Introduction to Logic 11

Last time:

- Un-natural Deduction
- Natural Deduction
- Introduction Rules
- Examples

This time:

- Natural Deduction Proof Rules
- Introduction Rules
- Elimination Rules
- Proof by Contradiction

Natural Deduction Proof Rules

- The system of Natural Deduction is a proof method that has some advantages over the axiomatic system and the tableaux method for propositional logic.
  - proofs are relatively easy to construct;
  - the proofs that result consist of a fairly natural sequence of steps
- The Natural Deduction inference rules attempt to capture frequently used patterns of reasoning or 'logical laws'.
- Broadly, the rules fall into two groups:
  1. **Introduction Rules:** i.e. rules that *introduce* connectives;
  2. **Elimination Rules:** i.e. rules that *eliminate* connectives.

Introduction Rules

$$\frac{A \quad B}{(A \wedge B)} \; \wedge I$$

$$\frac{A}{(A \vee B)} \; \vee I \qquad\qquad \frac{B}{(A \vee B)} \; \vee I$$

$$\begin{array}{c} \cancel{A} \\ \vdots \\ B \end{array}$$

$$\frac{B}{(A \rightarrow B)} \; \rightarrow I$$

$$\frac{\bot}{A} \; \bot$$

Note:

$$\begin{aligned} (A \leftrightarrow B) \quad &\equiv \quad ((A \rightarrow B) \wedge (B \rightarrow A)) \\ \neg A \quad &\equiv \quad (A \rightarrow \bot) \end{aligned}$$

Elimination Rules

• Let us consider now the rules for eliminating connectives.

Conjunction Elimination ($\wedge$ E):

• Consider the following pattern of reasoning:

> *Suppose that you know that* $(A \wedge B)$ *is* **true** , *then it is safe to infer that A (or B) must be true.*

• Expressing this is the notation of the system of Natural Deduction, gives the following *two* rules of inference:

$$\frac{(A \wedge B)}{A}\ \wedge E \qquad\qquad \frac{(A \wedge B)}{B}\ \wedge E$$

Implication Elimination ($\rightarrow$ E):

- Consider the following pattern of reasoning

> *Suppose you know that* $(A \rightarrow B)$ *is* **true** *and also that* A *is* **true** *. In this case, it is safe to infer that* B *is* **true** *.*

- In the system of natural deduction, this may be notated as:

$$\frac{(A \rightarrow B) \quad A}{B} \rightarrow E$$

Question: *Where have we seen this rule before?*

Example: $\{(p \rightarrow q), (q \rightarrow r)\} \vdash (p \rightarrow r)$

$$\dfrac{\dfrac{\dfrac{(p \rightarrow q) \quad \cancel{p}}{q} \rightarrow E \qquad (q \rightarrow r)}{r} \rightarrow E}{(p \rightarrow r)} \rightarrow I$$

Disjunction Elimination($\vee$ E)

- The rule for eliminating a disjunction ($\vee$) is a little trickier to understand.

  Suppose you know that $(A \vee B)$ *is* **true** *. Suppose also that from the assumption that* A *is* **true** *you can reach a conclusion that* C *is* **true** *; and from the assumption that* B *is* **true** *, you can reach that same conclusion, that* C *is* **true**
  In this case, it is safe to infer that C *is* **true** *.*

- The rule is essentially that of analysis by cases:
  - whichever case we consider (A or B) we can show that C must be **true** ;
  - so we can conclude that C follows from $(A \vee B)$
- Like implication introduction, this rule is not straightforward to represent.

- Diagrammatically, the rule of Disjunction Elimination appears as follows:

$$\cfrac{(A \vee B) \qquad \begin{matrix} \cancel{A} \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \cancel{B} \\ \vdots \\ C \end{matrix}}{C} \; \vee E$$

- Here is an example of its use:

$$\cfrac{((p \wedge q) \vee q) \qquad \cfrac{\cfrac{\cancel{(p \wedge q)}}{p}\;\wedge E}{(p \vee q)}\;\vee I \qquad \cfrac{\cancel{q}}{(p \vee q)}\;\vee I}{(p \vee q)} \; \vee E$$

- So:

$$\{((p \wedge q) \vee q)\} \vdash (p \vee q)$$

Proof By Contradiction (reductio ad absurdum)

- We now have introduction and elimination rules for each of the binary connectives: $\wedge$, $\vee$ and $\rightarrow$.
- We have not yet considered negation: $\neg$.
- Consider the following method of reasoning:

  > *Suppose that we wish to prove that some statement A holds. Assume rather that $\neg A$ holds. If we can now show that this assumption leads to a contradiction, then it is safe to conclude that $\neg A$ cannot hold. In other words, A must hold.*

- This proof method is know as **Proof by Contradiction**, or **reductio ad absurdum (RAA)**.

- As a diagram, this proof rule **RAA** may be represented as follows:

$$\dfrac{\begin{array}{c}\cancel{\neg A}\\ \vdots\\ \bot\end{array}}{A}\ RAA$$

- The following example illustrates the use of RAA. We show:

$$\{\neg(\neg p \vee q)\} \vdash p$$

$$\dfrac{\dfrac{\dfrac{\cancel{\neg p}}{(\neg p \vee q)}\ \vee I \qquad \neg(\neg p \vee q)}{\bot}\ \rightarrow E}{p}\ RAA$$

- **NB:** *This proof also makes use of the fact that in this system,* $\neg A$ *is simply an abbreviation for* $(A \rightarrow \bot)$.

Elimination Rules

$$\frac{(A \wedge B)}{A}\ \wedge E \qquad\qquad \frac{(A \wedge B)}{B}\ \wedge E$$

$$\frac{(A \rightarrow B) \quad A}{B} \rightarrow E$$

$$\frac{(A \vee B) \qquad \begin{matrix} \cancel{A} \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \cancel{B} \\ \vdots \\ C \end{matrix}}{C}\ \vee E$$

$$\frac{\begin{matrix} \neg\cancel{A} \\ \vdots \\ \perp \end{matrix}}{A}\ RAA$$

Example

- We show that

$$\vdash (p \vee \neg p)$$

- The proof proceeds as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg \not p^{(1)}}{(p \vee \neg p)}\ \vee I \qquad \neg(p \not\vee \neg p)^{(2)}}{\bot}\ \to E}{p}\ RAA}{(p \vee \neg p)}\ \vee I \qquad \neg(p \not\vee \neg p)^{(2)}}{\bot}\ \to E}{(p \vee \neg p)}\ RAA$$

Remarks

- There are just *two* assumption introduced in this proof
- In the end, the proof is perhaps not quite as 'natural' as we would like!

Summary

- The system of Natural Deduction has introduction and elimination rules for connectives.
- Elimination rules for conjunction and implication are straightforward. Implication elimination, in particular, is familiar as the rule Modus Ponens.
- The elimination rule for disjunction corresponds to a method of 'reasoning by cases'
- The system also has a rule formalizing the famous *proof by contradiction* or *reductio ad absurdum*.