calculating’ with (sets of) statements.
  - The classical axiomatic presentation – logic as a formal, deductive system;
  - The method of Semantic Tableaux;
  - The system of Natural Deduction.

Example (Tableaux Method)

$$\{(p \to q), \neg q\} \models \neg p$$

$$(p \to q)$$

$$\neg q$$

$$\neg\neg p$$

$$\begin{array}{cc} \underline{\neg p} & \underline{q} \end{array}$$

- Tableaux is closed, so entailment holds.

Example (Natural Deduction)

$$\{(p \to q), \neg q\} \vdash \neg p$$

$$\dfrac{\dfrac{\dfrac{p \to q \quad \cancel{p}}{q}\ {\to}E \qquad \neg q}{\bot}\ {\to}E}{\neg p}\ RAA$$

- The relation $\vdash$ captures a notion of *deduction* or *proof*.
- It is the syntactic (formal) counterpart of the semantic relation $\models$.
- Consequently, we should expect that:

$$G \models A \textit{ if and only if } G \vdash A$$

Limitations of the Propositional Calculus

- Consider the following argument:

  All lecturers are happy.
  Bill is a lecturer
  So, Bill is happy.

Question: *Is the reasoning here sound (i.e. does the conclusion follow from the premises)?*

Question: *How might the argument be represented in propositional logic?*

- testing validity using a semantic tableau

$$(p \wedge q) \rightarrow r$$

$$\begin{array}{c}
\neg((p \wedge q) \rightarrow r) \\
| \\
(p \wedge q) \\
\neg r \\
| \\
p \\
q
\end{array}$$

- The tableaux for $\neg((p \wedge q) \rightarrow r)$ does not close.
- That means that $(p \wedge q) \rightarrow r$ is *not* a tautology.
- That in turn means that the argument is *not* valid!

The Structure of Propositions

- Consider the argument again:

  All lecturers are happy.
  Bill is a lecturer
  So, Bill is happy.

- **Insight:** We need some way of representing the *structure* of the elementary propositions.

- Propositions involve:
  - **named individuals** that the propositions are 'about':

    e.g. *Bill, Brighton, Logic*,...
  - **properties** of these individuals:

    e.g. *is_happy*, *is_a_city*, *is_a_lecturer*, ...
  - **relations** between individuals:

    e.g. *teaches*, *lives_in*,....

Question: *Is there anything else involved?*

- Consider the premise

Bill is a lecturer

- This statement:

  1. expresses a proposition 'about' the individual *Bill*'; and
  2. asserts that the individual has the property *is_a_lecturer*

- Rather than use a simple propositional variable (p say), we might represent this by:

  b has the property L

  In fact, we are going to write:

  $$L(b)$$

  where:

  – b stands for the individual called "*Bill*"; and

  – L stands for the property expressed by "*is_a_lecturer*".

- Likewise, we might represent the conclusion of the argument

Bill is happy

as follows:

$$H(b)$$

- But what about the first premise?

All lecturers are happy

- Note that:
  - this is a *generalization*;
  - it is not about a particular individual, but a whole group.

How can we represent general statements of this kind?

- Paraphrasing a little:

  For all individuals, if he/she is a lecturer, then he/she is happy

- Or perhaps:

  For all x, if x is a lecturer, then x is happy.

- This might be written more succinctly as:

  $$\textit{For all } x,\ (L(x) \rightarrow H(x))$$

  In fact, we are going to write:

  $$\forall x.(L(x) \rightarrow H(x))$$

- Here, the symbol $\forall$ is know as the **universal quantifier** and can be read as "for all".

- So now the whole argument can be notated:

$$\frac{\begin{array}{l}\forall x.(L(x) \rightarrow H(x)) \\ L(b)\end{array}}{H(b)}$$

- The notation introduced informally here is the **First Order Predicate Calculus (FOPC)**.
- Predicate Logic is more expressive than simple Propositional Logic.
- We will explore this new logic in the remainder of this course.

Summary

- Propositional logic allows us to represent simple propositions/arguments.
- We have explored the language from the point of view of its meaning and form.
- Propositional logic has limitations – there are some valid arguments that we cannot represent.
- There is more to the structure of propositions than simple boolean combinations of 'atomic' propositions.
- Propositions are about individuals (or sets thereof) and their properties.
- We need a new language for representing this structure: the language of predicate logic.