Naïve Synthetic Domain Theory – A Logical Approach

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Abstract

We present two different approaches to Synthetic Domains. First we give a synthetic, axiomatic version of extensional PERs which [Freyd et al. 92] introduced model theoretically. Second we rephrase the theory of replete objects (cf. [Hyland 91], [Taylor 91]). If the category of predomains is small and internally complete two *logical* characterizations of repletion are possible.

Closure properties of the corresponding predomains and domains are stated and finally the most important proofs of traditional domain theory are sketched in the synthetic setting. So this paper might serve as an introduction into Synthetic Domain Theory also for readers unfamiliar with topos or category theory.

1 Introduction

1.1 The history of Synthetic Domain Theory

In the first half of the 80's Dana Scott in several lectures proposed and advertised that the best logical frame for developing his domain theory - an extensional theory of computation - is intuitionistic higher order logic or intuitionistic set theory. The advantage of an intuitionistic setting is that one may consistently assume that all functions between domains are continuous and therefore all endofunctions on domains have least fixpoints. Actually these continuity assumptions are not only consistent but there exist very beautiful topos-theoretic models with a very rich structure. Nevertheless till the end of the 80's there was not much activity in developing domain theory axiomatically in an intuitionistic setting. A notable exception is the Ph. D. Thesis of G. Rosolini [Rosolini 86] who introduced the basic notion of Synthetic Domain Theory : the r.e. subobject classifier Σ. This line of research was taken up again by W. Phoa in his Ph. D. Thesis [Phoa 90]. In 1989 an approach to a synthetic theory of domains was suggested independently by J.M.E. Hyland and P. Taylor, respectively (cf. [Hyland 91], [Taylor 91]). The main idea of their approach is to take for predomains the objects of the least full reflective subcategory of the ambient category of intuitionistic sets containing the r.e. subobject classifier Σ. All this work is based on rather tough category theory and it is not immediately clear how it can be rephrased in purely logical terms. But this is indispensible for providing an axiomatic theory of domains which can be given as a tool to computer scientists for supporting their aims of verifying functional programs. Surely both Hyland and Taylor have strongly advertised the use of the socalled "internal language of a topos" which amounts to jumping back and forth between the diagramatic language of category theory and the language of higher order logic. Actually the method of employing the "internal language of a topos" is more an art than a fully developed scientific method and requires a lot of skill.

Therefore the starting point of our work is to develop Synthetic Domain Theory in a purely logical way using the diagramatic language only for the purpose of illustrating our arguments.

This approach is what we call "naive Synthetic Domain Theory". The paradigm is A. Kock's presentation of Synthetic Differential Geometry in the first part of his book [Kock 81]. Later on R. Lavendhomme wrote a whole book on Synthetic Differential Geometry fully avoiding the language of category theory. He called this "Naive Synthetic Differential Geometry" and that is the sense of "naive" we have employed in the title of our paper. It is probably impossible - at least for us - to achieve such a high level of clarity as has become standard by the work of A. Kock and R. Lavendhomme. But impressed and influenced by their work we try to make at least a first step towards *Naive Synthetic Domain Theory.*

1.2 Motivation

The aim of Synthetic Differential Geometry is to replace the formulation of differential calculus in terms of the "ϵ-δ-language" by a formulation in terms of "infinitesimals". Analogously the aim of Synthetic Domain Theory is to replace the traditional formulation of domain theory in terms of sets endowed with an order-theoretic or topological structure by a new formulation where domains are simply sets with a certain property *providing* the required topological structure.

Already in traditional Domain Theory there is a 1-1-correspondence between Scott open subsets of a domain D and continuous maps from D to Σ where Σ is the 2-element lattice. That means that one can "externalize" the notion of open set. So the starting point of Synthetic Domain Theory is to assume a distinguished "archetypical domain" Σ which is described axiomatically and to associate with an arbitrary set X its "natural topology" by defining the open sets of X as the functions from X to Σ. The computational intuition behind these "open sets" is that they correspond to *semidecidable predicates* which constitute the most general form of experiment which can be applied to a computational object. The objects $\top$ and $\perp$ of Σ correspond to the propositions expressing termination and nontermination, respectively. Thus Σ is considered as the subset of the set Ω of propositions that intuitively corresponds to "Σ^0_1"-sentences.

As semidecidable predicates are closed under conjunction and existential quantification over N one puts axiomatically that Σ is closed under the respective logical operations (cf. 2.1.1). As will become clear subsequently we need some technical assumptions (Phoa's axioms) about the structure of $\Sigma \to \Sigma$ (cf. 2.1.2).

Moreover, we have to ensure that the natural topology on a *domain* D (as given by Σ) satisfies the conditions required in the definition of Scott topology, i.e. any open set P must satisfy

$$x \in P \ \wedge \ x \sqsubseteq y \Rightarrow y \in P$$

and for any ascending chain $(x_n)_{n \in N}$

$$\sup_n x_n \in P \Rightarrow \exists n{:}N.\, (x_n \in P)$$

Of course, we have to explain what $\sqsubseteq$ and sup mean. The first condition forces one to define $x \sqsubseteq y$ as $\forall P{:}D \to \Sigma.\, x \in P \Rightarrow y \in P$ (cf. Definition 2.2.1). To satisfy the second condition we simply define sup by the condition $\sup_n x_n \in P \iff \exists n{:}N.\, (x_n \in P)$ (cf. Definition 2.2.3). Without further requirements this supremum is not necessarily unique unless any object of a domain is determined by the results of all possible experiments applied to it (i.e. its observational behaviour). This will be ensured by the definition 3.1.2 of Σ-posets.

The main aspect of traditional Domain Theory is to approximate the intutive notion of *computability* by the notion of *continuity.* Therefore when working classically one has to distinguish all the time between arbitrary and continuous functions. The distinguishing feature of the synthetic approach is to assume that *all functions between domains are continuous.* Relative to the axioms described above Taylor proved that this is equivalent to Scott's axiom (cf. Section 2.1.3) which claims continuity for all function from $(N \to \Sigma)$ to Σ (cf. [Taylor 91]). More precisely, Scott's axiom is needed for proving that there is a chain domain $\overline{\omega}$ (i.e. a constructive version of the cpo

$\omega+1$). Continuity of functions between domains follows already from the previous axioms and the definition of the supremum (cf. 2.2.7).

The aim of this paper is to demonstrate that the core of ordinary domain theory, i.e. least fixpoint theorem, fixpoint induction, initial fixpoints of functors etc. (cf. Section 6) can be developed in a fairly detailed way from the concise axioms of *Synthetic Domain Theory* within the few pages of this paper.

The paper is organized as follows: After some logical preleminaries and the axioms of Synthetic Domain Theory presented in Section 2, Sections 3 and 4 describe the approaches of [Freyd et al. 92] and [Taylor 91] & [Hyland 91], respectively. The first one will make heavy use of Markov's principle, the latter can avoid it. Section 5 then establishes a relation between them, stating that assuming Markov's principle every replete object is a Σ-cpo. Section 6 explains how on top of both variants the main results of classical domain theory can be proved quite easily.

2 Logical Preliminaries of Synthetic Domain Theory

We formulate the axioms of Synthetic Domain Theory in a topos logic (cf. [Lambek, Scott 86]) i.e. a higher-order intuitionistic logic with axiom of unique choice, arithmetic and subtype formation. Let N denote the natural numbers and Ω the type of propositions.

Recall that therefore we have extensionality on functions, i.e.

$$\forall f, g{:}A \to B.\ (\forall a{:}A. fa = ga) \Rightarrow f = g$$

and on propositions

$$\forall p, q{:}\Omega.\ (p = q) \Longleftrightarrow (p \Leftrightarrow q)$$

We also recall the Axiom of *Unique Choice* which allows us to turn functional relations in functions and will be used frequently in the proofs: Let A, B be arbitrary types and P be a predicate over $A \times B$

(AC!) $(\forall x{:}A.\ \exists! y{:}B.\ P(x, y)) \Rightarrow \exists f{:}A \to B.\ \forall a{:}A.\ P(a, fa)$

2.1 The Axioms of SDT

In this subsection we state formally all the axioms motivated in Section 1.2.

2.1.1 Definition of Σ

Definition 2.1.1 Let $\Sigma \subseteq \Omega$ such that

(a) $\top, \bot \in \Sigma$

(b) If $p, q \in \Sigma$ then $p \wedge q,\ p \vee q \in \Sigma$

(c) $\forall f{:}N \to \Sigma.\ (\exists n{:}N. fn) \in \Sigma$

Note that $p \vee q$ in (b) is a special case of (c).

2.1.2 Phoa's Axioms

Phoa's Axioms state that $\Sigma^\Sigma \cong \{(p,q) \in \Sigma \times \Sigma \mid p \Rightarrow q\}$. Roughly speaking this is necessary for assuring that Σ itself is a domain.

(PHOA1) $\forall f{:}\Sigma \to \Sigma.\ f\bot \Rightarrow f\top$

(PHOA2) $\forall f,g{:}\Sigma \to \Sigma.\ (f\bot = g\bot) \wedge (f\top = g\top) \Rightarrow f = g$

(PHOA3) $\forall p,q{:}\Sigma.\ (p \Rightarrow q) \Rightarrow \exists f{:}\Sigma \to \Sigma.\ f\bot = p \wedge f\top = q$

2.1.3 Scott's Axiom

Scott's Axiom ensures continuity and the existence of fixpoints.

(SCOTT) $\forall P{:}\Sigma^N \to \Sigma.\ P(\lambda x{:}N.\top) \Rightarrow (\exists n{:}N.P(step\, n))$

where $step\ n{:}N \to \Sigma$ is defined as follows: $step\, n\, m := \begin{cases} \top & \text{if } m < n \\ \bot & \text{otherwise} \end{cases}$

So these are the very only axioms we'll need, the whole bunch of theorems of domain theory (or at least the relevant part) will be developed successively on top of these axioms in intuitionistic higher-order logic with subtyping. Note that we cannot use classical logic since this would contradict the requirement that all functions are continuous in our approach. In fact, classical logic would provide a contradiction to PHOA1: Using classical logic we could for any $p{:}\Sigma$ derive $p = \bot \vee p = \top$ und thus we could prove $\forall x{:}\Sigma.\ \exists! y{:}\Sigma.(x = \bot \Rightarrow y = \top) \wedge (x = \top \Rightarrow y = \bot)$. With AC! we get a function $f : \Sigma \to \Sigma$ with $f(\bot) = \top$ and $f(\top) = \bot$ contradicting PHOA1. We can even contradict SCOTT, since we can prove classically $\forall f{:}\Sigma^N.\ \exists p{:}\Sigma.\ ((f = \lambda x{:}N.\top) \wedge (p = \top)) \vee (\neg(f = \lambda x{:}N.\top) \wedge (p = \bot))$.

2.2 Some global definitions and important lemmas

In this section we define the observational inequality and equality where observations are only allowed to have values in Σ. We prove that on Σ^X the pointwise inequality is equivalent to the observational one and that all functions are monotonic w.r.t. the observational inequality.

Definition 2.2.1 (Phoa) Let X be any type and $x, y{:}X$.

(i) $x \sqsubseteq y$ iff $\forall P{:}X \to \Sigma.\ (P\,x \Rightarrow P\,y)$.

(ii) $x =_{obs} y$ iff $x \sqsubseteq y \wedge y \sqsubseteq x$

(iii) $x \sqsubseteq_{link} y$ iff $\exists h{:}\Sigma \to X.\, h\bot = x \wedge h\top = y$.

(iv) If the relations $\sqsubseteq_{link}$ and $\sqsubseteq$ coincide, i.e. $\forall x, y{:}X.\ x \sqsubseteq_{link} y \Leftrightarrow x \sqsubseteq y$ then X is called *linked*

Lemma 2.2.1 $\forall p, q{:}\Sigma.\ p \Rightarrow q$ iff $p \sqsubseteq q$.

Proof: "$\Rightarrow$": Assume $p \Rightarrow q$. Let f be the function we get by PHOA3 and $P{:}\Sigma \to \Sigma$ be arbitrary. Then $P \circ f{:}\Sigma \to \Sigma$ and by PHOA1 we get $(P \circ f)(\bot) \Rightarrow (P \circ f)(\top)$ and thus $Pp \Rightarrow Pq$.

"$\Leftarrow$": Take for P the identity on Σ. □

Lemma 2.2.2 For an arbitrary X we have $x \sqsubseteq_{link} y$ implies $x \sqsubseteq y$.

PROOF: Suppose $x \sqsubseteq_{link} y$, then there exists $h{:}\Sigma \to X$ with $h\,\bot = x$ and $h\,\top = y$. Let $P{:}X \to \Sigma$ be arbitrary then $P \circ h : \Sigma \to \Sigma$. Therefore by PHOA1 we have $P\,x = (P \circ h)\,\bot \Rightarrow (P \circ h)\,\top = P\,y$. □

THEOREM 2.2.3 For an arbitrary X and $f, g \in \Sigma^X$ the following conditions are equivalent:

(i) $f \sqsubseteq_{link} g$

(ii) $f \sqsubseteq g$

(iii) $\forall x{:}X.\,(f\,x \Rightarrow g\,x)$

PROOF: (i) ⇒ (ii): 2.2.2.
(ii) ⇒ (iii): Define $\mathrm{eval}_x := \lambda h{:}\Sigma^X.\,h\,x : \Sigma^X \to \Sigma$. Thus $f(x) = \mathrm{eval}_x(f) \Rightarrow \mathrm{eval}_x(g) = g(x)$.
(iii) ⇒ (i): If $f\,x \Rightarrow g\,x$ for all $x \in X$ then according to PHOA3 there exists for all $x \in X$ a $p_x{:}\Sigma \to \Sigma$ with $p_x\bot = f\,x$ and $p_x\top = g\,x$. Due to PHOA2 this p_x is unique. By AC! we get a function $p : X \to \Sigma^\Sigma$ with $p\,x\bot = f\,x$ and $p\,x\top = g\,x$. But then $h = \lambda s{:}\Sigma.\,\lambda x{:}X.\,p\,x\,s : \Sigma \to \Sigma^X$ with $h\,\bot = f$ and $h\,\top = g$. □

THEOREM 2.2.4 (*Monotonicity*) For any X, Y and $f{:}X \to Y$ if $x \sqsubseteq x'$ then $f(x) \sqsubseteq f(x')$.

PROOF: If $x \sqsubseteq x'$ then $\forall P{:}Y \to \Sigma.\ (P \circ f)(x) \Rightarrow (P \circ f)(x')$ i.e. $f(x) \sqsubseteq f(x')$. □

DEFINITION 2.2.2 Let us define:

$$\overline{\omega} := \{p \in \Sigma^N \mid \forall n, m{:}N.\,(p\,n \wedge m < n \Rightarrow p\,m)\}$$

$$\omega := \{p \in \overline{\omega} \mid \exists n{:}N.\,p(n) = \bot\}$$

Moreover let $inc : \omega \to \overline{\omega}$ denote the obvious inclusion.

LEMMA 2.2.5 If $Y \xrightarrow{e} \Sigma^X \xrightarrow{r} Y$ with $r \circ e = id$ then for any $y_1, y_2{:}Y$ we have $y_1 \sqsubseteq y_2$ iff $\forall x{:}X.\,e\,y_1\,x \Rightarrow e\,y_2\,x$, i.e. the inequality is pointwise.

PROOF: "⇒": obvious. "⇐": Assume $\forall x{:}X.\,e\,y_1 x \Rightarrow e\,y_2\,x$. Then $e\,y_1 \sqsubseteq e\,y_2$ and therefore $y_1 = r(e(y_1)) \sqsubseteq r(e(y_2)) = y_2$. □

Remark: As $\overline{\omega} \xrightarrow{e} \Sigma^N \xrightarrow{p} \overline{\omega}$ with e the obvious embedding and $p(f) = \lambda n{:}N.\,(\forall x \le n.f(x))$ is a retraction this holds in particular for $\overline{\omega}$.

LEMMA 2.2.6 Let $H : \overline{\omega} \to \Sigma$ then $H(\lambda n{:}N.\,\top) \Rightarrow \exists m{:}N.\ H(\mathit{step}\ m)$.

PROOF: Follows immediately from 2.2.5 and SCOTT. □

We define a notion of supremum for ascending chains and chain completeness of objects. We get that every function on chain complete types is Scott continuous.

DEFINITION 2.2.3 Let $a {\in} AC(X)$ where $AC(X) := \{f{:}N \to X \mid \forall n{:}N.f(n) \sqsubseteq f(n+1)\}$ is the type of ascending chains. By sup_X we denote an binary relation on $AC(X) \times X$ where $\mathit{sup}_X(a, x)$ holds iff $\forall P{:}\Sigma^X.\ P\,x \Leftrightarrow \exists n{:}N.\ P(a\,n)$.

X is called *chain complete* if for any $a{:}AC(X)$ there exists an $x \in X$ such that $\mathit{sup}_X(a, x)$.

THEOREM 2.2.7 (*Scott continuity*) Any function $f{:}A \to B$ is Scott continuous, i.e. $\forall a{:}AC(A).\ \forall x{:}A.\ \mathit{sup}_A(a, x) \Rightarrow \mathit{sup}_B(f \circ a, f(x))$.

PROOF: Suppose an $x {\in} A$ and for any $P {\in} \Sigma^A$ it holds that $P(x) \Leftrightarrow \exists n{:}N.\,P(a(n))$. So if we assume a $Q {\in} \Sigma^B$ and let $P := Q \circ f$ then we can conclude that $Q(f(x)) \Leftrightarrow \exists n{:}N.\,Q(f(a(n)))$, i.e. $\mathit{sup}_B(f \circ a, f(x))$. Implicitely we have used monotonicity of f (2.2.4) to guarantee $f \circ a \in AC(B)$. □

3 Extensional Σ-cpos as Predomains

In this Section we axiomatize the ExPER model of [Freyd et al. 92]. In contrast to the work of [Phoa 90] we don't require linkedness for ensuring that the inequality is pointwise (for extensional PERS) but we require the Σ-posets to be $\neg\neg$-closed subobjects and together with Markov's principle this already implies linkedness.

(MP) $\forall p{:}\Sigma.\neg\neg p \Rightarrow p$

3.1 Σ-posets and Σ-cpos

Using (MP) we can prove that

Lemma 3.1.1 For any X the equality on Σ^X is $\neg\neg$-closed.

Proof: $\neg\neg$-closed predicates are closed under negation, conjunction and implication (in fact it already suffices that the conclusion is $\neg\neg$-closed), and universal quantification over an arbitrary set. With (MP) and the closure properties one can easily prove that the equality on Σ^X is $\neg\neg$-closed. $\square$

Definition 3.1.1 A mono $A \overset{m}{\hookrightarrow} B$ is called $\neg\neg$-closed iff $\forall b{:}B.\ (\neg\neg\exists a{:}A.\ a = b) \Rightarrow \exists a{:}A.a = b$. The subobject A is then called a $\neg\neg$-closed subobject, written $A \subseteq_{\neg\neg} B$.

A predicate $P{:}X \to \Omega$ is called $\neg\neg$-closed iff $\forall x{:}X.\neg\neg P(x) \Rightarrow P(x)$. E.g. for a $\neg\neg$-closed predicate its characteristic map is a $\neg\neg$-closed mono.

Definition 3.1.2 X is called a Σ-poset iff $\eta_X{:}X \to \Sigma^{\Sigma^X}$ with $\eta_X(x) = \lambda p{:}\Sigma^X.p\,x$ is a $\neg\neg$-closed mono.

This defines a predicate that tells us whether a set X with its associated equality $=_X$ is a Σ-poset. The condition that η_X is a mono ensures that the observational equality coincides with the given equality on X.

It is possibile to develop domain theory if we drop the condition that η_X is $\neg\neg$-closed, if one requires that $\sqsubseteq_{link}$ and $\sqsubseteq$ coincide. W. Phoa developed this approach in his thesis [Phoa 90].

Corollary 3.1.2 If X is a Σ-poset then $X \cong \{p \in \Sigma^{\Sigma^X} \mid \neg\neg\exists x{:}X.\ \eta_X(x) = p\}$.

Proof: $X \overset{\eta_X}{\to} \{p \in \Sigma^{\Sigma^X} \mid \neg\neg\exists x{:}X.\ \eta_X(x) = p\}$ is an iso because it is mono and epi by definition. $\square$

Lemma 3.1.3 For any X the function η_X reflects $\sqsubseteq$.

Proof: Suppose $\eta_X(x) \sqsubseteq \eta_X(x')$. We have to show $x \sqsubseteq x'$.
Let $p{\in}X \to \Sigma$ then define $\hat{p}{:}\Sigma^{\Sigma^X} \to \Sigma := \lambda F{:}\Sigma^{\Sigma^X}.Fp$. Then $p\,x = \hat{p}(\eta_X x)$ for all $x{\in}X$ which by assumption implies $\hat{p}(\eta_X x')$ which equals $p\,x'$. Thus $p\,x \Rightarrow p\,x'$ and therefore $x \sqsubseteq x'$. $\square$

Next we show that $\sqsubseteq$ and $\sqsubseteq_{link}$ are equivalent for Σ-posets.

Theorem 3.1.4 (Phoa) Any Σ-poset is linked.

Proof: "$\Rightarrow$"2.2.2.
"$\Leftarrow$"Assume $x \sqsubseteq y$. Then $\eta(x) \sqsubseteq \eta(y)$ and by 2.2.3 $\eta(x) \sqsubseteq_{link} \eta(y)$. Thus there exists $h{:}\Sigma \to \Sigma^{\Sigma^X}$ with $h(\bot) = \eta(x)$ and $h(\top) = \eta(y)(*)$. As $\exists x{:}X.\, \eta(x) = h(s)$ is $\neg\neg$-closed we can prove $\forall s{:}\Sigma.\, \exists x{:}X.\, \eta(x) = h(s)$ by case analysis on s by (*). Thus as η is a mono $\forall s{:}\Sigma.\, \exists! x{:}X.\, \eta(x) = h(s)$. By AC! we get a $k{:}\Sigma \to X$ with $\forall s{:}\Sigma.\, \eta(k(s)) = h(s)$. Therefore $\eta(x) = \eta(k(\bot))$ and $\eta(y) = \eta(k(\top))$. As η is a mono we have $x = k(\bot)$ and $y = k(\top)$ and thus $x \sqsubseteq_{link} y$. □

Due to the definition we know that every Σ-poset Y is a $\neg\neg$-closed subobject of Σ^{Σ^Y}, i.e. a $\neg\neg$-closed subobject of Σ^X for some X. We prove now that every $\neg\neg$-closed subobject of Σ^X is a Σ-poset and that Σ-posets are therefore closed under $\neg\neg$-closed subobjects.

Theorem 3.1.5 (Σ-*poset Representation Theorem*) A is a Σ-poset iff A is a $\neg\neg$-closed subobject of Σ^X for some X.

Proof: "$\Rightarrow$": Since A is a Σ-poset it is by 3.1.2 a $\neg\neg$-closed subobject of Σ^{Σ^A}.

"$\Leftarrow$": (i) η_A is a mono: If $\eta_A(a_1) = \eta_A(a_2) \in \Sigma^{\Sigma^A}$ then for all $x \in X$ we define $\mathrm{eval}_x := \lambda a{:}A.\, a\, x \in \Sigma^A$ and we have $a_1\, x = \mathrm{eval}_x(a_1) = \eta_A(a_1)(\mathrm{eval}_x) = \eta_A(a_2)(\mathrm{eval}_x) = \mathrm{eval}_x(a_2) = a_2\, x$. Thus $a_1 = a_2$.

(ii) We still have to show $\forall P{:}\Sigma^{\Sigma^A}.\, (\neg\neg\exists a{:}A.\ \eta_A a = P) \Rightarrow \exists a{:}A.\ \eta_A a = P$.
Assume a and P such that $\eta_A(a) = P$ then $a\, x = \eta_A(a)(\lambda a{:}A.\, a\, x) = P(\lambda a{:}A.\, a\, x)$ for all $x \in X$. Thus $a = \lambda x{:}X.\, P(\lambda a{:}A.\, a\, x)$. Let $t := \lambda P{:}\Sigma^{\Sigma^A}.\, \lambda x{:}X.\, P(\lambda a{:}A.\, a\, x)$ of type $\Sigma^{\Sigma^A} \to \Sigma^X$. Then we have $t(P) = a \in A$ and $\eta_A(t(P)) = P$. Discharging our assumptions we get $\forall P{:}\Sigma^{\Sigma^A}.\, (\exists a{:}A.\, \eta(a) = P) \Rightarrow t(P) \in A \ \wedge\ \eta(t(P)) = P$. As $A \subseteq_{\neg\neg} \Sigma^X$ and equality on Σ^{Σ^A} is $\neg\neg$-closed we have $\forall P{:}\Sigma^X.\, \neg\neg(\exists a{:}A.\, \eta(a) = P) \Rightarrow \exists a{:}A.\, \eta(a) = P$. □

Theorem 3.1.6 For any Σ-poset A if $A \subseteq_{\neg\neg} \Sigma^X$ then for $a_1, a_2{:}A$ it holds that $a_1 \sqsubseteq a_2$ iff $\forall x{:}X.\, a_1\, x \Rightarrow a_2\, x$.

Proof: "$\Rightarrow$": obvious. "$\Leftarrow$": If $\forall x{:}X.\, a_1\, x \sqsubseteq a_2\, x$ then by 3.1.4 there exists $h{:}\Sigma \to \Sigma^X$ with $h\,\bot = a_1$ and $h\,\top = a_2$. It remains to show that $\forall s{:}\Sigma.\, h(s) \in A$. But as A is $\neg\neg$-closed it is sufficient to check $h(\bot), h(\top) \in A$ which holds by assumtion and then apply 3.1.4 again. □

Corollary 3.1.7 The equality on a Σ-poset X is $\neg\neg$-closed.

Proof: By 3.1.1 and the definition of Σ-posets. □

Corollary 3.1.8 Any $\neg\neg$-closed mono between Σ-posets reflects $\sqsubseteq$.

Proof: Let A, B be Σ-posets and $A \subseteq_{\neg\neg} B$ then by 3.1.5 there is a X such that $A \subseteq_{\neg\neg} B \subseteq_{\neg\neg} \Sigma^X$ and thus $A \subseteq_{\neg\neg} \Sigma^X$. so due to 3.1.6 we are done.□

Remark: Obviously Σ-posets are closed under $\neg\neg$-closed subobjects.

Definition 3.1.3 X is a Σ-cpo (or an *extensional predomain*) iff X is a chain complete Σ-poset.

Remark: For a chain complete Σ-poset X the supremum of an ascending chain is necessarily unique, since η_X is a mono and so by AC! we get a function $sup_X{:}AC(X) \to X$ such that $\forall P{:}\Sigma^X.\, P(sup\ a) \Leftrightarrow \exists n{:}N.\, P(a\, n)$. Scott continuity then means $\forall a{:}AC(A).\, f(sup\, a) = sup\,(f \circ a)$.

3.2 The chain poset

In this section we want to prove that $\overline{\omega}$ is a Σ-cpo. We will prove a representation theorem for Σ-cpos 3.2.5 in order to do that. According to that theorem a Σ-poset A is a Σ-cpo iff for any $a{:}AC(A)$ the supremum $sup_A a$ in Σ^X (where it always exists) is already in A.

$\overline{\omega}$ is a Σ-poset, since it is a $\neg\neg$-closed-subobject of Σ^N. Note that ω is no Σ-poset due to the occurence of an existential quantifier in its definition. If one substitutes this quantifier by a negated universal one then one could define a Σ-poset.

It is an easy exercise for the reader to prove that $step\, n \in \omega$ and moreover $step\ n\, m \Leftrightarrow m < n$.

For proving the representation theorem for Σ-cpos we need the following lemmata.

Lemma 3.2.1 $\forall p{:}\omega.\ \neg\neg\exists n{:}N.\,((p\,n = \bot) \wedge (\forall m{:}N.\, m < n \Rightarrow p\,m))$

Proof: Assume $\neg\exists n{:}N.((p\,n = \bot) \wedge (\forall m{:}N.\, m < n \Rightarrow p\,m = \top))$ i.e. $\forall n{:}N.\,(\forall m{:}N.m < n \Rightarrow p\,m = \top) \Rightarrow \neg(p\,n = \bot)$ iff $\forall n{:}N.\,(\forall m{:}N.\, m < n \Rightarrow p\,m) \Rightarrow \neg\neg p\,n$ iff $\forall n{:}N.\,(\forall m{:}N.\, m < n \Rightarrow p\,m) \Rightarrow p\,n$ due to MP. But this by induction implies $\forall n{:}N.\, p\,n$. But as $p \in \omega$ we have $\exists n{:}N.\, p\,n = \bot$ thus $\bot = \top$ which implies $\bot$. □

Lemma 3.2.2 If P is $\neg\neg$-closed then $\forall f{:}\omega.\, P(f)$ iff $\forall n{:}N.\, P(step\, n)$.

Proof: It suffices to prove $\forall f{:}\omega.\ \neg\neg(\exists n{:}N.f = step\ n)$. Therefore, suppose $f \in \omega$, then by 3.2.1 $\neg\neg\exists n{:}N.\ ((f\,n = \bot) \wedge (\forall m{:}N.m < n \Rightarrow f\,m))$. But we can show $\exists n{:}N.\,((f\ n = \bot) \wedge (\forall m{:}N.\, m < n \Rightarrow f\ m)) \Rightarrow \exists n{:}N.\, f = step\ n$ by EXT and case analysis w.r.t. $<$ and thus we are done. □

Lemma 3.2.3 If P is $\neg\neg$-closed then $\forall f{:}\overline{\omega}.\, P(f)$ iff $(\forall n{:}N.\, P(step\, n)) \wedge P(\lambda n{:}N.\, \top)$.

Proof: Follows immediately from $\forall f{:}\overline{\omega}.\ \neg\neg((f \in \omega) \vee (\forall n{:}N.f\ n))$ and 3.2.2. □

Lemma 3.2.4 For any $a{\in}AC(\Sigma^X)$ there is an $H_a : \overline{\omega} \to \Sigma^X$ such that $H_a(step\, n) = a\,n$ and $H_a(\lambda x{:}N.\, \top) = \lambda x{:}X.\, \exists n{:}N.\, a\,n\,x$.

Proof: Define $H_a := \lambda p{:}\overline{\omega}.\, \lambda x{:}X.\exists n{:}N.\,(p\,n\ \wedge a\,(n+1)\,x) \vee a(0)$ and the rest of the proof is straightforward. □

The following Theorem is a generalization of a result of [Freyd et al. 92], which states the Representation Theorem below for the ExPER model whereas we can prove it from the axioms and MP.

Theorem 3.2.5 (Σ-*cpo Representation Theorem*)
A is a Σ-cpo iff there exists an X such that $A \subseteq_{\neg\neg} \Sigma^X$ and $(\lambda x{:}X.\, \exists n{:}N.\, a\,n\,x){\in}A$ for all $a{\in}AC(A)$.

Proof: Let $a{\in}AC(A)$ and $\tilde{a} := (\lambda x{:}X.\ \exists n{:}N.\ a\,n\,x){\in}\Sigma^X$.

"$\Rightarrow$": If A is a Σ-cpo then for $X := \Sigma^A$ we know by 3.1.5 that $A \subseteq_{\neg\neg} \Sigma^X$ and we have $sup\,(a)(P) \Leftrightarrow P(sup\,(a)) \Leftrightarrow \exists n{:}N.\, P(a\,n) \Leftrightarrow \exists n{:}N.\,(a\,n)(P) \Leftrightarrow \tilde{a}(P)$.

"$\Leftarrow$": Suppose $A \subseteq_{\neg\neg} \Sigma^X$ and $\tilde{a} \in A$ whenever $a \in AC(A)$. By 3.1.5 A is a Σ-poset. By pointwise inequality (3.1.6) $a\,n \sqsubseteq \tilde{a}$ for all $n{:}N$. It remains to show that for all $P{\in}\Sigma^A$ if $P\,\tilde{a}$ then $\exists n{:}N.\, P(a\,n)$.

Consider the function H_a of 3.2.4. First we show that $\forall p{:}\overline{\omega}.\, H\,p \in A$. Since A is $\neg\neg$-closed it is sufficient to prove $H(step\, n) \in A$ for all $n{\in}N$ and $H(\lambda k{:}N.\, \top){\in}A$ (using 3.2.3 and 3.2.2). But

$H(step\, n) = a\, n \in A$ for all $n{\in}N$ and $H(\lambda k{:}N.\top) = \tilde{a} \in A$. Now suppose $P{\in}\Sigma^A$ with $P\,\tilde{a} = \top$. Then $P \circ H \in \overline{\omega} \to \Sigma$ and $(P \circ H)(\lambda k{:}N.\top) = P(H(\lambda k{:}N.\top)) = P(\tilde{a}) = \top$. By 2.2.6 there exists an $n{\in}N$ such that $(P \circ H)(step\, n) = \top$. But then for this n we have $P(a\, n) = P(H(step\, n)) = \top$. $\square$

Remarks:

1. As a consequence for any Σ-cpo $A \subseteq_{\neg\neg} \Sigma^X$ and any $a{\in}AC(A)$ we have that $sup_A(a) = \lambda x{:}X.\exists n{:}N.\, a\, n\, x$.
2. From the Representation Theorem the reader may easily deduce that $\overline{\omega}$ is a Σ-cpo. The proof that $\overline{\omega}$ is a Σ-cpo can already be done by 3.2.4 without making any use of MP.

3.3 Characterization of Σ-cpos by orthogonality

In this section we will show that X is a Σ-cpo iff X is a Σ-poset and any $f : \omega \to X$ uniquely extends to an $\overline{f} : \overline{\omega} \to X$, i.e. X is orthogonal to the inclusion $inc : \omega \to \overline{\omega}$.

$$\begin{tikzcd} \omega \arrow[rr, "inc"] \arrow[dr, "f"'] & & \overline{\omega} \arrow[dl, dashed, "\exists!"] \\ & X & \end{tikzcd}$$

Lemma 3.3.1 For any Σ-poset X and $F, G : \overline{\omega} \to X$ we have that $F \circ step = G \circ step$ iff $F = G$. The same holds for any $F, G : \omega \to X$.

Proof: "$\Leftarrow$": obvious. "$\Rightarrow$": By 3.1.7 the predicate $F(p) = G(p)$ is $\neg\neg$-closed. For showing $\forall p{:}\overline{\omega}.\, F(p) = G(p)$ by 3.2.3 it is sufficient to show $\forall n{:}N.\, F(step\, n) = G(step\, n)$ and $F(\lambda n{:}N.\top) = G(\lambda n{:}N.\top)$. The first condition is guaranteed by assumption. For proving the second consider that $\lambda n{:}N.\top = sup\,(step)$ and therefore for an arbitrary $P{\in}X \to \Sigma$ we have $P(F(\lambda n{:}N.\top)) = \top$ iff $\exists n{:}N.\, P(F(step\, n)) = \top$ iff $\exists n{:}N.\, P(G(step\, n)) = \top$ iff $P(G(\lambda n{:}N.\top))$.
The proof for $F, G : \omega \to X$ is analogue but uses 3.2.2 instead of 3.2.3. $\square$

Lemma 3.3.2 Let X be a Σ-poset.

(i) For any $a \in AC(X)$ there exists a unique $\overline{a}{:}\omega \to X$ with $\overline{a} \circ step = a$.

(ii) Let X be a Σ-cpo. For any $a \in AC(X)$ there exists a unique $\overline{a}{:}\overline{\omega} \to X$ with $\overline{a} \circ step = a$.

(iii) Let X be a Σ-cpo. $X^{inc} : X^{\overline{\omega}} \to X^{\omega}$ is an iso.

Proof: (i) Consider the function $H_{\eta_X \circ a}$ of 3.2.4. First we prove that the image of $H_{\eta_X \circ a}$ lies in $X \subseteq_{\neg\neg} \Sigma^{\Sigma^X}$. By 3.2.2 it suffices to show $H_{\eta_X \circ a}(step\ n) \in X$ for any $n{\in}N$. But due to 3.2.4 we have that $H_{\eta_X \circ a}(step\ n) = \eta_X(a(n))$ and omitting the inclusion map η_X we get $a(n) \in X$. So let $\overline{a} = H_{\eta_X \circ a}|_{\omega}$. This proves also that $\overline{a}(step\, n) = a\, n$ and uniqueness follows from 3.3.1.

(ii) The same as (i) only that we don't have to restrict $H_{\eta_X \circ a}$ to ω.

(iii) We have to prove that for any $f{:}\omega \to X$ there exists a unique $\overline{f}{:}\overline{\omega} \to X$ with $\overline{f} \circ inc = f$. Let $f{:}\omega \to X$ then $a := f \circ step$. a is monotone since $\forall x{:}N.\, step\, n\, x \sqsubseteq step\,(n+1)\, x$. By 3.3.2(ii) there exists a unique $\overline{a}$ such that $\overline{a}(step\, n) = f(step\, n)$, i.e. by 3.3.1 we get $\overline{a}|_{\omega} = f$, which is equivalent to $\overline{a} \circ inc = f$. So $\overline{f}$ is $\overline{a}$. $\square$

THEOREM 3.3.3 X is a Σ-cpo iff X is a Σ-poset and $X^{inc} : X^{\overline{\omega}} \to X^{\omega}$ is an isomorphism.

PROOF: "$\Rightarrow$": 3.3.2(iii).

"$\Leftarrow$": By 3.2.5 we only have to show that $sup\, a \in X$ for any $a{\in}AC(X)$. Define $sup\ a := (X^{inc^{-1}} \circ \overline{a})\,(\lambda n{:}N.\top)$ where $\overline{a}$ is the function we get by 3.3.2(i). Let us verify that $sup\, a$ really is the supremum: Therefore assume $P{\in}\Sigma^X$. $\exists n{:}N.\, P(a\, n) \iff \exists n{:}N.\, P(\overline{a}(step\, n)) \iff \exists n{:}N.\, P((X^{inc^{-1}} \circ \overline{a})(step\, n)) \iff P((X^{inc^{-1}} \circ \overline{a})(\lambda n{:}N.\top))$ since "$\Leftarrow$"is proved by 2.2.6 and "$\Rightarrow$"is trivial. $\Box$

3.4 Closure properties of Σ-posets and Σ-cpos

In this section some important closure properties of Σ-cpos are derived.

LEMMA 3.4.1 Let X be a type and $(A_x)_{x\in X}$, $(B_x)_{x\in X}$ families of Σ-posets. If $A_x \subseteq_{\neg\neg} B_x$ for all $x{\in}X$ then $(\Pi x{:}X.\, A_x) \subseteq_{\neg\neg} (\Pi x{:}X.\, B_x)$.

PROOF: $(\lambda f{:}(\Pi x{:}X.\, B_x).\ \forall x{:}X.\, f(x) \in A_x)$ is a $\neg\neg$-closed predicate. $\Box$

THEOREM 3.4.2 Σ-posets are closed under (i) equalizers and (ii) arbitrary products.

PROOF: (i): Suppose A, B are Σ-posets and $f, g{:}A \to B$. Since $f(a) = g(a)$ is $\neg\neg$-closed (due to 3.1.7) we conclude by the Representation Theorem 3.1.5 that $\{a \in A \mid f(a) = g(a)\}$ is a Σ-poset.

(ii): Assume $(A_x)_{x\in X}$ is a family of Σ-posets. As $A_x \subseteq_{\neg\neg} \Sigma^{\Sigma^{A_x}}$ by 3.1.2 we conclude by 3.4.1 that $\Pi x{:}X.\, A_x \subseteq_{\neg\neg} \Pi x{:}X.\, \Sigma^{\Sigma^{A_x}} \cong \sum x{:}X.\, \Sigma^{A_x} \to \Sigma$. By 3.1.5 we are done. $\Box$

LEMMA 3.4.3 The $\sqsubseteq$ ordering on products of Σ-posets is pointwise.

PROOF: Let A_x be a family of Σ-posets and $f, g : \Pi x{:}X.\, A_x$.
"$\Rightarrow$": Assume $f \sqsubseteq g$ and let $x{\in}X$. Define $Q := \lambda h{:}(\Pi x{:}X.\, A_x.)\ P(h\, x)$. Now by assumption we know that $Q(a) \Rightarrow Q(b)$ i.e. $P(a\, x) \Rightarrow P(b\, x)$.
"$\Leftarrow$": Assume that for all $x \in X$ we have $f\, x \sqsubseteq g\, x$. Then due to the coincidence of the inequalities (3.1.4) we have $\forall x{:}X.\, \exists! f{:}\Sigma \to A_x.\, f\, \bot = a\, x \wedge f\, \top = b\, x$. By AC! there exists an $F{:}(\Pi x{:}X.\, (\Sigma \to A_x))$ such that $\forall x{:}X.\, F\, x\, \bot = a\, x \wedge F\, x\, \top = b\, x$. Then for $G = \lambda s{:}\Sigma.\, \lambda x{:}X.\, F\, x\, s$ we have $G\, \bot = a \wedge G\, \top = b$. Thus by 3.1.4 we have $a \sqsubseteq b$. $\Box$

THEOREM 3.4.4 Σ-cpos are closed under (i) equalizers and (ii) arbitrary products where the supremum in products is componentwise.

PROOF: (i): Suppose A, B are Σ-cpos and $f, g : A \to B$. Define $\overline{A} := \{a \in A \mid f(a) = g(a)\}$. As $\overline{A}$ is a $\neg\neg$-closed subobject of A and A is a $\neg\neg$-closed subobject of Σ^X for some X (3.4.2,3.1.5) it is sufficient to show that $\overline{A}$ is closed under taking suprema in Σ^X (cf. 3.2.5) i.e. we have to prove that for any $a{:}AC(\overline{A})$ $(\lambda x{:}X.\, \exists n{:}N.\, a\, n\, x) \in \overline{A}$. Since A is a Σ-cpo it suffices to show $sup\,_A a \in \overline{A}$, i.e. $f(sup\, a) = g(sup\, a) \overset{2.2.7}{\Leftrightarrow} sup\,(f \circ a) = sup\,(g \circ a) \Leftarrow f \circ a = g \circ a \overset{EXT}{\Leftarrow} \forall n{:}N.\ f(a\, n) = g(a\, n) \Leftarrow a \in AC(\overline{A})$.

(ii): Assume X is a type and $(A_x)_{x\in X}$ is a family of Σ-cpos. Due to 3.4.2(ii) we know that $(\Pi x{:}X.\, A_x) \subseteq_{\neg\neg} \sum x{:}X.\, \Sigma^{A_x} \to \Sigma$. By the Representation Theorem 3.2.5 we only must prove that $sup_{\sum x:X.\,\Sigma^{A_x}} \to \Sigma(a) \in \Pi x{:}X.\, A_x$ for any $a{\in}AC(\Pi x{:}X.\, A_x)$. But $sup_{\sum x:X.\,\Sigma^{A_x} \to \Sigma}(a) = \lambda p{:}(\sum x{:}X.\, \Sigma^{A_x}).\, \exists n{:}N.\ (a\, n\, p) = \lambda x{:}X.\, \lambda y{:}\Sigma^{A_x}.\, \exists n{:}N.\ a\, n\, x\, y = (\lambda x{:}X.\, sup\,_{A_x}(\lambda n{:}N.\, a\, n\, x)) \in \Pi x{:}X.\, A_x$ since any A_x is a Σ-cpo. $\Box$

4 Replete Objects as Predomains

Now we drop MP and define predomains as the socalled Σ-replete objects due to the ideas of [Hyland 91] and [Taylor 91]. This means that we are going to prove the orthogonality theorem (3.3.3) with more categorical techniques defining our domains as the least full, reflective subcategory containing Σ of the ambient category. Notice that we have quotients and disjoint sums available in our language.

4.1 About Factorization Systems

DEFINITION 4.1.1 A map e is called Σ-epi iff Σ^e is a mono where $A^f(g) = g \circ f$, i.e. e is Σ-epi iff $\forall f, g{:}B \to \Sigma.\ f \circ e = g \circ e \Rightarrow f = g$.

A map e is called Σ-equable iff Σ^e is an iso.

DEFINITION 4.1.2 Let $e : A \to B$ and $m : C \to D$ be morphisms. $e \perp m$ (e orthogonal to m) iff for all $f{:}A \to C$ and $g{:}B \to D$ such that $g \circ e = m \circ f$ there exists a unique function $h : B \to C$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

DEFINITION 4.1.3 m is called extremal mono iff $e \perp m$ for all Σ-epi e.

DEFINITION 4.1.4 A pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$ is called a factorization system iff

1. both $\mathcal{M}$ and $\mathcal{E}$ are closed under composition and contain all isos.
2. Any map f has a $(\mathcal{E}, \mathcal{M})$-factorization, i.e. $f = m \circ e$ for $m \in \mathcal{M}$ and $e \in \mathcal{E}$.
3. $\forall e{:}\mathcal{E}.\ \forall m{:}\mathcal{M}.\ e \perp m$ (e orthogonal m)

It is easy to verify that Σ-epis are closed under composition and contain all isos. Moreover, one can show that an extremal mono is really a mono (as all epis are Σ-epis).

LEMMA 4.1.1 For any $f{:}X \to Y$ there exists a Σ-epi e and a mono m such that $f = m \circ e$ and such that for any other such factorization (e', m') there exists a (unique) h such that the following diagram commutes.

$$\begin{tikzcd} & Z \arrow[dr, "m"] & \\ X \arrow[ur, "e"] \arrow[dr, "e'"'] & & Y \\ & Z' \arrow[ur, "m'"'] \arrow[uu, dashed, "\exists! h"] & \end{tikzcd}$$

This factorization is obviously unique (up to isomorphism) and is called *maximal.*

The maximal factorization has the universal property also with respect to factorizations $f = g \circ e$ where e is Σ-epi and g is arbitrary.

PROOF: Let $f : X \to Y$ then define

$$Z(f) :=_{def} \{P{\in}\mathcal{P}(Y) \mid \forall x{:}X.\, P(f\,x) \;\wedge\; \forall p, q : \Sigma^{\{y:Y \mid P(y)\}}.\, (\forall x{:}X.\, p(f\,x) = q(f\,x)) \Rightarrow p = q\}$$

and

$$Z :=_{def} \{y{\in}Y \mid \exists P{:}Z(f).\, P(y)\} \text{ i.e. } Z = \bigcup Z(f).$$

Now let m be the inclusion of Z into Y and $e{:}X \to Z$ is defined as $e(x) := f(x)$ for all $x{\in}X$. Obviously $f = m \circ e$. Moreover, e is Σ-epi. To verify this assume $p(e(x)) = q(e(x))$ for all $x{\in}X$ and $p, q{\in}Z \to \Sigma$. We must prove that $p(z) = q(z)$ for an arbitrary $z{\in}Z$. However, by definition of Z there is a $P{\in}Z(f)$ such that $P(z)$. So it suffices to prove $p|_{\{y\in Y \mid P(y)\}} = q|_{\{y\in Y \mid P(y)\}}$. But this follows from the definition of $Z(f)$ since $\forall x{:}X.\, p|_{\{y\in Y \mid P(y)\}}(x) = q|_{\{y\in Y \mid P(y)\}}(x)$ by assumption. Now assume there exist a Σ-epi $e'{:}X \to Z'$ and a mono $m'{:}Z' \to Y$ such that $f = m' \circ e'$. Since $Im(m') \in Z(f)$ we have $Z' \cong Im(m') \subseteq Z$ and we can take for h the inclusion $Im(m')$ into Z. h is obviously unique since m is a mono.
For the more general case where $f = g \circ e'$ and g arbitrary we take the epi-mono factorization of $g = m'' \circ e''$ and apply the lemma to the factorization $f = m'' \circ (e'' \circ e')$ such that we get a unique h with $h \circ e'' \circ e' = e$ and $m \circ h = m''$. It is then easily verified that the unique mediating arrow is $h \circ e''$. □

LEMMA 4.1.2 Let $m : A \to B$ be mono, then
m is extremal iff $\forall e{:}A \to Y.\, \forall f{:}Y \to B.\, f \circ e = m \wedge e\,\Sigma\text{-epi} \Rightarrow \exists e'{:}Y \to A.\ m \circ e' = f$.

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "id"'] & Y \arrow[d, "f"] \arrow[dl, dashed, "e'"'] \\
A \arrow[r, "m"] & B
\end{tikzcd}$$

PROOF: "⇒": The right hand side condition is a special case of orthogonality.

"⇐": Let f', e' be the pushout cocone of f, e in the following diagram. Assume $g \circ e = m \circ f$.

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \arrow[ddr, "g"] & \\
A \arrow[r, shift right, "e'"'] \arrow[drr, "m"'] & C \arrow[l, dashed, shift right, "e''"'] \arrow[dr, "u"] & \\
 & & B
\end{tikzcd}$$

Since pushouts preserve Σ-epis and e is a Σ-epi, e' is a Σ-epi and therefore there exists by assumption an $e''{:}C \to A$ such that $e'' \circ e' = id$ and $m \circ e'' = u$. Define $h := e'' \circ f'$. We must prove that h is the diagonal fill in: $h \circ e = e'' \circ f' \circ e = e'' \circ e' \circ f = f$ and $m \circ h = m \circ e'' \circ f' = u \circ f' = g$. The morphism h is unique since m is a mono. □

Lemma 4.1.3 If $f = m \circ e$ is the maximal factorization for f then m is an extremal mono.

Proof: Let $(e{:}X \to Z, m)$ be the maximal factorization for $f{:}X \to Y$. By 4.1.2 it is sufficient to show that for any g and $e' : Z \to Y'$ Σ-epi and $m = g \circ e'$ there exists an e'' such that $m \circ e'' = g$.

$$\begin{tikzcd}
 & & Y' \arrow[dll, dashed, bend right, "e''"'] \arrow[dr, "g"] & \\
X \arrow[r, "e"'] & Z \arrow[ur, "e'"] \arrow[rr, "m"'] & & Y
\end{tikzcd}$$

Due to 4.1.1 there exists a unique function $e'' : Y' \to Z$, such that the above diagram commutes, i.e. $m \circ e'' = g$. $\square$

Corollary 4.1.4 Any map factors as a Σ-epi followed by an extremal mono.

Proof: 4.1.1 and 4.1.3. $\square$

Theorem 4.1.5 Σ-epis and extremal monos form a factorization system.

Proof: Let us check the conditions of Definition 4.1.4. 1. It is an easy exercise to prove that extremal monos and Σ-epis are closed under composition and contain all isos. 2. By 4.1.4 any map factors as required. 3. By Definition of extremal mono. $\square$

Remark: The Σ-epi-extremal-mono factorization is unique up to isomorphism by definition of extremal mono.

4.2 The Replete Objects

Definition 4.2.1 (Hyland) A is called Σ-replete iff for any Σ-equable map f the map A^f is an iso, where $A^f(g) = g \circ f$.

Remarks:

1. If A is Σ-replete and e is Σ-equable then we have that $f \circ e = A^e(f) = A^e(g) = g \circ e \Rightarrow f = g$ since A^e is an iso.
2. Σ^X is Σ-replete for any X because for any Σ-equable $f{:}A \to B$ we have that $(\Sigma^X)^f : (\Sigma^X)^B \to (\Sigma^X)^A$ is an iso iff $(\Sigma^f)^X : (\Sigma^B)^X \to (\Sigma^A)^X$ is an iso if $\Sigma^f : \Sigma^B \to \Sigma^A$ is an iso if f is Σ-equable.

Lemma 4.2.1 $f{:}X \to Y$ is Σ-equable iff f is a Σ-epi and for some $g{:}Y \to \Sigma^{\Sigma^X}$ we have $g \circ f = \eta_X$.

Proof: "$\Rightarrow$": f Σ-equable implies f Σ-epi. For g choose $\Sigma^{(\Sigma^f)^{-1}} \circ \eta_Y$ since Σ^f is an iso by assumption.
"$\Leftarrow$": We must prove that any function $p : X \to \Sigma$ can be uniquely extended to a function $\overline{p} : Y \to \Sigma$ such that $\overline{p} \circ f = p$.
Uniqueness follows from the fact that f is a Σ-epi. Define $\overline{p}(y) :=_{def} (g\,y)\,p$. But now $\overline{p}(f(x)) = (g(f(x)))\,p = (\eta_X\,x)\,p = p(x)$. Thus $\overline{p} \circ f = p$. $\square$

Theorem 4.2.2 X is Σ-replete iff η_X is an extremal mono.

Proof: "⇒": Let X be Σ-replete. We factorize $\eta_X = m_X \circ e_X$ where $e_X{:}X \to R(X)$ is Σ-epi and $m_X{:}R(X) \to \Sigma^{\Sigma^X}$ an extremal mono (by 4.1.5).

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dr, "\eta_X"'] & R(X) \arrow[d, "m_X"] \\
 & \Sigma^{\Sigma^X}
\end{tikzcd}$$

By 4.2.1 we know that e_X is Σ-equable therefore $id_X{:}X \to X$ extends uniquely to a function $i{:}R(X) \to X$ such that $i \circ e_X = id_X$.

$$\begin{tikzcd}
X \arrow[rr, "e_X"] \arrow[dr, "id_X"'] & & R(X) \arrow[dl, dashed, "i"] \\
 & X &
\end{tikzcd}$$

Moreover, this implies $\eta_X = m_X \circ e_X = m_X \circ e_X \circ i \circ e_X$. As Σ^{Σ^X} is Σ-replete and since e_X is Σ-epi we get $m_X = m_X \circ e_X \circ i$ and since m_X is a mono $e_X \circ i = id_{R(X)}$, Thus e_X is an iso and therefore η_X is an extremal mono since m_X is extremal.

"⇐": Suppose η_X is an extremal mono. Let $f{:}Y \to Z$ be Σ-equable and $g{:}Y \to X$.

$$\begin{tikzcd}
 & Z \arrow[d, "h"] \arrow[dl, dashed] \\
Y \arrow[ur, "f"] \arrow[r, "\eta_Y"] \arrow[d, "g"'] & \Sigma^{\Sigma^Y} \arrow[d, "\Sigma^{\Sigma^g}"] \\
X \arrow[r, "\eta_X"] & \Sigma^{\Sigma^X}
\end{tikzcd}$$

By 4.2.1 there exists a $h{:}Z \to \Sigma^{\Sigma^Y}$ such that $\eta_Y = h \circ f$. As f is Σ-equable and therefore Σ-epi and η_X is an extremal mono by 4.1.5 and Definition 4.1.2 there exists a unique $\overline{g}{:}Z \to X$ such that $\overline{g} \circ f = g$ and $\eta_X \circ \overline{g} = \Sigma^{\Sigma^g} \circ h$. To show that X^f is an iso we must only prove that this $\overline{g}$ is unique ! Therefore, suppose that for some $g'{:}Z \to X$ we have that $g' \circ f = g$. Then $\eta_X \circ g' \circ f = \eta_X \circ g = \Sigma^{\Sigma^g} \circ \eta_Y = \Sigma^{\Sigma^g} \circ h \circ f$. Since f is Σ-equable and Σ^{Σ^X} is Σ-replete we get $\eta_X \circ g' = \Sigma^{\Sigma^g} \circ h = \eta_X \circ \overline{g}$. Since η_X is a mono we finally get $g' = \overline{g}$. □

Lemma 4.2.3 For any X the object $R(X)$ is defined as the intermediate object in the Σ-epi-extremal-mono factorization of η_X, i.e. if $\eta_X = e \circ m$ with $e{:}X \to Z$ then $R(X) := Z$. $R(X)$ is Σ-replete for any X.

Proof:

$$\begin{tikzcd}
X \arrow[rr, "e_X"] \arrow[d, "\eta_X"'] & & R(X) \arrow[d, "\eta_{R(X)}"] \arrow[dll, "m_X"'] \\
\Sigma^{\Sigma^X} \arrow[rr, "\sim", "\Sigma^{\Sigma^{(e_X)}}"'] & & \Sigma^{\Sigma^{R(X)}}
\end{tikzcd}$$

We know that the square and the upper triangle commute and that $\Sigma^{\Sigma^{(e_X)}}$ is an iso since e_X is Σ-equable by 4.2.1. If we can show that the lower triangle commutes then we know that $\eta_{R(X)} = \Sigma^{\Sigma^{e_X}} \circ m_X$ and since $\Sigma^{\Sigma^{e_X}}$ is an iso and m_X is an extremal mono $\eta_{R(X)}$ is an extremal mono too.
Now $\Sigma^{\Sigma^{e_X}} \circ m_X \circ e_X = \Sigma^{\Sigma^{e_X}} \circ \eta_X = \eta_{R(X)} \circ e_X$. Since $\Sigma^{\Sigma^{R(X)}}$ is Σ-replete and e_X is Σ-equable we get $\Sigma^{\Sigma^{e_X}} \circ m_X = \eta_{R(X)}$. □

Theorem 4.2.4 A is Σ-replete iff any Σ-equable $e{:}A \to B$ is a split mono and η_A is a mono.

Proof: "⇒": If A is Σ-replete then η_A is extremal mono by 4.2.2. If $f{:}A \to B$ is Σ-equable then by 4.2.1 there is a g with $\eta_A = g \circ e$ and e is a Σ-epi. But η_A is extremal so there exists an $e' : B \to A$ such that $e' \circ e = id$, so e is a split mono.
"⇐": Suppose $\eta_A = g \circ e$ with e Σ-epi so by 4.2.1 it is Σ-equable and by assumption it is a split mono, i.e. $e' \circ e = id$. Since therefore $\eta \circ e' \circ e = g \circ e$ holds we have that $\eta_A \circ e' = g$ since e is Σ-epi and by 4.1.2 we can conclude that η_A is extremal and thus by 4.2.2 A is Σ-replete. □

Lemma 4.2.5 If an extremal mono $m : A' \to A$ factors into a Σ-epi $e : A' \to A''$ and $f : A'' \to A$ and if $\eta_{A''}$ is mono then e is an iso.

Proof: By 4.1.2 we already have that there is a $p : A'' \to A'$ such that $p \circ e = id_{A'}$. This implies $e \circ p \circ e = e \Rightarrow \eta_{A''} \circ e \circ p \circ e = \eta_{A''} \circ e$. But since e is Σ-epi and $\Sigma^{\Sigma^{A''}}$ Σ-replete we get $\eta_{A''} \circ e \circ p = \eta_{A''}$ and since $\eta_{A''}$ is monic we arrive at $e \circ p = id_{A''}$.□

Lemma 4.2.6 $m : X \to Z$ is an extremal mono iff e is an iso whenever $m = m' \circ e$ where $m' : Y \to Z$ mono and $e : X \to Y$ Σ-epi.

Proof: "⇒": By 4.1.2 we already have that there is a $p : Y \to X$ such $m' = m \circ p$. This implies $m' \circ e \circ p = m \circ p = m'$. But since m' is a mono we get $e \circ p = id_Y$. Moreover $m' \circ e = m$ implies $m \circ p \circ e = m$ and since m is mono this gives us $p \circ e = id_X$.

"⇐": Let $m = m' \circ e$ where m' extremal mono and e Σ-epi. This factorization exists due to 4.1.5. But by assumption e is an iso, so m is extremal as m' is. □

Corollary 4.2.7 (Taylor) X is Σ-replete iff for all Σ-equable $f : X \to Y$ with η_Y a mono, i.e. Y fulfills the weak Leibniz property, f is an iso.

Proof: "⇒": By 4.2.2 we know that η_X is an extremal mono and since f is Σ-equable there is a $g : Y \to \Sigma^{\Sigma^X}$ such that $\eta_X = g \circ f$. By 4.2.6 we can conclude that f is an iso.

"⇐": Consider the Σ-replete object $R(X)$ in the extremal factorization of η_X, i.e. $X \overset{e_X}{\to} R(X) \overset{m_X}{\to} \Sigma^{\Sigma^X}$. Then e_X is Σ-equable and by assumption it is an iso. But then η_X is an extremal mono since m_X is one and by 4.2.2 X is Σ-replete. □

Corollary 4.2.8 Any extremal subobject $X \overset{m}{\to} A$ of a Σ-replete A is Σ-replete.

Proof: Let $e : X \to Y$ be Σ-equable then there exists a unique $g : Y \to A$ such that $m = g \circ e$ since A is Σ-replete. If we assume that Y fulfills the weak Leibniz property then by 4.2.5 e is an iso. Thus by 4.2.7 X is Σ-replete. □

Theorem 4.2.9 Let $\iota : 2 \to \Sigma$ denote the obvious inclusion. Then the relations $\sqsubseteq$ and $\sqsubseteq_{link}$ coincide on X iff $\iota \perp \eta_X$.

Proof: "$\Rightarrow$": Consider the following diagram.

$$\begin{tikzcd}
2 \arrow[r, "\iota"] \arrow[d, "g"'] & \Sigma \arrow[d, "f"] \arrow[dl, dashed, "h"'] \\
X \arrow[r, "\eta_X"'] & \Sigma^{\Sigma^X}
\end{tikzcd}$$

Let $h(\bot) = g(0)$ and $h(\top) = g(1)$. Then by definition $h \circ \iota = g$. Moreover, $\eta_X(h(\bot))\eta_X(g(0)) = f(\iota(0)) = f(\bot)$. The same equation holds for $\top$ so we also get $\eta_X \circ h = f$.monotonic Uniqueness follows from the fact that η_X is a mono.

"$\Leftarrow$": Assume $x_1 \sqsubseteq x_2$. Then by monotonicity $\eta_X(x_1) \sqsubseteq \eta_X(x_2)$ and by 2.2.3 $\eta_X(x_1) \sqsubseteq_{link} \eta_X(x_2)$. So we have a $f{:}\Sigma \to \Sigma^{\Sigma^X}$ such that $f(\bot) = \eta_X(x_1)$ and $f(\top) = \eta_X(x_2)$.

$$\begin{tikzcd}
2 \arrow[r, "\iota"] \arrow[d, "{[x_1, x_2]}"'] & \Sigma \arrow[d, "f"] \arrow[dl, dashed, "g"'] \\
X \arrow[r, "\eta_X"'] & \Sigma^{\Sigma^X}
\end{tikzcd}$$

Now $f \circ \iota = \eta_X \circ [x_1, x_2]$. As $\iota : 2 \to \Sigma$ is a Σ-epi and X is Σ-replete we know by assumption that there exists a unique $g{:}\Sigma \to X$ such that $g(\bot) = g(\iota(0)) = [x_1, x_2](0) = x_1$ and $g(\top) = g(\iota(1)) = [x_1, x_2](1) = x_2$. Thus $x_1 \sqsubseteq_{link} x_2$.
The other direction holds by 2.2.2. □

4.3 Chain Completeness of Replete Objects

First we will show that $step{:}N \to \omega$ is a Σ-epi. for this purpose we need some auxiliary definitions.

Definition 4.3.1 Recall the definitions of ω and $\overline{\omega}$ from Definition 2.2.2.

$\underline{n} := \{m{:}N \mid m < n\}$

$\hat{n} := \{f{:}\overline{\omega} \mid f(n) = \bot\}$

$\iota{:}2 \to \Sigma$ with $\iota(0) = \bot$ and $\iota(1) = \top$

$e_2(n) : \underline{n+1} \to 2^{\underline{n}}$
$p_2(n) : 2^{\underline{n}} \to \underline{n+1}$
$e_\Sigma(n) : \hat{n} \to \Sigma^{\underline{n}}$
$p_\Sigma(n) : \Sigma^{\underline{n}} \to \hat{n}$ defined as follows

$$
\begin{aligned}
&e_2(n)(m)(k) := \begin{cases} 1 & \text{if } k < m \\ 0 & \text{otherwise} \end{cases} \\
&p_2(n)(s) := \text{the least } m \text{ such that } s(m) = 0 \vee m \geq n \\
&e_\Sigma(n)(f) := f|_{\underline{n}} \\
&p_\Sigma(n)(s)(m) := \exists k{:}N.\, m \leq k < n \wedge s(k)
\end{aligned}
$$

LEMMA 4.3.1 For all $n{\in}N$ it holds that

(i) $\iota^{\underline{n}} : 2^{\underline{n}} \to \Sigma^{\underline{n}}$ is a Σ-epi.

(ii) $p_2(n) \circ e_2(n) = id_{\underline{n+1}}$ and $p_\Sigma(n) \circ e_\Sigma(n) = id_{\hat{n}}$

(iii) $step \circ p_2(n) = p_\Sigma(n) \circ \iota^{\underline{n}}$

(iv) $step : \underline{n+1} \to \hat{n}$ is a Σ-epi.

PROOF: (i) by induction on n using PHOA2. (ii),(iii) by straightforward computation. (iv): Suppose $f, g : \hat{n} \to \Sigma$ with $f \circ step = g \circ step$. Then by (iii) we have $f \circ p_\Sigma(n) \circ \iota^{\underline{n}} = f \circ step \circ p_2(n) = g \circ step \circ p_2(n) = g \circ p_\Sigma(n) \circ \iota^{\underline{n}}$. By (i) the morphism $\iota^{\underline{n}}$ is a Σ-epi and therefore $f \circ p_\Sigma(n) = g \circ p_\Sigma(n)$. As by (ii) $p_\Sigma(n)$ is a split epi by we have $f = g$. $\Box$

THEOREM 4.3.2 The map $step : N \to \omega$ is a Σ-epi.

PROOF: Suppose $F, G : \omega \to \Sigma$ with $F(step\, m) = G(step\, m)$ for all $m{\in}N$. Thus we have for all $n, m{\in}N$ with $m \leq n$ that $F|_{\hat{n}}(step\, m) = G|_{\hat{n}}(step\, m)$ and by 4.3.1(iv) we get $\forall n{:}N.\, \forall f{:}\omega.\, F|_{\hat{n}}(f) = G|_{\hat{n}}(f)$ which entails $\forall f{:}\omega.\, F(f) = G(f)$ as $\omega = \{f{:}\overline{\omega} \mid \exists n{:}N.\, f \in \hat{n}\}$. $\Box$

Next we show that ω is dense in $\overline{\omega}$ and therefore the *step* functions are dense also in $\overline{\omega}$.

THEOREM 4.3.3 The inclusion $inc : \omega \to \overline{\omega}$ is a Σ-epi.

PROOF: Suppose that $F, G : \overline{\omega} \to \Sigma$ and $F(h) = G(h)$ for all $h{\in}\omega$. Then we have to show that $\forall f{:}\overline{\omega}.\, F(f) \Leftrightarrow G(f)$. But if we let $\cap$ denote the pointwise $\wedge$, then $F(f) \Leftrightarrow F(f \cap \lambda n{:}N.\, \top) \Leftrightarrow$ (2.2.6) $\exists n{:}N.\, F(f \cap step\, n)$ which by assumption is equivalent to $\exists n{:}N.\, G(f \cap step\, n) \Leftrightarrow G(f \cap \lambda n{:}N.\, \top)$ $\Leftrightarrow G(f)$. $\Box$

COROLLARY 4.3.4 The map $step : N \to \overline{\omega}$ is a Σ-epi.

PROOF: By 4.3.2 and 4.3.3 $\Box$

LEMMA 4.3.5 Any monotonic $f : N \to \Sigma$ extends to a unique $\overline{f} : \overline{\omega} \to \Sigma$ with $\overline{f} \circ step = f$.

PROOF: Uniqueness follows from 4.3.4.

We define $\overline{f}$ in terms of f. $\overline{f} :=_{def} \lambda s{:}\overline{\omega}.\, (\exists m{:}N.\, s(m) \wedge f(m+1)) \vee f(0)$. The condition $\overline{f} \circ step = f$ holds since for any $n{\in}N$ we have $\overline{f}(step\, n) = (\exists m{:}N.\, step\, n\, m\ \wedge f(m+1)) \vee f(0) = (\exists m{:}N.\, (m < n) \wedge f(m+1)) \vee f(0) = f(n)$ since f is monotonic. $\Box$

COROLLARY 4.3.6 Any monotonic $f : N \to \Sigma$ extends to a unique $\overline{f} : \omega \to \Sigma$ with $\overline{f} \circ step = f$.

PROOF: Extend F to $\overline{\omega}$ by 4.3.5 and then restrict to ω. Uniqueness follows from 4.3.2. $\Box$

Lemma 4.3.7 For any monotonic $f : N \to \Sigma^X$ there exists a unique $\overline{f} : \overline{\varpi} \to \Sigma^X$ such that $\overline{f} \circ \mathit{step} = f$.

Proof: By using 4.3.5 pointwise on Σ^X. $\sqsubseteq$ and pointwise inequality coincide due to 2.2.3. $\Box$

Theorem 4.3.8 Let X be Σ-replete and $f : N \to X$ be monotonic, then there exists a unique $\overline{f} : \overline{\varpi} \to X$ such that $\overline{f} \circ \mathit{step} = f$.

Proof:

$$\begin{tikzcd}
N \arrow[r, "\mathit{step}"] \arrow[d, "f"'] & \overline{\varpi} \arrow[dl, dashed, "\overline{f}"'] \arrow[d, dashed, "\tilde{f}"] \\
X \arrow[r, "\eta_X"'] & \Sigma^{\Sigma^X}
\end{tikzcd}$$

Since f is monotonic $\eta_X \circ f$ is monotonic. By 4.3.7 an $\tilde{f} : \overline{\varpi} \to \Sigma^{\Sigma^X}$ exists such that the rectangle above commutes. Now the unique $\overline{f}$ exists by orthogonality since step is a Σ-epi (4.3.4) and η_X is an extremal mono since X is Σ-replete (4.2.2). $\Box$

Theorem 4.3.9 Let $f \in AC(X)$ where X is Σ-replete, then for the unique $\overline{f}$ with $\overline{f} \circ \mathit{step} = f$ we have $\overline{f}(\lambda n{:}N.\, \top)) = \mathit{sup}\ f$, i.e. any Σ-replete object is chain complete.

Proof: Assume an $P{:}\Sigma^X$ then we must prove: $P(\overline{f}(\lambda n{:}N.\, \top)) \iff \exists m{:}N.\, P(f(m))$. Now define $H := P \circ f$ and use 2.2.6 for the "$\Rightarrow$"-direction. The other direction holds by monotonicity. $\Box$

4.4 Closure Properties of Replete Objects

Theorem 4.4.1 Σ-replete objects are closed under arbitrary products.

Proof: Let $e : X \to Y$ be an Σ-equable map and $(A_i)_{i \in I}$ be a family of Σ-replete objects. Assume $g : X \to \Pi i{:}I.\, A_i$ and $g_i = \pi_i \circ g$. Since any A_i is Σ-replete there exist unique $h_i : Y \to A_i$ such that $h_i \circ f = g_i$ and therefore $\pi_i \circ g = \pi_i \circ (h_i \circ f) = (\pi_i \circ h_i) \circ f$. Uniqueness of $(h_i)_{i \in I}$ follows by the uniqueness of any h_i. $\Box$

Theorem 4.4.2 Σ-replete objects are closed under equalizers.

Proof: Assume E is the equalizer of $g_1, g_2 : A \to B$. Let $f{:}X \to Y$ be Σ-equable and $k{:}X \to E$. To show that E is Σ-replete we must find a unique $h{:}Y \to E$ such that $h \circ f = k$. Now since A is Σ-replete there exists a $k'{:}Y \to A$ such that $k' \circ f = e \circ k$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "k"'] & Y \arrow[dl, dashed, "h"'] \arrow[d, "k'"] & \\
E \arrow[r, "e"'] & A \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & B
\end{tikzcd}$$

k' equalizes g_1 and g_2 since for $i = 1, 2$ we have that $g_i \circ k' \circ f = g_i \circ e \circ k$ and therefore $g_1 \circ k' = g_2 \circ k'$ as B is Σ-replete. Thus, there exists a unique $h{:}Y \to E$ such that $e \circ h = k'$. But now we get $e \circ k = k' \circ f = e \circ h \circ f$ which implies $k = h \circ f$ since e is a mono as it is an equalizer. $\Box$

4.5 Characterizing Repletion by Initiality

Definition 4.5.1 Let *PreDom* denote the category of replete objects.

If the category of replete objects *PreDom* is small and internally complete (cf. [Robinson 89]) then we can construct the *weakly initial* object $W(X)$ in $X \downarrow PreDom$. Then by the fact that the repletion $R(X)$ of X is initial in $X \downarrow PreDom$ we get three alternative explicit characterizations of $R(X)$.

Theorem 4.5.1 Let $X \stackrel{e_X}{\to} R(X) \stackrel{m_X}{\to} \Sigma^{\Sigma^X}$ be the repletion of X. Then $(R(X), e_X)$ is initial in $X \downarrow PreDom$.

Proof: Let $X \stackrel{e_X}{\to} R(X) \stackrel{m_X}{\to} \Sigma^{\Sigma^X}$ as in 4.2.3 and $f \in X \downarrow PreDom$. We must find a unique $h : R(X) \to A$ such that $f = h \circ e_X$. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dd, "f"'] & R(X) \arrow[r, "m_X"] \arrow[ddl, dashed, "\exists! h"'] & \Sigma^{\Sigma^X} \arrow[dd, "\Sigma^{\Sigma^f}"] \\
& & \\
A \arrow[rr, "\eta_A"'] & & \Sigma^{\Sigma^A}
\end{tikzcd}$$

Thus we have that η_A is extremal as A is Σ-replete and e_X is Σ-epi. By definition of extremal mono there exists a $h : R(X) \to A$. Uniqueness follows from the fact that η_A is a mono. □

Definition 4.5.2 For any X we define

- $W(X) :=_{def} \Pi_{A \in PreDom} A^{A^X}$
- $\epsilon_X \;:=_{def} \lambda x{:}X.\, \lambda A{:}PreDom.\, \lambda f{:}A^X.\, f(x) \in X \to W(X)$
- Let i_X be the unique function $R(X) \to W(X)$ given by the initiality of $R(X)$.
- $j_X :=_{def} \lambda y{:}W(X).\, y(\Sigma) \in W(X) \to \Sigma^{\Sigma^X}$
- $p_X :=_{def} \lambda y{:}W(X).\, y\, R(X)\, e_X \in W(X) \to R(X)$

Lemma 4.5.2 (X, ϵ_X) is weakly initial in $X \downarrow PreDom$.

Proof: Let $f : X \to A$. We have to find a map $\tilde{f} : W(X) \to A$ such that $\tilde{f} \circ \epsilon_X = f$. Define $\tilde{f} := \lambda y{:}W(X).\, y\, A\, f$ and we have $\tilde{f}(\epsilon_X(x)) = \epsilon_X(a)\, A\, f = f(x)$. □

Lemma 4.5.3 $p_X \circ \epsilon_X = e_X$.

Proof: $p_X(\epsilon_X(x)) = \epsilon_X(x)\, R(X)\, e_X = e_X(x)$. □

Lemma 4.5.4 $j_X \circ \epsilon_X = \eta_X$.

PROOF: $j_X(\epsilon_X(x)) = \epsilon_X(x)(\Sigma) = \lambda p{:}\Sigma^X.\, p(x) = \eta_X(x)$. □

This situation can be visualized by the following diagram:

$$\begin{tikzcd}
 & R(X) \arrow[d, shift right, "i_X"'] \\
X \arrow[ur, "e_X"] \arrow[r, "\epsilon_X"] \arrow[dr, "\eta_X"'] & W(X) \arrow[u, shift right, "p_X"'] \arrow[d, "j_X"'] \\
 & \Sigma^{\Sigma^X}
\end{tikzcd}$$

THEOREM 4.5.5 $j_X \circ i_X$ is the extremal mono of the (Σ-epi, extremal mono) factorization of η_X.

PROOF: Let m_X denote the extremal mono of the factorization. Then $m_X \circ e_X = \eta_X = j_X \circ i_X \circ e_X$ (4.5.4). Since Σ^{Σ^X} is Σ-replete and e_X is Σ-epi we get $m_X = j_X \circ i_X$. □

LEMMA 4.5.6 $p_X \circ i_X = id_{R(X)}$.

PROOF: $p_X \circ i_X = id_{R(X)}$ iff $p_X \circ i_X \circ e_X = e_X$ (since e_X is Σ-epi) iff $p_X \circ \epsilon_X = e_X$ which holds by Lemma 4.5.3. □

LEMMA 4.5.7 i_X is the equalizer of $i_X \circ p_X$ and $id_{W(X)}$.

PROOF: From Lemma 4.5.6 we know that $i_X = i_X \circ p_X \circ i_X$ and therefore i_X equalizes $i_X \circ p_X$ and $id_{W(X)}$.
Now assume there is an $f{:}Z \to W(X)$ such that $f = i_X \circ p_X \circ f$. Then for $p_X \circ f$ we have that $i_x \circ (p_X \circ f) = f$. As i_X is a mono the uniqueness for $p_X \circ f$ follows. □

DEFINITION 4.5.3 Let $e \in X \to Y$. Then let $h \in e \to e$ be an abbreviation for $h : Y \to Y$ such that $h \circ e = e$.

THEOREM 4.5.8 *(First Characterization of R(X))*
$R(X) = \{y \in W(X) \,|\, \forall h : \epsilon_X \to \epsilon_X.\, h(y) = y\}$

PROOF: By 4.5.7 we already know that $R(X) = \{y \in W(X) \,|\, (i_X \circ p_X)(y) = y\}$. Let $h_X := i_X \circ p_X \in \epsilon_X \to \epsilon_X$ and $A(X) := \{y \in W(X) \,|\, \forall h{:}\epsilon_X \to \epsilon_X.\, h(y) = y\}$. Now obviously $A(X) \subseteq R(X)$. It remains to show the other direction $R(X) \subseteq A(X)$. So let $h \in \epsilon_X \to \epsilon_X$. Since $h \circ \epsilon_X = \epsilon_X$ we have $h \circ i_x \circ e_X = \epsilon_X = i_X \circ e_X$ which implies $h \circ i_x = i_x$ since $W(X)$ is Σ-replete and e_X is Σ-epi. Thus $h \circ h_X = h \circ i_X \circ p_X = i_X \circ p_X = h_X$ (*) . For any $y \in W(X)$ consequently $h(y) = (h \circ h_X)(y) \stackrel{(*)}{=} h_X(y) = y$ holds. □

LEMMA 4.5.9 Let $m : A \to B$ be a regular mono in *PreDom* such that $A = \{y \in B \,|\, fy = gy\}$ for some $f, g : B \to C$. Then we have that $A = \{y \in B \,|\, \forall p, q{:}\Sigma^B.\, p \circ m = q \circ m \Rightarrow p(y) = q(y)\}$.

PROOF: As C is Σ-replete we know that η_C is a mono, so $A = \{y \in B \,|\, \eta_C(f\,y) = \eta_C(g\,y)\} = \{y \in B \,|\, \forall h{:}\Sigma^C.\; f\,y\,h = g\,y\,h\}$. So it remains to show for any $y \in B$ that $\forall p, q{:}\Sigma^B.\, p \circ m = q \circ m \Rightarrow p(y) = q(y)$ iff $\forall h{:}\Sigma^C.\; h\,y\,f = h\,y\,g$. Assume a $y \in B$. "$\Leftarrow$": Assume $p \circ m = q \circ m$. This implies $p(m(x)) = q(m(x))$ for any $x \in A$. But by assumption $\forall h{:}\Sigma^C.\; h\,y\,f = h\,y\,g$ iff $\exists x{:}A.\, m(x) = y$ and therefore $p(y) = q(y)$.
"$\Rightarrow$": Let $h \in \Sigma^C$ and define $f_h\,y := h\,y\,f$ and $g_h\,y := h\,y\,g$. Since m equalizes f_h and g_h we know that $f_h\,y = g_h\,y$ by assumption. Thus $h\,y\,f = h\,y\,g$. □

Theorem 4.5.10 *(Second Characterization of R(X))*
$R(X) = \{ y \in W(X) \mid \forall p, q{:}\Sigma^{W(X)}.\, p \circ \epsilon_X = q \circ \epsilon_X \Rightarrow p(y) = q(y) \}$

Proof: By 4.5.7 i_X is an equalizer in *PreDom*. Therefore by 4.5.9 we get that $R(X) = \{ y \in W(X) \mid \forall p, q{:}\Sigma^{W(X)}.\, p \circ i_X = q \circ i_X \Rightarrow p(y) = q(y) \}$. But as $p \circ i_X = q \circ i_X$ iff $p \circ i_X \circ e_X = q \circ i_X \circ e_X$ (as Σ is Σ-replete) iff $p \circ \epsilon_X = q \circ \epsilon_X$ we conclude that $R(X) = \{ y \in W(X) \mid \forall p, q{:}\Sigma^{W(X)}.\, p \circ \epsilon_X = q \circ \epsilon_X \Rightarrow p(y) = q(y) \}$. □

Now for the third characterization we are indebted to Martin Hyland and his talks on an informal meeting on synthetic domain theory in Genova in autumn 93. The idea is to consider as *locales* the limit preserving functors $F : PreDom \to PreDom$, as open sets the functor $\mathcal{O}_X \equiv (\_)^X$ and as the points of a "locale" F the natural transformations $F \stackrel{\cdot}{\to} Id$. What we will get is that repletion is soberification in this modified sense, i.e. $R(X) \cong Nat(\mathcal{O}_X, Id)$.

Lemma 4.5.11 Any natural transformation in $Nat(\mathcal{O}_X, Id)$ is the equalizer of $(\varphi\, u)\, A\, B\, f =_{def} (u\, B) \circ f^X$ and $(\psi\, u)\, A\, B\, f =_{def} f \circ (u\, A)$ such that $\psi, \varphi {\in} W(X) \to \Pi_{A,B \in PreDom} \Pi_{B^A} B^{A^X}$.

Proof: By the definition of natural transformations □

Theorem 4.5.12 $R(X) \cong Nat(\mathcal{O}_X, Id)$ for an arbitrary X.

Proof: let $\alpha(\tau) =_{def} \tau_{R(X)}(e_X) : Nat(\mathcal{O}_X, Id) \to R(X)$ and $\beta(x)_A(f) =_{def} \tilde{f}(x)$ where $\tilde{f} {\in} R(X) \to A$ is the unique map $e_x \to f$ in $X{\downarrow}PreDom$.
$\alpha(\beta(x)) = \beta(x)_{R(X)}(e_X) = \tilde{e_X}(x) = id_{R(X)}(x) = x$ since $id_{R(X)} \circ e_x = e_x$.
Now consider the diagram:

$$\begin{tikzcd}
R(X)^X \arrow[r, "\tau_{R(X)}"] \arrow[d, "\tilde{f}^X"'] & R(X) \arrow[d, "\tilde{f}"] \\
A^X \arrow[r, "\tau_A"] & A
\end{tikzcd}$$

We have that $\beta(\alpha(\tau))_A(f) = \tilde{f}(\alpha(\tau)) = \tilde{f}(\tau_{R(X)}(e_X)) =$ (naturality) $\tau_A(\tilde{f}^X(e_X)) = \tau_A(f)$ by definition of $\tilde{f}$. □

Corollary 4.5.13 *(Third Characterization of R(X))*
Let $\psi, \varphi {\in} W(X) \to \Pi_{A,B \in PreDom} \Pi_{B^A} B^{A^X}$ such that $(\varphi\, u)\, A\, B\, f =_{def} (u\, B) \circ f^X$ and $(\psi\, u)\, A\, B\, f =_{def} f \circ (u\, A)$ like in 4.5.11. Then $R(X) = \{ y \in W(X) \mid \phi(y) = \psi(y) \}$.

Proof: By Lemma 4.5.11 and Theorem 4.5.12. □

5 On the Relation of Σ-cpos and Σ-replete objects

The aim of this section is to show that a Σ-replete object is a Σ-cpo assuming MP.

Definition 5.1 Let $\mathcal{E}_{\neg\neg}$ denote the class of $\neg\neg$-dense maps e i.e.
$e{:}X \to Y \in \mathcal{E}_{\neg\neg}$ iff $\forall y{:}Y.\, \neg\neg\exists x{:}X.\, e(x) = y$.

Lemma 5.1 Any map factors into a $\neg\neg$-dense map and a $\neg\neg$-closed mono.

Proof: Let $f{:}X \to Y$. Then define $Z := \{y{:}Y \mid \neg\neg\exists x{:}X.\, y = f(x)\}$. Then $e{:}X \to Z$ is defined as $e(x) = f(x)$ and $m{:}Z \to Y$ is defined as $m(y) = y$. $\Box$

Lemma 5.2 Under Markov's principle MP any $\neg\neg$-dense map is a Σ-epi.

Proof: Let $f, g : B \to \Sigma$. Assume $e : A \to B$ is $\neg\neg$-dense, i.e. $\forall b{:}B.\, \neg\neg\exists a{:}A.\, e(a) = b$. Now we have $b = e(a) \Rightarrow f(b) = g(b)$ as $f(e(a)) = g(e(a))$. Therefore also $(\exists a{:}A.\, b = e(a)) \Rightarrow f(b) = g(b)$ and as by MP equality on Σ is $\neg\neg$-closed we have $\neg\neg(\exists a{:}A.\, b = e(a)) \Rightarrow f(b) = g(b)$. By assumption we finally get $\forall b{:}B.\, f(b) = g(b)$ and thus $f = g$ by EXT. $\Box$

Theorem 5.3 A map m: $A \to B$ is a $\neg\neg$-closed mono iff $\mathcal{E}_{\neg\neg} \bot m$.

Proof: "$\Rightarrow$": We have to find the h in the following diagram

$$\begin{tikzcd}
X \arrow[r, "e \in \mathcal{E}_{\neg\neg}"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\exists! h"'] \\
A \arrow[r, "m"'] & B
\end{tikzcd}$$

If $y = e(x)$ then $\exists! a{:}A.\, m(a) = g(y)$ as by taking $a := f(x)$ this is unique as m is a mono by assumption. Thus $\neg\neg(\exists x{:}X.\, y = e(x)) \Rightarrow \neg\neg\exists! a{:}A.\, m(a) = g(y)$. But as m is $\neg\neg$-closed and e is $\neg\neg$-dense we get $\forall y{:}Y.\, \exists! a{:}A.\, m(a) = g(y)$. By AC! there exists a function $h{:}Y \to A$ with $\forall y{:}Y.\, \exists! a{:}A.\, m(h(y)) = g(y)$. Thus $m \circ h = g$ and as m is a mono h is unique with this property and $h \circ e = f$ as $m \circ h \circ e = g \circ e = m \circ f$.

"$\Leftarrow$": If $\mathcal{E}_{\neg\neg} \bot m$ let $m = m' \circ e'$ be a factorization in a $\neg\neg$-dense e' followed by a $\neg\neg$-closed mono m' that exists by 5.1. Then since $m \circ id = m' \circ e'$ by orthogonality there exists an h with $h \circ e' = id$ and $m \circ h = m'$. From $m' \circ e' \circ h = m \circ h \circ e' \circ h = m \circ h = m'$ we conclude $e' \circ h = id$ since m' is a mono. Thus h is an iso and therefore m is a $\neg\neg$-closed mono.$\Box$

Corollary 5.4 Assume that Markov's Principle MP holds. Then any Σ-replete object X is a Σ-cpo.

Proof: By 4.2.2 we have that η_X is an extremal mono so by 5.2 and 5.3 X is a Σ-poset. By 4.3.9 it is chain complete.$\Box$

However, the converse is not true, i.e. not any Σ-cpo is a Σ-replete object as has been showed by Martin Hyland [Hyland 93].

Theorem 5.5 If X is Σ-replete then it is linked.

Proof: By 4.2.2 η_X is an extremal mono, therefore $\iota : 2 \to \Sigma \bot \eta_X$ and by 4.2.9 it is linked. $\Box$

6 Synthetic Domains

We define bottom elements for predomains and prove that every endofunction on domains has a unique least fixpoint. From that we can deduce fixpoint induction and prove e.g. Plotkin's Lemma which is useful when characterizing the fixpoint of domain equations. Note that for all this it is immaterial whether for predomains we choose Σ-cpos or Σ-replete objects.

Definition 6.1 A predomain A is called a *domain* (or *focal*) iff there exists a (neccessarily unique) $\perp_A \in A$ such that $\forall a{:}A. \perp_A \sqsubseteq a$.

A function $f{:}A \to B$ between domains is called strict iff $f(\perp_A) = \perp_B$.

Theorem 6.1 Let A be a domain. Then any $f{:}A \to A$ has a least fixpoint. Since this fixpoint is unique by AC! we get $fix_A{:}(A \to A) \to A$ where $f(fix(f)) = fix(f) \wedge (\forall y{:}A.(fy \sqsubseteq y) \Rightarrow fix(f) \sqsubseteq y)$.

Proof: First, we define a function $a{:}N \to A$ by $a(0) = \perp_A$, $a(n+1) = f(a\,n)$. As f is monotonic we can easily prove $a \in AC(A)$ by induction. We already have by 2.2.7 that $sup\,(f \circ a) = f(sup\,a)$ therefore it remains to show $sup\,a = sup\,(f \circ a)$. If $P{:}\Sigma^A$ then we have $P(sup\,a)$ iff $\exists n{:}N.\,P(a(n))$ iff (since a is ascending) $\exists n{:}N.\ P(a(n+1))$ iff $\exists n{:}N.\ P(f(a\,n))$ iff $\exists n{:}N.\ P((f \circ a)\,n)$ iff $P(sup\,(f \circ a))$. Thus $sup\,a$ is a fixpoint of f.

Now suppose that x is another fixpoint of f, i.e. $f\,x \sqsubseteq x$ then by a simple induction one can show that $\forall n{:}N.\ a(n) \sqsubseteq x$, thus for all $P{:}\Sigma^A$ if $P(sup\ a)$ then $P(a(n))$ for some $n{:}N$ and therefore $P(x)$. Thus $sup\ a \sqsubseteq x$ and by definition of the supremum $sup\ a = x$. □

Corollary 6.2 Domains are closed under (i) equalizers with strict maps and (ii) arbitrary products.

Proof: (i) 3.4.4(i) and 4.4.2 respectively. Strictness implies that $\perp$ is contained in the equalizing object. (ii) By 3.4.4(ii) and 4.4.1 respectively. Define $\perp$ pointwise as $\lambda x.\,\perp$.□

Since domains have the expected closure properties one can proceed to develop domain theory as usual.

Definition 6.2 Let D be a domain, $f{\in}AC(D)$, $P{:}D \to \Omega$.
Then P is called *admissible* iff $(\forall n{:}N.\ P(f\,n)) \Rightarrow P(sup_D\,f)$.

Corollary 6.3 (*Fixpoint induction*) Let D be a domain, $P{:}D \to \Omega$ admissible, $f{:}D \to D$. Then $P(\perp_D) \wedge (\forall d{:}D.\ P(d) \Rightarrow P(f(d))) \Rightarrow P(fix_D(f))$.

Proof: Assume $P(\perp_D)$ and $\forall d{:}D.\,P(d) \Rightarrow P(f(d))$. We know by 6.1 that $fix_D(f) = sup_D\,a$ where $a{:}AC(D), a(0) = \perp_D$ and $a(n+1) = f(a\,n)$. Then by admissibility of P we only have to show $\forall n{:}N.\,P(a\,n)$ which can be easily verified by induction on n. □

Because of our previous work we can internalize the type of domains with strict functions Dom_{strict}, provided that it is a small internally complete subcategory, and therefore functors on Dom_{strict}. This allows us to compute least fixpoints of such functors:

Theorem 6.4 For any functor $T : Dom_{strict}^{op} \times Dom_{strict} \to Dom_{strict}$ there exists a domain D together with an isomorphism $\alpha : T(D,D) \to D$ such that id_D is the least fixpoint of the functional $\lambda h{:}D \to D.\,\alpha \circ T(h,h) \circ \alpha^{-1}$.

Proof: An internal functor is locally continuous due to 2.2.7. Now perform the traditional inverse limit construction [Plotkin 83]. This is possible due to the fact that both kinds of domains (Σ-cpos and Σ-replete objects) are closed under equalizers and products. □

7 Conclusions

We have seen that it is possible to develop the main results of traditional domain theory synthetically, i.e. naively without worrying about continuity, within a couple of pages. We have also seen that all this can be done axiomatically without any knowledge about topos theory or category theory. (But we do not claim that this knowledge is superfluous.)

The logic we use is not completely formal since we are dealing with subtypes informally. So one may ask what is the adequate logical basis for our theory, e.g. higher order intuitionistic logic with subtypes, intuitionistic set theory or constructive type theory with assumptions on proof irrelevance, extensionality etc.

We have developed domain theory without the notion of *compact objects*. But from a pragmatic point of view the question arises: do we need them? If yes, what are convincing examples and how can we formalize them in Synthetic Domain Theory.

Paul Taylor recently tried to give sufficient conditions for sobriety of Σ-replete objects. Up to now no convincing answer has been given to the question in which models Σ-replete and sober objects are the same. Eugenio Moggi [Moggi 93] suggested to consider filter spaces as model for Synthetic Domain Theory as they seems to have a lot of nice properties, e.g. that sobrification is repletion, but this has to be investigated more closel

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A Appendix: Open Questions on SDT

1. Is "repletion = sobrification (á la Rosolini/Taylor)" ?
   We know that this can be rephrased equivalently:

   - η_X is regular. i.e. the equalizer of $\Sigma^{\Sigma^X} \xrightarrow{\Sigma^{\Sigma^{\eta_X}}} \Sigma^{\Sigma^{\Sigma^X}}$ and $\Sigma^{\Sigma^X} \xrightarrow{\eta_{\Sigma^{\Sigma^X}}} \Sigma^{\Sigma^{\Sigma^X}}$ where $(\Sigma^{\Sigma^-}, \eta_X, \Sigma^{\eta_X})$ is the continuation monad.
   - Does for any $h : \Sigma^Y \to \Sigma^X$ such that the following diagram commutes

$$\begin{tikzcd} \Sigma^{\Sigma^{\Sigma^Y}} \arrow[r, "\Sigma^{\Sigma^h}"] \arrow[d, "\Sigma^{\eta_Y}"'] & \Sigma^{\Sigma^{\Sigma^X}} \arrow[d, "\Sigma^{\eta_X}"] \\ \Sigma^Y \arrow[r, "h"'] & \Sigma^X \end{tikzcd}$$

     exist an $f : X \to Y$ such that $\Sigma^f = h$?

2. Are extremal monos and regular monos the same? (*which is sufficient for sobrification=repletion.*) or equivalently: Are regular monos in *PreDom* closed under composition ?

3. Does m extremal mono imply Σ^m epi ? *(this is sufficient for 2.)* This is not true in $\mathcal{E}ff$, so in which models does it hold (in Filter Spaces) ?

4. Does *PreDom* small internally complete imply that the equality on Σ is $\neg\neg$-closed? (*that would mean that one has to restrict to Sep($\mathcal{E}$) (cf. 9.) for modelling polymorphism* ***and*** *SDT*).

5. Is the modified realizability topos $\mathcal{M}od$ with $\Sigma = \{p \in \Omega \mid \exists f{:}N^N. p \leftrightarrow \neg\neg\exists n{:}N. f(n) = 0\}$ with "$\neg\neg$-relaxed" axioms for Σ a model of SDT (so we suppose) ?

6. Is there a model (where obvioulsy MP cannot be valid) such that there exists a Σ-replete object which is not a Σ-cpo? *(probably $\mathcal{M}od$ with Σ as in 5.)*

7. Pragmatic: Where do we reason about compact elements in program verification ?

8. How to axiomatize algebraic domains *AlgPreDom* in SDT and is *AlgPreDom* small internally complete ? (Counterexample !)

9. Characterize the quasi-toposes which are equivalent to *Sep*($\mathcal{E}$) for some topos $\mathcal{E}$?