Introduction to Logic 10

Last time:

- Tableaux and Valuations
- Justifying the Tableaux Rules
- Inconsistency and Entailment
- Testing Validity of Arguments

This time:

- Un-natural Deduction
- Natural Deduction
- Introduction Rules
- Examples

Un-natural Deduction

- We have seen how logic can be viewed as a formal system of deduction consisting of:
  1. a *language* for expressing propositions;
  2. a set of *axioms*;
  3. a set of *rules of inference*
- We can furnish a precise definition of the notion of a proof (in a formal system):

Defintion: (Proof) A proof in a formal system is a sequence of sentences

$$A_1, A_2, \ldots, A_n$$

where each A_i $(1 \leq i \leq n)$ is either:

1. an *axiom*; or
2. a *direct consequence* of two earlier sentences A_j and A_k $(j, k < i)$

For example, in Lectures 6 and 7 we saw how the Propositional Calculus could be formalized as an axiomatic system.

- This systems had three axiom schemas and a single rule of inference (Modus Ponens);
- We have noted that proofs constructed within this system are not particularly 'natural':
  - They are hard to construct;
  - The use of axioms is not intuitive
  - The individual proof steps do not appear to correspond to steps in 'informal' proofs or argumentation;

Is it possible to formulate some system of deduction that is more 'natural' than this?

- The method of Semantic Tableaux has some merit:
  - it is easier to use than the axiomatic systems (i.e. constructing tableaux is a relatively straightforward, rule-governed process);
  - the tableaux derivation rules have a straightforward semantic interpretation;
- In other ways however, the method is not as ‘natural’ as we might like:
  - the use of tableaux to establish *inconsistency* is not particularly intuitive;
  - the individual derivation rules do not correspond well to steps in informal proofs or reasoning.

Natural Deduction

- People seem to use a variety of methods for constructing informal arguments or proofs in natural language.
- Even mathematicians do not generally proceed from axiom systems of the kind we have seen for the Propositional Calculus.
- Informal proofs exhibit ‘patterns of reasoning’ like the following:

$$\frac{\begin{array}{l}\textit{if Logic is fun, then Bill is happy}\\ \textit{Logic is fun}\end{array}}{\textit{Bill is happy}}$$

- This instance of Modus Ponens seem quite natural.

Could we formulate a system of deduction based *entirely* on ‘natural laws’ such as the above?

- The new formal system of Natural Deduction will consist of the following components:
  1. The language of Propositional Logic
  2. Various rules of inference:
     - Introduction rules;
     - Elimination rules;
- Note that in constrast to the axiomatic system that we saw earlier, this formal system has *no* axioms.
- Also, rather than a single rule of inference (Modus Ponens) it has *many* such rules.
- The natural deduction rules are intended to express frequently used patterns of reasoning.
- The rules come in two varieties:
  - rules that produce complex statements from smaller statements by *introducing* connectives; and
  - rules that produce simpler statements from complex statements by *eliminating* connectives.

Introduction Rules

• The introduction rules are so-called because they are used to *introduce* connectives.

Conjunction Introduction (∧I)

• This rule captures the following informal 'pattern of reasoning':

> *If you know that A is* **true** *and that B is* **true** , *then it is valid to conclude that* $(A \wedge B)$ *is* **true** .

• In the system of Natural Deduction, this rule is represented diagrammatically as follows:

$$\frac{A \quad B}{(A \wedge B)} \wedge I$$

• The rule has two premisses A and B, and produces a conclusion $(A \wedge B)$, that has $\wedge$ as its principal connective.

Disjunction Introduction ($\vee$I)

• This rule corresponds to the following informal pattern of reasoning:

> *If you know that A is* **true** *, then you can conclue that* $(A \vee B)$ *is* **true** *(for* any *sentence B).*

• This rule of disjunction introduction actually corresponds to two rules of inference in the system of Natural Deduction:

$$\frac{A}{(A \vee B)} \vee I \qquad\qquad \frac{B}{(A \vee B)} \vee I$$

• The introduction rules for the connectives $\wedge$ and $\vee$ may seem rather trivial.

• a more interesting rule is **Implication Introduction**: the so-called **Method of Conditional Proof**.

Implication Introduction ($\rightarrow$ I):

• This is a method for introducing the conditional or implication connective $\rightarrow$.

• the method of conditional proof corresponds to the following line of argumentation:

> *Under the assumption that statement A is* **true** *, it is possible to reason to the conclusion that statement B is* **true** *.*
>
> *As the conclusion B rests on the assumption A, it is valid to conclude that* $(A \rightarrow B)$ *is* **true** *.*

• It is a little harder to represent this rule diagrammatically

• The rule of **implication introduction** does not correspond neatly to a single proof step.

• The reasoning in conditional proof concerns the overall structure of (part of) a proof.

• Implication introduction ($\rightarrow$ I) is represented as follows:

$$\cfrac{\begin{array}{c}\cancel{A}\\ \vdots\\ B\end{array}}{(A \rightarrow B)} \rightarrow I$$

Note:

- The intermediate conclusion B rests on the assumption A. However, the final conclusion $(A \rightarrow B)$ does *not* rest on A!
- The assumption A must be *cancelled* or *discharged* once we draw the final conclusion $(A \rightarrow B)$.
- We cross out the assumption ($\cancel{A}$) to remind ourselves that $(A \rightarrow B)$ does not depend on A.

- We now have rules for introducing the connectives $\wedge$, $\vee$ and $\rightarrow$.
- Note that we have not provided introduction rules for $\neg$ and $\leftrightarrow$:
  - treatment of $\neg$ will be deferred until next lecture;
  - we will not consider $\leftrightarrow$ since, e.g.,

$$(A \leftrightarrow B) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

- The final rule included here is simply called $\perp$ (*falsum*).

Falsum ($\perp$):

- In essense this rule states:

  Anything follows from falsum (i.e. from an absurdity or contradiction).

- The rule is notated as:

$$\frac{\perp}{A}\ \perp$$

Example

- Using just the introduction rules of the system of natural deduction, we will show that:

$$\vdash (p \rightarrow (p \vee q))$$

- The proof proceeds as follows:

$$\dfrac{\dfrac{\cancel{p}}{(p \vee q)}\ \vee I}{(p \rightarrow (p \vee q))}\ \rightarrow I$$

- Note that the assumption p has been cancelled.
- Thus the conclusion $(p \rightarrow (p \vee q))$ does not rest on any assumptions.
- This means that $(p \rightarrow (p \vee q))$ is a theorem.

Example

• We will show that

$$\{p\} \vdash (r \to ((p \vee q) \wedge r))$$

• The proof proceeds as follows:

$$\dfrac{\dfrac{\dfrac{p}{(p \vee q)}\ \vee I \qquad \cancel{r}}{((p \vee q) \wedge r)}\ \wedge I}{(r \to ((p \vee q) \wedge r))}\ \to I$$

• Note that in this case, only the assumption r has been cancelled.

• The proof still contains an undischarged assumption p.

• This means that the final statement rests on the assumption p (though not on r).

Summary

- The axiomatic system of Propositional Logic is not particularly intuitive or ‘natural’
- The method of Semantic Tableaux is more easy to apply, but does not correspond well with informal methods of proof or argumentation
- The System of Natural Deduction is an attempt to formalize reasoning in a way that captures commonly used ‘patterms of reasoning’.
  - There are no axioms...
  - ...but many rules of inference
- The inference rules fall into two groups: Introduction Rules and Elimination Rules.