General Synthetic Domain Theory – A Logical Approach†

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Synthetic Domain Theory (SDT) is a version of Domain Theory where "all functions are continuous". Following the original suggestion of Dana Scott several approaches to SDT have been developed which are *logical* or *categorical*, *axiomatic* or *model-oriented* in character and which are either *special*ised towards Scott domains or aim at providing a *general* theory axiomatising the structure common to the various notions of domains studied so far.

In (RS93; Reu95) there has been developed a logical and axiomatic version of SDT which is special in the sense that it captures the essence of Domain Theory *à la Scott* but rules out *e.g.* Stable Domain Theory as it requires order on function spaces to be pointwise. In this article we will give a *logical* and *axiomatic* account of a *general* SDT aiming to grasp the structure common to all notions of domains.

As in (RS93; Reu95) the underlying logic is a sufficiently expressive version of *constructive type theory*. We start with a few basic axioms giving rise to a core theory on top of which we study various notions of predomains as *e.g.* complete and well-complete S-spaces (LS96), define the appropriate notion of domain and verify the usual induction principles of domain theory.

Although each domain carries a logically definable "specialization order" we avoid order-theoretic notions as much as possible in the formulation of axioms and theorems. The reason is that the order on function spaces cannot be required to be pointwise as this would rule out the model of stable domains à la Berry.

The consequent use of logical language – understood as the internal language of some categorical model of type theory – avoids the irritating coexistence of the *internal* and the *external* view pervading purely categorical approaches. Therefore, the paper is aimed at providing an *elementary introduction to Synthetic Domain Theory*, albeit requiring some knowledge of basic type theory.

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Contents

1. Introduction

1.1. *Origins of Synthetic Domain Theory*

In various lectures in the 80's Dana Scott has strongly promoted the idea of using intuitionistic higher order logic or set theory as an adequate logical framework for axiomatising Domain Theory as an *Extensional Theory of Computation.*

The key idea of traditional Domain Theory à la Scott is to approximate the notion of *computability* by the notion of *continuity.* This amounts to a systematic axiomatic generalisation of the well-known Myhill-Shepherdson-Theorem saying that *all effective operations are continuous w.r.t. the finite information topology* (see *e.g.* (Rog67)). When working in classical set theory – where *functions* are identified with *functional relations* – one has to live with the coexistence of full and continuous function spaces of domains (which both carry a domain structure). Although one may have classical theories of domains with continuous function spaces only as *e.g.* LCF (Pau87; Reg94) in such theories one cannot have the intuitively appealing *Axiom of Unique Choice (AUC)* stating that any functional relation is traced by a continuous function.

But it is consistent with *intuitionistic* logic to claim that *all functions between domains are continuous* and even that *all functions between domains are computable*. Moreover, there are *intended models* for this situation, namely the various *realisability models*. These might be considered as the standard model built relative to some *partial combinatory algebra* (*pca*) providing an untyped *model of computation*. A prominent example is the Kleene algebra of Gödel numbers of partial recursive functions.

Nevertheless till the end of the 80's there was not much of activity seriously taking up Scott's suggestion of developing Domain Theory in a purely axiomatic way in an intuitionistic setting. A notable exception is the Ph.D. Thesis of Giuseppe Rosolini (Ros86) who introduced the basic notion of Synthetic Domain Theory : the "r.e. subobject classifier" Σ. Using Σ one may define for every intuitionistic set a relation of "observational inequality" corresponding to the "information ordering" of Classical Domain Theory. But as opposed to Classical Domain Theory this "information ordering" appears as a derived notion and not as a structure component supporting the view that "domains are certain sets" and *not* a "certain kind of structure" be it order-theoretic or topological.

This line of research was taken up again by Wesley Phoa in his Ph.D. Thesis (Pho90) where he gave a detailed account of Domain Theory in the Effective Topos, *i.e.* the model built on Kleene realisability.

Whereas these approaches were model-based in 1989, independently, Martin Hyland (Hyl91) and Paul Taylor (Tay91) have introduced two different, but essentially equivalent, formulations of a synthetic theory of domains. Their idea was to define *predomains* as so-called Σ-*replete objects*, *i.e.* the objects of the least full reflective subcategory (of the ambient category of intuitionistic sets) containing the r.e. subobject classifier Σ. This property of Σ-repleteness can be shown to be equivalent to closure under "all generalised limit processes", *cf.* (Str97). Both Hyland and Taylor strongly advertised the use of the so-called "internal language" of a topos or a model of type theory which amounts to jumping back and forth between the diagrammatic language of category theory and the language of higher order logic. Unfortunately, in their development it is not always clear whether their definitions and theorems have to be understood in the internal or in the external sense.

1.2. *The Logical Approach to SDT*

The starting point of our Logical Approach to SDT is that *an axiomatic theory of domains has to be developed in a purely logical way* if it should be suitable as a *logical framework for the verification of functional programs.*

Of course, this does not mean to avoid categorical notions and diagrams at all but to understand them in the sense of the internal language. In order to make this possible we have to assume the existence of a *full internal subcategory* Set of the ambient category establishing a model of the type theory in use. From a type-theoretic point of view such a full internal subcategory is nothing but an appropriate *type-theoretic universe* Set.

Fortunately, realisability models – even over arbitrary partial combinatory algebras – provide such universes, namely the full internal subcategory of partial equivalence relations (*per*s). Moreover, these universes are even *impredicative* in the sense that they are closed under arbitrary (internal) products of families whose index set need not be itself an element of the universe Set. However, impredicativity of Set is not needed for developing a substantial part of general SDT though it is, of course, necessary to require that the universe Set be closed under products of families whose index set is in Set, *i.e.* that Set is a predicative universe. But the pleasant aspect of impredicativity of Set is that (pre)domains will provide a model of *polymorphic* λ*-calculus*.

However, for sheaf models of SDT, see *e.g.* (FR96), there arises the problem that it is not obvious whether they contain even only predicative universes (one definitely knows that Grothendieck toposes – due to their set-like nature – cannot contain nontrivial impredicative universes). This seems to have the consequence that for sheaf models not all concepts and arguments can be expressed in the internal language which certainly is a drawback when one desires SDT to provide a convenient *logic* of domains. But we do not worry too much about this problem because we consider the realisability models as the intended models of our axiomatisation anyway. The reason is that in realisability models functions between data types are precisely the algorithmic ones – in the sense of "algorithmic" as given by the underlying partial combinatory algebra.

In our previous draft paper (RS93) we have coined the term "Naive Synthetic Domain Theory" for the logical approach to SDT in analogy with the term "Naive Synthetic Differential Geometry" as introduced by Lavendhomme in his lecture notes (Lav96). An earlier published account of "Naive Synthetic Differential Geometry" can be found in the first part of Kock's book (Koc81). There he develops an impressive part of differential calculus in the informal language of intuitionistic higher order logic or set theory from a few axioms (inconsistent with classical logic) without any reference to specific models of the axioms.

In this paper, of which an extended abstract has already appeared in (RS97), our aim is to give an analogous account of Naive SDT though we probably will not achieve that extent of clarity and elegance which has become standard by the work of Kock and Lavendhomme. Due to the fact that SDT is more categorical in nature than Synthetic Differential Geometry we often will use the diagrammatic language of category theory but we emphasise that all diagrams have to be understood as living in the universe Set.

However, see (Reu95) for a "diagram-free" formulation of a synthetic theory of Domains á la Scott. This theory has been fully formalised and verified mechanically checked using (an extension of) the LEGO Proof Checker.

1.3. *The Key Ideas Underlying Synthetic Domain Theory*

The basic idea underlying Synthetic Differential Geometry is to replace the "analytic" formulation of differential calculus in terms of the "ε-δ-language" by a "synthetic" formulation in terms of "infinitesimals" whose existence is ensured by appropriate axioms which, however, contradict classical logic. Analogously, the basic idea of Synthetic Domain Theory is to replace the traditional "analytic" formulation of domain theory in terms of sets endowed with some order-theoretic or topological structure by a new "synthetic" formulation where domains are simply *sets with certain properties* that *guarantee* (analogues of) the required order-theoretic or topological structure.

1.3.1. *Special Synthetic Domain Theory à la Scott* The starting point for achieving such a "synthetic" formulation is the observation that already classical domain theory there is a one-to-one-correspondence between Scott open subsets of a domain D and the continuous maps from D to Σ, the 2-element lattice also known as *Sierpinski space*. Thus, the *topological structure of all domains* is *concentrated into one single domain* Σ as for an arbitrary domain D the open subsets of D can be recovered as the collection of all maps from D to Σ. Thus, the topology on D is not provided as *some additional structure* but is already determined by the category itself by the collection of maps from D to Σ (forming a domain Σ^D).

In the Kleene realisabilty model these "open sets" appear as the *semi-decidable predicates* constituting the most general form of experiment applicable to a computational object. Here the objects $\top$ and $\bot$ of Σ correspond to the propositions expressing *termination* and *divergence* of computations, respectively. In other words, Σ can be considered as the set of those propositions corresponding to Σ^0_1-sentences. As semi decidable predicates on a set X are closed under finite conjunctions and disjunctions and existential quantification over $\mathbb{N}$ they can be considered as a so-called "natural topology" on X. Notice, however, that the "open sets of the natural topology" on X will in general *not be closed under arbitrary unions* but only under unions of $\mathbb{N}$-indexed families.

Now the starting point of Synthetic Domain Theory is to postulate *axiomatically* a distinguished set Σ associating with every set X its "natural topology" whose "open sets" are the functions from X to Σ. This situation was axiomatised by Taylor in (Tay91) and has subsequently been developed in (RS93; Reu95).

1.3.2. *General Synthetic Domain Theory* The assumption that the Σ-predicates on a set X, *i.e.* the $p : X \to \Sigma$, form a weak topology on X is an assumption which is typical for domain theory à la Scott. But already the requirement that Σ-predicates are closed under finite disjunctions entails the existence of the Σ-predicate

$$por_\Sigma \triangleq \lambda u, v : \Sigma.\, u \vee v$$

which *cannot be computed sequentially*. The requirement of closure under finite disjunctions would rule out all models where functions between domains are *sequential*. But Longley has shown in his Thesis (Lon94) that in the realisabilty model over the Böhm tree model of untyped λ-calculus the Σ-predicate por_Σ does not exist (essentially because all λ-definable functions are sequential).

Thus there is a need for a *more general axiomatic setting* that does not rule out interesting models but is strong enough for developing the relevant part of basic domain theory needed for the verification of functional programs.

A fairly general categorical axiomatisation has been developed by Rosolini in (Ros95). However, it does not employ the internal language and gives only a core theory. It is not clear under which additional axioms *e.g.* the domain theoretic induction principles can be derived from his axioms. Universes are not used either.

A detailed account of domain theory in *arbitrary realisability models* has been given by John Longley and Alex Simpson in (LS96). Their exposition is *purely model based* and refrains from developing their theory axiomatically. In (Sim96) Simpson develops a version of the well-complete S-spaces in intuitionistic set theory which is more reminiscent of our approach, still we use type theory instead of set theory.

Our logical approach to *general synthetic domain theory* is heavily based on and inspired by the above mentioned work. One could say that we aim at an *axiomatic logical reformulation* of the work of Longley and Simpson inspired by the categorical axiomatisation of Rosolini. Our axioms have been chosen in a way that on the one hand they are much more restrictive than in (Ros95) and on the other hand are valid in all realisability models as considered in (LS96). The main purpose of our logical approach is to demonstrate that the *consequent use of the internal type-theoretic language* facilitates a simpler presentation avoiding both complicated external category-theoretic arguments as in (Ros95) and complicated explicit constructions in realisability models as in (LS96). This is also achieved in (Sim96) using intuitionistic set theory. Our type theoretic formulation – at least in our opinion – still appears to be simpler and allows for comparison to other variants as (Reu95; RS93). How the general setting can be refined to extensional S-spaces will, however, be treated in a followup article due to space limitations.

We now give a brief sketch of the ideas underlying our approach to general synthetic domain theory based on the above mentioned works.

First of all one supposes to be given a set S together with a distinguished element $\top \in S$ such that for all $u, v \in S$ we have that $u = v$ iff $u = \top \Leftrightarrow v = \top$. This allows to consider S as a subobject of Prop modulo "$\Leftrightarrow$" via the embedding $\mathsf{def} : S \to \mathsf{Prop}$ defined as $\mathsf{def}(u) \triangleq u = \top$. One furthermore assumes that there is an object $\perp \in S$ with $\neg\mathsf{def}(\perp)$.

General SDT has to be "order-free" in the sense that one should avoid using the information ordering – which by the way is always definable – as much as possible because in arbitrary realisability models one cannot expect it to be well-behaved for the following reasons (*cf.* (LS96)).

1 The order on function spaces need not be pointwise.
2 Not all ascending $\mathbb{N}$-indexed chains will have limits.

But for ensuring the existence of canonical fixpoint for all endomaps $f : A \to A$ we need some notion of "ω-chain". This notion must be (in general) more restrictive than the notion of ascending $\mathbb{N}$-indexed chain but has to include the so-called "Kleene chains" $(f^n(\perp))_{n\in\mathbb{N}}$ for all endomaps $f : A \to A$.

Now the key idea of (Ros95; LS96) was to define ω-chains in A as maps $a : \omega \to A$ where ω is the least fixpoint of the lifting functor L derived from the dominance S. Then the closure of a set A under "limits of ω-chains" is expressed by the requirement that for any $a : \omega \to A$ there exists a *unique extension* $\overline{a} : \overline{\omega} \to A$ of a along the canonical map $\iota : \omega \to \overline{\omega}$ from the initial fixpoint ω of L to the terminal fixpoint $\overline{\omega}$ of L (which is required to exist).

Diagrammatically the situation looks as follows:

$$\begin{tikzcd} \omega \arrow[r, "\iota"] \arrow[d, "a"'] & \overline{\omega} \arrow[dl, dashed, "\overline{a}"] \\ A & \end{tikzcd}$$

Accordingly, the *pendant* of Scott continuity, *i.e.* the property that any map between domains preserves suprema of ascending chains, is now the property that for any $f : A \to B$ and $a : \omega \to A$ we have $\overline{f \circ a} = f \circ \overline{a}$.

The intuition underlying this reformulation of *completeness* and *continuity* is the following. In analogy with the category of posets and monotonic maps the set ω corresponds to the *ordinal* ω and $\overline{\omega}$ corresponds to the ordered set obtained from ω by adding a greatest element ∞ (*i.e.* the ordinal $\omega + 1$). The *limit* of an ω-chain a is then defined as $\overline{a}(\infty)$. As $\overline{f \circ a} = f \circ \overline{a}$ we then get

$$\overline{f \circ a}(\infty) = (f \circ \overline{a})(\infty) = f(\overline{a}(\infty))$$

i.e. that f preserves limits of ω-chains.

It has been shown in (Ros95) that applying this construction to the Kleene chain for an endomap $f : A \to A$ gives rise to a canonical fixpoint of f.

Notice that *prima facie* the Kleene chain for an endomap f is a map $c_f : \mathbb{N} \to A$ and not an ω-chain $a_f : \omega \to A$. However, one can define a_f as the unique L-algebra morphism from the initial L-algebra ω to the L-algebra $\alpha \circ Lf$ where $f : A \to A$ and $\alpha : LA \to A$ is a *focal* L-algebra structure on A, *i.e.* $\alpha \circ \eta_A = id_A$. The focal L-algebra structure provides A with the required domain structure. This map a_f is related to the Kleene chain $c_f(n) = f^n(\bot)$ via the canonical map $\mathsf{step} : \mathbb{N} \to \omega$ as $a_f \circ \mathsf{step} = c_f$.

In our generalised setting the analogue of the traditional ascending $\mathbb{N}$-indexed chains are those $c : \mathbb{N} \to A$ for which there exists an extension along step, *i.e.* for which there exists a (unique) $a : \omega \to A$ with $a \circ \mathsf{step} = c$. It has been shown in (LS96) that there are realisability models where there exist ascending $\mathbb{N}$-indexed chains which cannot be extended along step.

1.4. *Survey of Contents*

In Section 2 we recall the type-theory on which we base our logical approach to general SDT and postulate some logical axioms which appear as natural though usually are not found in constructive type theories.

The notions of $\mathcal{L}$-completeness and S-separatedness are introduced in Section 3 and their closure properties are proved.

In Section 4 we give a type-theoretic (re)formulation of the so-called Dominance Axiom (originally introduced by Rosolini for toposes). From this we derive the lifting monad L and show the existence of an initial L-algebra $\phi : L\omega \to \omega$ and a terminal L-coalgebra

$v : \overline{\omega} \to L\,\overline{\omega}$. Furthermore, we characterise those $\mathcal{L}$ for which the $\mathcal{L}$-complete sets are closed under lifting.

In Section 5 the notion of complete set is introduced providing a synthetic analogue of closure under "limits of ω-chains". Further we study two extremal classes of predomains, *i.e.* classes of complete separated sets closed under lifting and containing the set $S = L\,1$, namely the well-complete separated sets of Longley and Simpson, and the replete sets of Hyland and Taylor.

Domains, *i.e.* predomains admitting a so-called "focal L-algebra structure" are studied in Section 6. This notion is aimed at providing a synthetic analogue of the existence of a bottom element. We introduce two last axioms from which we derive that focal L-algebra structure on domains is unique.

In Section 7 we present a synthetic analogue of Kleene's fixpoint construction providing any endomap of a domain with a canonical fixpoint.

Analogues of the common induction principles for these canonical fixpoints are verified in Section 8. For this purpose we introduce an appropriate synthetic version of the classical notion of "admissible predicate". In particular, we verify the principle of Park Induction from which it follows that for a map $f : A \to A$ on a domain A its canonical fixpoint is actually the least fixpoint w.r.t. the information order on A.

In Section 9 we discuss (a variant of) Markov's Principle which is equivalent to the requirement that equality on domains is $\neg\neg$-closed and, therefore, is valid in all realisability models. We show that in presence of Markov's Principle some useful forms of classical reasoning are valid for $\neg\neg$-closed predicates and, moreover, one of our axioms gets derivable.

Finally, in Section 10 domain equations are treated. In particular, following a suggestion of Simpson, we show how one may solve domain equations if the type-theoretic universe Set is impredicative, *i.e.* closed under arbitrary dependent products.

2. The Logic

The logic underlying our formulation of SDT is the Extended Calculus of Constructions (ECC) as described in (Luo94) and implemented within the LEGO system (see (LP92)). The Extended Calculus of Constructions is a constructive type theory with an *impredicative* universe Prop of propositions together with an infinite hierarchy of predicative universes $\mathsf{Type}_0 \in \mathsf{Type}_1 \in \cdots \in \mathsf{Type}_n \in \ldots$ with $\mathsf{Prop} \in \mathsf{Type} := \mathsf{Type}_0$.

For our purposes we need the following extension of ECC.

Axiom 1. The universe Prop of propositions is "proof-irrelevant", *i.e.*

$$\forall P : \mathsf{Prop}.\, \forall p, q{:}P.\ p = q \quad .$$

Axiom 2. There is a universe $\mathsf{Set} \in \mathsf{Type}$ with $\mathsf{Prop} \subseteq \mathsf{Set}$ but *not* $\mathsf{Prop} \in \mathsf{Set}$. Furthermore, Set contains the inductive type $\mathbb{N}$ of natural numbers. The universe Set may be impredicative but need not.

In the following the word "set" refers to objects of the universe Set.

Axiom 3. Functions between sets are extensional, *i.e.*

$$\forall A{:}\mathsf{Set}.\, \forall B{:}A \to \mathsf{Set}.\, \forall f,g : \Pi x{:}A.\, B(x).\ (\forall a{:}A. f(a) = g(a)) \Rightarrow f = g \quad .$$

Axiom 4. Functional relations are required to be traced by functions, *i.e.* we postulate the Axiom of Unique Choice (AUC)

$$\begin{aligned}&\Pi A : \mathsf{Set}.\, \Pi B : A \to \mathsf{Set}.\, \Pi P : \Pi x{:}A.\, B(x) \to \mathsf{Prop}.\\ &\quad (\forall x{:}A.\, \exists!\, y{:}B(x).\, P(x,y)) \to \textstyle\sum f : \Pi x{:}A.\, B(x).\, \forall a{:}A.\, P(a, f(a)) \ .\end{aligned}$$

It has been shown in (Str91; Luo90) that ECC can be interpreted in the category of realisability sets over an arbitrary partial combinatory algebra. For interpreting the extension above in arbitrary realisability models one has to interpret the universe Set as the collection of partial equivalence relations and Prop as the sub-collection of those partial equivalence relations R where aRa and bRb implies aRb (*i.e.* there is at most one equivalence class). See (Reu95) for detailed description of these models and the verifications of Axioms 3 and 4.

Notice that the type theory described above is closely related to the internal language of toposes, see *e.g.* (LS80; Pho90). But it is different from topos logic in the following two aspects. On the one hand in a topos there need not exist a universe Set containing the natural numbers object $\mathbb{N}$. On the other hand our type theory as opposed to topos logic does not support the following "extensionality principle for propositions"

$$\forall P : \mathsf{Prop}.\, (P \Leftrightarrow Q) \Rightarrow P = Q$$

claiming that equivalent propositions are equal. It turns out that in our subsequent development of SDT this principle is never needed. And that is good so as one does not know whether it is consistent with the above axioms.

Notice that due to the assumption that the universe Prop is proof-irrelevant we have subset types as in topos logic. If A is a set and $P : A \to \mathsf{Prop}$ then we put

$$\{\, x \in A \mid P(x) \,\} := \textstyle\sum x{:}A.\, P(x)$$

where the inclusion of $\{\, x \in A \mid P(x) \,\}$ into A is given by projection on the first component, *i.e.*

$$\lambda z : \{\, x \in A \mid P(x) \,\}.\, \pi_0(z)$$

which is injective as for $z, z' \in \{\, x \in A \mid P(x) \,\}$ we have $\pi_1(z) = \pi_1(z')$ provided $\pi_0(z) = \pi_0(z')$ as $P(\pi_0(z)) = P(\pi_0(z'))$ contains at most one element by Axiom 1.

Following the practice of topos theory we will use informal set-theoretic notation instead of the formal type theoretic language which makes sense as we have subset types and set comprehension available. Thus by *abus de langage* we (almost always) omit the inclusion maps $\pi_0 : \{\, x \in A \mid P(x) \,\} \to A$ for subset types.

The only restriction compared to the internal language of a topos is that one must not consider equivalent predicates as equal.

Furthermore, in the following all diagrams have to be understood as living in some of the universes $\mathsf{Set}, \mathsf{Type}_0, \ldots, \mathsf{Type}_n, \ldots$ considered as *full internal subcategories* of the ambient category establishing a model of the type theory in use.

3. $\mathcal{L}$-Completeness and Separatedness

3.1. $\mathcal{L}$-*Completeness*

The "order-free" analogue of closure under suprema of ω-chains will be that for any ω-chain $a : \omega \to A$ there is a unique extension $\overline{a} : \overline{\omega} \to A$ of a along $\iota : \omega \to \overline{\omega}$, *i.e.* $a = \overline{a} \circ \iota$.

This and related closure properties can be formulated quite elegantly employing the following notion of *orthogonality* (going back to Kelly and Freyd).

Definition 3.1. A map $f : X \to Y$ is called *orthogonal* to a map $g : Z \to U$, abbreviated as $f \perp g$, iff for all $h : X \to Z$ and $k : Y \to U$ with $k \circ f = g \circ h$ there is a unique map $\alpha : Y \to Z$ (called "fill-in") with $\alpha \circ f = h$ and $g \circ \alpha = k$, *i.e.* diagrammatically

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "k"] \arrow[dl, dashed, "\alpha"] \\ Z \arrow[r, "g"'] & U \end{tikzcd}$$

If $U = 1$ then we write $f \perp Z$ for $f \perp !_Z$, *i.e.* when any $h : X \to Z$ uniquely extends to a map $\overline{h} : Y \to Z$ along f

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[dl, dashed, "\overline{h}"] \\ Z & \end{tikzcd}$$

Using the notion of orthogonality we define an abstract notion of completeness relative to a class $\mathcal{L}$ of maps in some universe.

Definition 3.2. Let $\mathcal{L}$ be a class of maps, *i.e.* a predicate on

$$\textstyle\sum X{:}U.\, \sum Y{:}U.\ (X \to Y)$$

where U is any of the universes $\mathsf{Set} \subseteq \mathsf{Type}_0 \subseteq \mathsf{Type}_1 \ldots$. A type A is called $\mathcal{L}$-complete iff $e \perp A$ for all $e \in \mathcal{L}$, *i.e.* $(X, Y, e) \in \mathcal{L}$ where $e : X \to Y$.

This is the general pattern that later will be instantiated by specific classes $\mathcal{L}$ giving rise to notions of predomain as *e.g.* complete or replete objects.

Remark 3.1. Note that for the rest of this paper we will mostly consider the case $U = \mathsf{Set}$ and, accordingly, speak about $\mathcal{L}$-complete sets, well-complete sets, complete sets etc. In principle, however, we could also use any other of the universes Type_n instead of Set.
In any case the important point is that *our type theory allows us to quantify over* $\mathcal{L}$ as $\mathcal{L}$ is contained in some universe.

Even without any further assumptions about $\mathcal{L}$ the $\mathcal{L}$-complete sets satisfy many closure properties. First we consider closure under dependent products.

Lemma 3.2. Dependent products preserve $\mathcal{L}$-completeness.

Proof. Let X be a type and $B : X \to \mathsf{Set}$ be a family of $\mathcal{L}$-complete sets. In order to show that $\Pi x{:}X.\,B(x)$ is $\mathcal{L}$-complete assume that $(Y, Z, e) \in \mathcal{L}$ and $a : Y \to \Pi x{:}X.\,B(x)$. We have to show that there exists a unique $\overline{a} : Z \to \Pi x{:}X.\,B(x)$ with $\overline{a} \circ e = a$.
Let $\tilde{a} \in \Pi x{:}X.\,B(x)^Y$ be defined as $\tilde{a}(x)(y) = a(y)(x)$. As all $B(x)$ are $\mathcal{L}$-complete we get that for all $x \in X$ there exists a unique $\overline{\tilde{a}_x} : Z \to B(x)$ with $\overline{\tilde{a}_x} \circ e = \tilde{a}_x$. Now $\overline{a} : Z \to \Pi x{:}X.\,B(x)$ is defined as $\overline{a}(y)(x) \triangleq \overline{\tilde{a}_x}(y)$.
For showing that $\overline{a} \circ e = a$ assume that $y \in Y$ and $x \in X$. Then we have

$$(\overline{a} \circ e)(y)(x) = \overline{a}(e(y))(x) = \overline{\tilde{a}_x}(e(y)) = \tilde{a}_x(y) = a(y)(x)$$

and therefore get $\overline{a} \circ e = a$.
For uniqueness assume that $b : Z \to \Pi x{:}X.\,B(x)$ with $b \circ e = a$. Then for all $x \in X$ and $y \in Y$ we have

$$b(e(y))(x) = (b \circ e)(y)(x) = a(y)(x) = \overline{a}(e(y))(x) \in B(x)$$

and, therefore, we get by $\mathcal{L}$-completeness of $B(x)$ that $b(z)(x) = \overline{a}(z)(x)$ for all $z \in Z$. As this holds for all $x \in X$ we finally get that $b = \overline{a}$. □

Next we characterise the complete subobjects of complete types.

Lemma 3.3. Let A be $\mathcal{L}$-complete and $m : P \rightarrowtail A$ be a subobject of A. Then P is $\mathcal{L}$-complete iff m is $\mathcal{L}$-*closed*, *i.e.* $e \perp m$ for all $e \in \mathcal{L}$.

Proof. $\Rightarrow$: Assume that P is $\mathcal{L}$-complete and $(X, Y, e) \in \mathcal{L}$. For showing that $e \perp m$ assume that $f \circ e = m \circ g$. By $\mathcal{L}$-completeness of P there exists a unique $\overline{g} : Y \to P$ with $\overline{g} \circ e = g$. Therefore, we also have

$$m \circ \overline{g} \circ e = m \circ g = f \circ e$$

from which it follows that $m \circ \overline{g} = f$ as A is $\mathcal{L}$-complete by assumption. If $\alpha : Y \to P$ with $\alpha \circ e = g$ and $f = m \circ \alpha$ then $\alpha = \overline{g}$ as m is monic. This proves the uniqueness of $\overline{g}$.
$\Leftarrow$: Assume that m is $\mathcal{L}$-closed. For completeness of P assume that $(X, Y, e) \in \mathcal{L}$ and $g : X \to P$. By $\mathcal{L}$-completeness of A there exists a unique $f : Y \to A$ with $f \circ e = m \circ g$. By $\mathcal{L}$-closedness of m we have $e \perp m$ and, therefore, there exists a map $\overline{g} : Y \to P$ with $\overline{g} \circ e = g$. For uniqueness of $\overline{g}$ assume that $h : Y \to P$ with $h \circ e = g$. Then we have

$$m \circ h \circ e = m \circ g = m \circ \overline{g} \circ e$$

which implies $m \circ h = m \circ \overline{g}$ by $\mathcal{L}$-completeness of A and therefore $h = \overline{g}$ as m is monic by assumption. □

Remark 3.4. The previous lemma gives justification to the following *abus de langage*: for subobjects $m : P \rightarrowtail A$ of an $\mathcal{L}$-complete set A we often say "P is an $\mathcal{L}$-closed subobject (of A)" instead of the more explicit "m is an $\mathcal{L}$-closed subobject (of A)" when P is $\mathcal{L}$-complete.

Next we prove that $\mathcal{L}$-complete sets are closed under equalisers.

Theorem 3.5. Let $f_1, f_2 : A \to B$ be maps between $\mathcal{L}$-complete types. If $m : E \rightarrowtail A$ is an equaliser of f_1 and f_2 then E is $\mathcal{L}$-complete, too.

Proof. Let $m : E \rightarrowtail A$ be an equaliser of f_1 and f_2. Let $(X, Y, e) \in \mathcal{L}$ and $g : X \to E$.

Then by $\mathcal{L}$-completeness of A there is a unique $h : Y \to A$ with $h \circ e = m \circ g$. Therefore, we get

$$f_1 \circ h \circ e = f_1 \circ m \circ g = f_2 \circ m \circ g = f_2 \circ h \circ e$$

which entails $f_1 \circ h = f_2 \circ h$ by $\mathcal{L}$-completeness of B. Thus, as m is an equaliser there is a map $\overline{g} : Y \to E$ with $m \circ \overline{g} = h$. Now we have

$$m \circ \overline{g} \circ e = h \circ e = m \circ g$$

from which it follows that $\overline{g} \circ e = g$ as m is monic.
For uniqueness of $\overline{g}$ assume that $k \circ e = g$. Then we get

$$m \circ k \circ e = m \circ g = m \circ \overline{g} \circ e$$

entailing $m \circ k = m \circ \overline{g}$ by $\mathcal{L}$-completeness of A and therefore, $k = \overline{g}$ as m is monic. □

3.2. *S-Spaces*

Next we define the notion of S-*space* as an abstract version of T_0-spaces where each point is determined uniquely by its collection of open neighbourhoods. This notion has been considered by Rosolini and later by Phoa (Pho90) in the (specific) context of the effective topos for the "r.e.-subobject-classifier" Σ.

In this subsection we assume S to be an arbitrary set, *i.e.* $S \in \mathsf{Set}$.

Notation. We write $S(\_)$ for $S^{(\_)}$ and $S^n(\_)$ for the n-times application of the functor $S(\_)$.

Definition 3.3. For any type X let the map $\varepsilon_X : X \to S^2(X)$ be defined as

$$\varepsilon_X(x) = \lambda p{:}S(X).\, p(x)$$

Remark 3.6. Notice that – when restricted to Set – ε is the unit of the so-called "continuation monad" $S^2(\_)$ on Set and, therefore, a natural transformation from the identity functor on Set to the functor $S^2(\_) : \mathsf{Set} \to \mathsf{Set}$.

Lemma 3.7. For any type X

$$S(\varepsilon_X) \circ \varepsilon_{S(X)} = id_{S(X)}$$

i.e. $\varepsilon_{S(X)}$ is a split mono.

Proof. Let $p \in S(X)$ and $x \in X$ then

$$(S(\varepsilon_X) \circ \varepsilon_{S(X)})(p)(x) = S(\varepsilon_X)(\varepsilon_{S(X)}(p))(x) = \varepsilon_{S(X)}(p)(\varepsilon_X(x)) = \varepsilon_X(x)(p) = p(x)\ .$$

□

We will call a set $X \in \mathsf{Set}$ an S-*space* iff every $x \in X$ is determined uniquely by $\varepsilon_X(x)$.

Definition 3.4. A set $X \in \mathsf{Set}$ is called an S-*space* or S-*separable* iff ε_X is monic.

We immediately get plenty of nontrivial examples of S-spaces.

Lemma 3.8. For all sets A the set $S(A)$ is an S-space. Thus, in particular, $S \cong S(1)$ is an S-space.
Moreover, if Set is impredicative then $S(X)$ is an S-space for all types X.

Proof. Immediate from Lemma 3.7. □

3.2.1. *A Representation Theorem for S-spaces* We will now prove that a set is an S-space iff it is a subset of some power of S.

Theorem 3.9. A set $A \in \mathsf{Set}$ is an S-space iff it is a subobject of $S(I)$ for some $I \in \mathsf{Set}$.

Proof. "$\Rightarrow$": If $\varepsilon_A : A \to S^2(A)$ is monic then A is a subobject of $S^2(A)$.
"$\Leftarrow$": Suppose that $m : A \rightarrowtail S(I)$ is monic. By naturality of ε we get that

$$\begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[d, tail, "m"'] & S^2(A) \arrow[d, "S^2(m)"] \\ S(I) \arrow[r, tail, "\varepsilon_{S(I)}"'] & S^3(I) \end{tikzcd}$$

where $\varepsilon_{S(I)}$ is monic by Lemma 3.7. Thus, ε_A is monic and A is is an S-space. $\square$

3.2.2. *Closure Properties of S-spaces* Some important closure properties of S-spaces follow immediately from the above Representation Theorem for S-spaces.

Corollary 3.10. S-spaces are closed under subobjects and, therefore, in particular under equalisers and retracts.

Proof. If A is an S-space then A is a subobject of $S(I)$ for some set I. If set B is a subobject of A then B is also a subobject of $S(I)$ and, therefore, B is also an S-space by Representation Theorem 3.9.
As equalisers and embedding parts of retracts are particular subobjects S-spaces are closed under equalisers and retracts. $\square$

Next we consider closure of S-spaces under dependent products. For this we first need the following lemmas.

Lemma 3.11. Let X be a type and $B : X \to \mathsf{Set}$ with $B(x)$ an S-space for all $x \in X$. Then for $f, g \in \Pi x{:}X.\, B(x)$ it holds that $f = g$ iff $\phi(f) = \phi(g)$ for all $\phi \in S(\Pi x{:}X.\, B(x))$, *i.e.* $\varepsilon_{\Pi x:X.\, B(x)}$ is a mono. Consequently, if $\Pi x{:}X.\, B(x)$ is (isomorphic to) a set then it is already (isomorphic to) an S-space.
Moreover, $\Pi x{:}X.\, B(x)$ appears as a subobject of $S(\sum x{:}X.\, S(B(x)))$ via the map

$$i_{X,B} : \Pi x{:}X.\, B(x) \rightarrowtail S(\textstyle\sum x{:}X.\, S(B(x)))$$

where $i_{X,B}(f)\,\langle x, p\rangle \triangleq p(f(x))$.

Proof. Let $Z \triangleq \Pi x{:}X.\, B(x)$.
Obviously, if $f = g \in Z$ then $\phi(f) = \phi(g)$ for all $\phi \in S(Z)$.
For the reverse direction assume that $\phi(f) = \phi(g)$ for all $\phi \in S(Z)$. Then for all $x \in X$ and $p \in S(B(x))$ for the map

$$P_{x,p} : Z \to S$$

with $P_{x,p}(h) \triangleq p(h(x))$ it holds that

$$p(f(x)) = P_{x,p}(f) = P_{x,p}(g) = p(g(x)) \ .$$

Thus, $f(x) = g(x)$ as $B(x)$ is an S-space by assumption. By extensionality for dependent

functions it follows that $f = g$. Thus, $\varepsilon_Z : Z \rightarrowtail S^2(Z)$ is monic, hence Z is (isomorphic to) an S-space if Z is (isomorphic to) a set.
Form above it also follows that $f = g \in Z$ if $P_{x,p}(f) = P_{x,p}(g)$ for all $x \in X$ and $p \in S(B(x))$. Therefore, $i_{X,B}$ is monic as $i_{X,B}(h)\,\langle x, p\rangle = P_{x,p}(h)$. □

Now from the lemma it follows that S-spaces are appropriately closed under dependent products.

Theorem 3.12. (1) If $A \in \mathsf{Set}$ and $B : A \to \mathsf{Set}$ with $B(x)$ an S-space for all $x \in A$ then $\Pi x{:}A.\, B(x)$ is an S-space, too.
(2) If Set is an impredicative universe, *i.e.* Set is closed under arbitrary dependent products, then for all types X and families $B : X \to \mathsf{Set}$ with $B(x)$ an S-space for all $x \in X$ the dependent product $\Pi x{:}X.\, B(x)$ is an S-space, too.

Proof. If $A \in \mathsf{Set}$ and $B : A \to \mathsf{Set}$ is a family of S-spaces then $\sum x{:}A.\, B(x) \in \mathsf{Set}$ and therefore $S(\sum x{:}A.\, B(x)) \in \mathsf{Set}$, too. As by Lemma 3.11 $\Pi x{:}A.\, B(x)$ is a subobject of $S(\sum x{:}A.\, S(B(x)))$ it follows from the representation theorem for S-spaces that $\Pi x{:}A.\, B(x)$ is an S-space.
If Set is an impredicative universe then $\Pi x{:}X.\, B(x) \in \mathsf{Set}$ for all types X and all families of S-spaces $B : X \to \mathsf{Set}$. Thus, by Lemma 3.11 it follows that $\Pi x{:}X.\, B(x)$ is an S-space. □

3.2.3. *A Representation Theorem for $\mathcal{L}$-complete S-spaces* Next we prove a representation theorem for $\mathcal{L}$-complete S-spaces saying that they are precisely the $\mathcal{L}$-complete subobjects of powers of S by a set.

Theorem 3.13. A set is an $\mathcal{L}$-complete S-space iff it is an $\mathcal{L}$-closed subobject of some power of S by a set.

Proof. $\Rightarrow$: If A is an $\mathcal{L}$-complete S-space then $\varepsilon_A : A \rightarrowtail S^2(A)$ is a mono which is $\mathcal{L}$-closed by Lemma 3.3 as A is $\mathcal{L}$-complete by assumption. Thus A is an $\mathcal{L}$-closed subobject of $S^2(A)$ via ε_A.
$\Leftarrow$: If $I \in \mathsf{Set}$ and $m : A \rightarrowtail S(I)$ is an $\mathcal{L}$-closed subobject of $S(I)$ then A is an S-space by the representation theorem for S-spaces and it is $\mathcal{L}$-complete by Lemma 3.3 as $S(I)$ is $\mathcal{L}$-complete and m is $\mathcal{L}$-closed by assumption. □

4. Initial and Terminal Lift Algebra

4.1. *The Dominance S*

In the previous section we have made no assumptions about S beyond being a set. Now we will postulate that S corresponds to a set of propositions providing a notion of "well-behaved" subobject. Following Rosolini, who first introduced this notion in his Thesis (Ros86) for the particular case of toposes, we will call S a *dominance.*

As opposed to toposes in our more general type-theoretic setting equivalent propositions need not be equal in general. Accordingly, we cannot consider S simply as a subset of Prop but we have to require that elements of S are equal iff their associated propositions are logically equivalent. That means that, morally, we consider S as a subobject of Prop *modulo logical equivalence* although this quotient itself will not exist *qua* type in general [†]. Moreover, we have, of course, to make explicit by an inclusion map how we consider S as embedded into (this fictitious quotient of) Prop. This will be achieved by associating with every $u \in S$ the proposition $u = \top$ where $\top$ is a distinguished element of S corresponding to the true proposition $\top = \top$.

Axiom 5. There is a distinguished set $S \in \mathsf{Set}$ together with distinguished elements $\top, \bot \in S$ such that

$$\neg(\bot = \top)$$

and

$$\forall u, v{:}S.\, [u = \top \Leftrightarrow v = \top] \Rightarrow u = v$$

from which it follows that the map $\mathsf{def} : S \to \mathsf{Prop}$ given by

$$\mathsf{def}(u) \triangleq (u = \top) \in \mathsf{Prop}$$

is an inclusion allowing S to be considered as a subset of Prop which reflects logical equivalence in Prop to equality in S.
Moreover, S is required to be a *dominance*, *i.e.* for all $u \in S$ and $v : S^{\mathsf{def}(u)}$ there exists a (necessarily unique) $u \angle v \in S$ with

$$\mathsf{def}(u \angle v) \Leftrightarrow \exists p{:}\mathsf{def}(u).\, \mathsf{def}(v(p))$$

providing a "dependent conjunction" on S.

Remark 4.1. Notice that $\angle$ provides us with an operation $\wedge_S : S \times S \to S$ given by

$$u \wedge_S v := u \angle \lambda p{:}\mathsf{def}(u).\ v$$

satisfying

$$\mathsf{def}(u \wedge_S v) \Leftrightarrow \mathsf{def}(u) \wedge \mathsf{def}(v)$$

Accordingly, we write $u \wedge v$ for $u \wedge_S v$.

The following lemma will be useful later when we discuss the notion of "information ordering".

Lemma 4.2. For all $u \in S$ we have $u = \bot$ iff $\neg\,(u = \top)$, *i.e.* $u = \bot$ iff $\neg\mathsf{def}(u)$.
Moreover, we have $\neg\neg(u = \top \vee u = \bot)$ for all $u \in S$.

Proof. By Axiom 5 we have for all $u \in S$ that

$$(\bot = \top \;\Leftrightarrow\; u = \top) \;\Leftrightarrow\; u = \bot$$

and therefore

$$\neg\,(u = \top) \;\Leftrightarrow u = \bot$$

as $\neg\,(\bot = \top)$ by Axiom 5.
For the second claim notice that $\neg(u = \top \vee u = \bot)$ is equivalent to $\neg(u = \top) \wedge \neg(u = \bot)$

[†] Typically in realisability models the quotient of Prop by logical equivalence does not exist as the latter is not $\neg\neg$-closed in general.

which in turn is equivalent to $(u = \bot) \wedge \neg(u = \bot)$ (by the previous argument). As $(u = \bot) \wedge \neg(u = \bot)$ is evidently contradictory we get that $\neg\neg(u = \top \vee u = \bot)$. $\square$

4.2. *Lifting*

From the dominance we will now construct a *lifting operation* which allows one to classify partial maps whose domain of definition is given by an S-valued predicate. Note that – just as for the notion of $\mathcal{L}$-completeness – we could consider lifting for types of all universes, but for the purposes of this paper it suffices to have available a lifting operation for sets.

Definition 4.1. The lifting functor $L : \mathsf{Set} \to \mathsf{Set}$ is defined as follows.
For $A \in \mathsf{Set}$

$$LA \triangleq \textstyle\sum u{:}S.\, A^{\mathsf{def}(u)}$$

and for $f : A \to B$ in Set

$$L(f) \triangleq \lambda z{:}LA.\, \langle \pi_0(z), \lambda p{:}\mathsf{def}(\pi_0(z)).\, f(\pi_1(z)(p))\rangle$$

Moreover, let η be the natural transformation from the identity functor on Set to L given by

$$\eta_A \triangleq \lambda x{:}A.\, \langle \top, \lambda p{:}\mathsf{def}(\top).\, x\rangle$$

for all $A \in \mathsf{Set}$.

Notice that $L0 \cong 1$ and $L1 \cong S$ where 0 is the empty set and 1 is the singleton set containing precisely the element $*$.
Convention: Following common practice in the sequel we will not distinguish between $a \in A \in \mathsf{Set}$ and the function $1 \to A$ sending $*$ to a.

Definition 4.2. For $A \in \mathsf{Set}$ let $\bot_A \in LA$ be defined as

$$\bot_A \triangleq \langle \bot, ?_A\rangle$$

where $?_A$ is the unique map from $\mathsf{def}(\bot) \cong 0$ to A and let

$$c_A \triangleq [\bot_A, \eta_A] : 1 + A \to LA$$

be the map sending $*$ to $\bot_A$ and $a \in A$ to $\eta_A(a)$.

Remark 4.3. Notice that for any $A \in \mathsf{Set}$ the map $\eta_A : A \to LA$ *classifies* partial maps into A whose domain of definition is given by an S-predicate. More precisely, for any $B \in \mathsf{Set}$, $p : B \to S$ and $f : B' := \{\, x \in B \mid \mathsf{def}(p(x))\,\} \to A$ there is a *unique* map $\overline{f} : B \to LA$ such that the following diagram is a pullback

$$\begin{tikzcd}
B' \arrow[r, "f"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, tail, "\eta_A"] \\
B \arrow[r, "\overline{f}"'] & LA
\end{tikzcd}$$

where $\overline{f}$ is given explicitly as

$$\overline{f}(y) = \langle\, p(y), \lambda u{:}\mathsf{def}(p(y)).\, f(\langle y, u\rangle)\,\rangle \quad .$$

4.3. *Initial L-Algebra and Terminal L-Coalgebra*

We now show that there exists an initial algebra and a terminal coalgebra for the lifting functor L in the sense of (Fre91; Fre92) where one can find a detailed discussion of these notions for arbitrary endofunctors on an arbitrary category.

An L-*algebra* is a map $\alpha : LA \to A$. An L-algebra morphism from $\alpha : LA \to A$ to $\beta : LB \to B$ is a map $h : A \to B$ with $\beta \circ Lh = h \circ \alpha$. As $LA = \sum u{:}S.\, A^{\mathsf{def}(u)}$ a map $\alpha : LA \to A$ gives rise to an S-indexed family $\alpha_u : A^{\mathsf{def}(u)} \to A$ for $u \in S$ where $\alpha_u(f) \triangleq \alpha(\langle u, f\rangle)$. Conversely, any S-indexed family $(\alpha_u : A^{\mathsf{def}(u)} \to A \mid u \in S)$ gives rise to a map $\alpha : LA \to A$ where $\alpha(\langle u, f\rangle) \triangleq \alpha_u(f)$. Under this correspondence L-algebras are nothing but algebras in Set for the signature where for any $u \in S$ there is a $\mathsf{def}(u)$-ary operation. Accordingly, a map $h : A \to B$ is an L-algebra morphism from $\alpha : LA \to A$ to $\beta : LB \to B$ iff

$$h(\alpha_u(a)) = \beta_u(h \circ a)$$

for all $u \in S$ and $a \in A^{\mathsf{def}(u)}$.

Dually, an L-coalgebra is a map $\alpha : A \to LA$ and an L-coalgebra morphism from $\alpha : A \to LA$ to $\beta : B \to LB$ is a map $h : A \to B$ such that $\beta \circ h = Lh \circ \alpha$. As $LA = \sum u{:}S.\, A^{\mathsf{def}(u)}$ a map $\alpha : A \to LA$ gives rise to a map $\alpha^{(1)} \triangleq \pi_0 \circ \alpha : A \to S$ and a family $\alpha^{(2)}_a \triangleq \pi_1(\alpha(a)) : A^{\mathsf{def}(\alpha^{(1)}(a))}$ for $a \in A$. Conversely, a map $\alpha^{(1)} : A \to S$ together with a family $(\alpha^{(2)}_a : A^{\mathsf{def}(\alpha^{(1)}(a))} \mid a \in A)$ gives rise to a map $\alpha : A \to LA$ with $\alpha(a) \triangleq \langle \alpha^{(1)}(a), \alpha^{(2)}_a\rangle$. Accordingly, a map $h : A \to B$ is an L-coalgebra morphism from $\alpha : A \to LA$ to $\beta : B \to LB$ iff the equalities

$$\alpha^{(1)}(a) = \beta^{(1)}(h(a)) \qquad\qquad h \circ \alpha^{(2)}_a = \beta^{(2)}_{h(a)}$$

hold for all $a \in A$.

Theorem 4.4. In Set there exist a terminal L-coalgebra $\upsilon : \overline{\omega} \to L\overline{\omega}$ and an initial L-algebra $\phi : L\omega \to \omega$ which is the least sub-L-algebra of υ^{-1}.

Proof. We first construct the terminal L-coalgebra $\upsilon : \overline{\omega} \to L\,\overline{\omega}$ following (Jib97). We define $\overline{\omega}$ as the subset of $S^{\mathbb{N}}$ consisting of all $f \in S^{\mathbb{N}}$ with $\mathsf{def}(f(n+1)) \Rightarrow \mathsf{def}(f(n))$ for all $n \in \mathbb{N}$. We define $\upsilon : \overline{\omega} \to L\,\overline{\omega}$ as

$$\upsilon(f) \triangleq \langle f(0), \lambda z{:}\mathsf{def}(f(0)).\ \lambda n{:}\mathbb{N}.\, f(n+1)\rangle$$

for every $f \in \overline{\omega}$.

Suppose $\alpha : A \to LA$ is some L-coalgebra. An L-coalgebra morphism from α to υ is given by a map $h : A \to \overline{\omega} \subseteq S^{\mathbb{N}}$ such that

$$\alpha^{(1)}(a) = h(a)(0)$$

$$h \circ \alpha^{(2)}_a = \lambda z{:}\mathsf{def}(\alpha^{(1)}(a)).\ \lambda n{:}\mathbb{N}.\, h(a)(n+1)$$

for all $a \in A$.

By twist of arguments such maps are in one-to-one-correspondence with maps $\tilde{h} : \mathbb{N} \to S^A$ satisfying the primitive recursive equations

$$\tilde{h}(0) = \alpha^{(1)}$$

$$\tilde{h}(n+1) = \lambda a{:}A.\, \alpha^{(1)}(a) \angle (\tilde{h}(n) \circ \alpha^{(2)}_a)$$

as it can be shown by induction over $\mathbb{N}$ that for any such $\tilde{h}$ it holds that $\lambda n{:}\mathbb{N}.\, \tilde{h}(n)(a) \in \overline{\omega}$ for all $a \in A$. Thus, there exists exactly one L-coalgebra morphism from α to υ as $\tilde{h}$ is uniquely determined by its defining primitive recursive equations.
Thus, $\upsilon : \overline{\omega} \to L\overline{\omega}$ is a terminal L-coalgebra. Therefore, the map υ is an isomorphism as this is the case for all terminal coalgebras, see *e.g.* (Fre91; Fre92). More explicitly, we have

$$\upsilon^{-1}(\langle u, o\rangle)(0) = \bot$$

$$\upsilon^{-1}(\langle u, o\rangle)(n+1) = u \angle \lambda p{:}\mathsf{def}(u).\, o(u)(n)$$

for all $u \in S$, $o \in \overline{\omega}^{\mathsf{def}(u)}$ and $n \in \mathbb{N}$.
We now construct an initial L-algebra as the least sub-L-algebra of the L-algebra $\upsilon^{-1} : L\overline{\omega} \to \overline{\omega}$. Again, this construction was originally given in (Jib97), however, our version of the proof makes use of our previous observation that L-algebras can be considered as algebras in the sense of universal algebra. But notice that this is possible only because we work in a type theory where propositions like $\mathsf{def}(u)$ are represented as the types of their proofs ‡.
Let ω be the least subset $P \subseteq \overline{\omega}$ closed under all operations υ_u^{-1}, *i.e.* $\upsilon_u^{-1}(f) \in P$ for all $u \in S$ and $f \in P^{\mathsf{def}(u)}$. Thus, υ^{-1} restricts to a map $\phi : L\omega \to \omega$. This L-algebra will be shown to be initial exploiting the fact that ω is an inductively defined subset of $\overline{\omega}$.
Let $\alpha : LA \to A$ be an L-algebra. As ϕ is the least sub-L-algebra of υ^{-1} there is at most one L-algebra morphism from ϕ to α. (This is an instance of the basic fact of universal algebra that algebra homomorphisms from the term generated algebra to an arbitrary algebra are necessarily unique.)
Next we show the existence of such a morphism. Let G be the least subset $R \subseteq \omega \times A$ such that for all $u \in S$, $o \in \omega^{\mathsf{def}(u)}$ and $g \in A^{\mathsf{def}(u)}$ it holds that $\langle \phi_u(o), \alpha_u(g)\rangle \in R$ whenever $\langle o(p), g(p)\rangle \in R$ for all $p \in \mathsf{def}(u)$. By definition of G it is the graph of an L-algebra morphism from ϕ to α provided G is the graph of a function from ω to A. By the Axiom of Unique Choice it suffices to prove that G is functional, *i.e.* that for all $f \in \omega$ there exists a unique $a \in A$ with $\langle f, a\rangle \in G$. We prove functionality of G by induction over ω.
Suppose that $u \in S$ and $o \in \omega^{\mathsf{def}(u)}$ such that for all $p \in \mathsf{def}(u)$ there is a unique $a_p \in A$ with $\langle o(p), a_p\rangle \in G$. By Axiom of Unique Choice there is a (unique) map $g \in A^{\mathsf{def}(u)}$ with $g(p) = a_p$, *i.e.* we have $\langle o(p), g(p)\rangle \in G$ for all $p \in \mathsf{def}(u)$. Now from the closure property of the inductively defined set G it follows that $\langle \phi_u(o), \alpha_u(g)\rangle \in G$. Thus, there exists an $a \in A$ with $\langle \phi_u(o), a\rangle \in G$.
For uniqueness suppose that $a \in A$ with $\langle \phi_u(o), a\rangle \in G$. Then – due to the inductive

‡ In the internal language of toposes this sort of reasoning cannot be expressed (so easily) as it does not support the principle of propositions–as–types

definition of G – $\langle \phi_u(o), a\rangle = \langle \phi_{u'}(o'), \alpha_{u'}(g')\rangle$ for some $u' \in S$, $o' \in \omega^{\mathsf{def}(u')}$ and $g' \in A^{\mathsf{def}(u')}$ with $\langle o'(p), g'(p)\rangle \in G$ for all $p \in \mathsf{def}(u')$. Thus, we have $\phi\langle u, o\rangle = \phi_u(o) = \phi_{u'}(o') = \phi\langle u', o'\rangle$. As ϕ is obtained as the restriction of the isomorphism υ^{-1} to ω and, therefore, ϕ is one-to-one it follows that $u = u'$ and $o = o'$. But then, for all $p \in \mathsf{def}(u)$ we have $\langle o(p), g'(p)\rangle \in G$ and also $\langle o(p), g(p)\rangle \in G$. From the induction hypothesis it follows that $g(p) = g'(p)$ for all $p \in \mathsf{def}(u)$. Thus, we have $g = g'$ and, therefore, also $a = \alpha_{u'}(g') = \alpha_u(g)$ establishing the uniqueness of $\alpha_u(g)$. □

As ϕ and υ are isomorphisms, see *e.g.* (Fre91; Fre92), the following definitions make sense.

Definition 4.3. The maps

$$\sigma \triangleq \phi \circ \eta_\omega : \omega \to \omega$$

$$\overline{\sigma} \triangleq \upsilon^{-1} \circ \eta_{\overline{\omega}} : \overline{\omega} \to \overline{\omega}$$

are called *successor maps* on ω and $\overline{\omega}$, respectively.

Definition 4.4. By initiality of ω there is a *unique* map $\iota : \omega \to \overline{\omega}$ with

$$\begin{tikzcd} L\omega \arrow[r, "\phi"] \arrow[d, "L\,\iota"'] & \omega \arrow[d, "\iota"] \\ L\overline{\omega} \arrow[r, "\upsilon^{-1}"'] & \overline{\omega} \end{tikzcd}$$

From these definitions and naturality of η it is obvious that

$$\overline{\sigma} \circ \iota = \iota \circ \sigma$$

as $\overline{\sigma} \circ \iota = \upsilon^{-1} \circ \eta_{\overline{\omega}} \circ \iota = \upsilon^{-1} \circ L\,\iota \circ \eta_\omega = \iota \circ \phi \circ \eta_\omega = \iota \circ \sigma$.

Definition 4.5. Whenever $\alpha : LA \to A$ is an L-algebra let $\mathsf{step}_\alpha : \mathbb{N} \to A$ be the unique map satisfying

$$\begin{tikzcd} 1+\mathbb{N} \arrow[rr, "{[0,s]}"] \arrow[d, "1+\mathsf{step}_\alpha"'] & & \mathbb{N} \arrow[d, "\mathsf{step}_\alpha"] \\ 1+A \arrow[r, "c_A"'] & LA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

where $s(n) = n+1$. The uniqueness of step_α follows from the fact that $[0, s]$ by definition is the initial algebra for the functor $1 + \_$.
We write step for $\mathsf{step}_\phi : \mathbb{N} \to \omega$ and $\overline{\mathsf{step}}$ for $\mathsf{step}_{\upsilon^{-1}} : \mathbb{N} \to \overline{\omega}$. Notice that $\overline{\mathsf{step}} = \iota \circ \mathsf{step}$.

Next we define an object $\infty \in \overline{\omega}$ intuitively corresponding to the "limit" of the sequence $(\mathsf{step}(n))_{n\in\mathbb{N}}$ in $\overline{\omega}$. This object $\infty \in \overline{\omega}$ will play a key role in the construction of canonical fixpoints for endofunctions on domains (*cf.* (CP92)).

Definition 4.6. Let $\infty : 1 \to \overline{\omega}$ be the unique map satisfying

$$\begin{array}{ccc} 1 & \xrightarrow{\eta_1} & L1 \\ {\scriptstyle\infty}\downarrow & & \downarrow{\scriptstyle L\infty} \\ \overline{\omega} & \xrightarrow[\upsilon]{} & L\overline{\omega} \end{array}$$

Lemma 4.5. The object ∞ is a fixpoint of $\overline{\sigma}$, *i.e.* $\overline{\sigma} \circ \infty = \infty$ and, moreover, ∞ is an equalizer of $id_{\overline{\omega}}$ and $\overline{\sigma}$.

Proof. That ∞ is a fixpoint can be seen as follows : $\overline{\sigma}\circ\infty = \upsilon^{-1}\circ\eta_{\overline{\omega}}\circ\infty = \upsilon^{-1}\circ L\infty\circ\eta_1$ (by naturality of η) $= \upsilon^{-1} \circ \upsilon \circ \infty = \infty$.
For showing that $\infty : 1 \to \overline{\omega}$ is an equalizer of $id_{\overline{\omega}}$ and $\overline{\sigma}$ assume that $f : B \to \overline{\omega}$ with $\overline{\sigma} \circ f = f$. We show that for the unique map $!_B : B \to 1$ we have $\infty \circ !_B = f$. For this it suffices to show that both f and $\infty \circ !_B$ are L-coalgebra morphisms from η_B to υ.
The map $!_B$ is an L-coalgebra morphism from η_B to η_1 by naturality of η and by definition ∞ is an L-coalgebra morphism from η_1 to υ. Thus, $\infty \circ !_B$ is an L-coalgebra morphism from η_1 to υ.
From $f = \overline{\sigma} \circ f = \upsilon^{-1} \circ \eta_{\overline{\omega}} \circ f$ it follows that $\upsilon \circ f = \eta_{\overline{\omega}} \circ f = Lf \circ \eta_B$ and, hence, f is an L-coalgebra morphism from η_B to υ. □

4.4. *Closure under Lifting*

In this subsection we will characterise those classes $\mathcal{L}$ of maps in Set such that $\mathcal{L}$-complete sets are closed under lifting. Moreover, we show S-spaces to be closed under lifting.

4.4.1. *Closure under Lifting for $\mathcal{L}$-complete Sets* We first give a necessary condition for a class $\mathcal{L}$ of maps in Set such that $\mathcal{L}$-complete sets are closed under lifting.

Lemma 4.6. If $\mathcal{L}$ is a class of maps in Set such that $\mathcal{L}$-complete sets are closed under lifting then $e \perp S$ for all e in $\mathcal{L}$.

Proof. As 1 is $\mathcal{L}$-complete and $\mathcal{L}$-complete sets are closed under lifting it follows that $L\,1$ is $\mathcal{L}$-complete. As $S \cong L\,1$ we get that S is $\mathcal{L}$-complete, *i.e.* we have $e \perp S$ for all e in $\mathcal{L}$. □

Maps e with $e \perp S$ were originally introduced and called S-isos by Martin Hyland in (Hyl91). As closure under lifting is sort of a minimal requirement for a class of predomains given as $\mathcal{L}$-complete sets the previous lemma says that one may restrict attention to classes $\mathcal{L}$ of S-isos. A detailed explanation of S-isos as "generalised limit process" has been given in (Str97). Accordingly, predomains will be classes of sets *closed* under a collection of "generalised limit process" as given by a class $\mathcal{L}$ of S-isos.

The notion of S-iso and some other useful notions are collected in the following definition.

Definition 4.7. A map $e : X \to Y$ in Set is called an *S-iso* iff $e \perp S$.
Accordingly, a map e in Set is called an *S-epi* iff S^e is monic, *i.e.* $p \circ e = q \circ e \Rightarrow p = q$ for all $p, q \in Y \to S$.

A mono $m : X \rightarrowtail Y$ is called an *S-mono* or *S-subobject of Y* iff there exists a (necessarily unique) map $\chi : Y \to S$ with $\chi(y) = \top \Leftrightarrow \exists x{:}X.y = m(x)$ for all $y \in Y$. Such a unique χ will be called *classifying map for m*.

Remark 4.7. Notice that S-monos are closed under composition as S is required to be a dominance. If $m : Y \rightarrowtail X$ and $n : Z \rightarrowtail Y$ are S-monos classified by $\varphi : X \to S$ and $\psi : Y \to S$, respectively. Then $m \circ n$ is classified by

$$\chi(x) \triangleq \varphi(x) \angle \lambda u{:}\mathsf{def}(\varphi(x)).\ \psi(\mathsf{y}(x,u))$$

where $\mathsf{y}(x,u)$ is the unique $y \in Y$ with $x = m(y)$.

The following lemma will be crucial for giving a necessary and sufficient characterisation of those $\mathcal{L}$ for which $\mathcal{L}$-complete sets are closed under lifting.

Lemma 4.8. If $\mathcal{L}$ is a class of S-isos and $A \in \mathsf{Set}$ then $L\,A$ is $\mathcal{L}$-complete iff $m^*e \perp A$ for all $e \in \mathcal{L}$ and S-subobjects m of the codomain of e.

Proof. "$\Rightarrow$": Assume that LA is $\mathcal{L}$-complete. Let $e : X \to Y$ be an S-iso in $\mathcal{L}$ and $m : Z \rightarrowtail Y$ be an S-subobject of Y. Let $n : m^*X \rightarrowtail X$ and $m^*e : m^*X \to Z$ be a pullback cone over e and m. Notice that n is also an S-mono as these are preserved by pullbacks along arbitrary morphisms.
We have to show that $m^*e \perp A$. For this purpose let $g : m^*X \to A$. We have to construct a unique extension $\overline{g}$ of g along m^*e.
As L classifies partial maps whose domain of definition is an S-subobject (*cf.* remark after Definition 4.2) there is a unique map $f : X \to L\,A$ such that n and g is a pullback cone over f and η_A. By completeness of LA there is a unique extension $\overline{f}$ of f along e. Now let $m' : P \rightarrowtail Y$ and $h : P \to A$ be a pullback cone over $\overline{f}$ and η_A, *i.e.* $\overline{f}$ classifies the partial map (m', h) from Y to A. Notice that m' is an S-mono as it arises as pullback of the S-mono η_A classified by $L(!_A)$.
We next show that m and m' are isomorphic subobjects of Y. Let $\chi : Y \to S$ be the classifying map for m. Then $\chi \circ e$ classifies n as n is the pullback of m along e. The map $L(!_A) \circ f$ is also a classifier for n as $L(!_A)$ classifies η_A and n arises as pullback of η_A along f. By uniqueness of classifying maps (for n) we get that $\chi \circ e = L(!_A) \circ f = L(!_A) \circ \overline{f} \circ e$. As S is $\mathcal{L}$-complete (due to the assumption that all maps in $\mathcal{L}$ are S-isos) we get that $\chi = L(!_A) \circ \overline{f}$. But as χ classifies m and $L(!_A) \circ \overline{f}$ classifies m' we get that m and m' are isomorphic subobjects of Y. Thus, without loss of generality we may assume that $m = m'$.
We now choose $\overline{g} \triangleq h$ as the desired extension of g along m^*e. That $g = \overline{g} \circ m^*e$ follows from the following equational reasoning as η_A is monic

$$\eta_A \circ g = f \circ n = \overline{f} \circ e \circ n = \overline{f} \circ m \circ m^*e = \eta_A \circ \overline{g} \circ m^*e\ .$$

The situation is visualized in the following diagram where all squares are pullbacks.

$$
\begin{tikzcd}
m^*X \arrow[r, "m^*e"] \arrow[dr, tail, "n"] \arrow[dd, tail, "g"'] & Z \arrow[dr, tail, "m"] \arrow[ddl, dashed, tail, "\overline{g}"] & \\
& X \arrow[r, tail, "e"'] \arrow[dd, tail, "f"] & Y \arrow[ddl, tail, "\overline{f}"] \\
A \arrow[dr, tail, "\eta_A"'] & & \\
& L\,A &
\end{tikzcd}
$$

It remains to show the uniqueness of $\overline{g}$. Suppose $h' \circ m^*e = g$. Let $f' : Y \to LA$ classify the partial map (m, h'). Then $f' \circ e$ classifies the partial map (n, g) and, therefore, by uniqueness of classifying maps it follows that $f = f' \circ e$. As e is an S-iso and LA is complete we get that $f' = \overline{f}$ and, therefore, the partial maps (m, h') and $(m, \overline{g}$ are isomorphic from which it follows that $\overline{g} = h'$.
"$\Leftarrow$": Assume that $m^*e \perp A$ for all $e : X \to Y$ in $\mathcal{L}$ and S-monos $m : Z \rightarrowtail Y$. We have to show that A is $\mathcal{L}$-complete.
Let $e : X \to Y$ be map in $\mathcal{L}$ and $f : X \to L\,A$. We are looking for a unique extension $\overline{f} : Y \to L\,A$ along e. Let $n : f^*A \rightarrowtail X$ and $g : f^*A \to A$ be a pullback cone over f and η_A. Notice n is an S-mono as it appears as pullback of the S-mono η_A. Let $\chi : X \to S$ be the classifying map for n. As S is $\mathcal{L}$-complete and e is an S-iso there exists a unique extension $\overline{\chi}$ of χ along e. Let m be the S-mono classified by $\overline{\chi}$ and $\alpha : f^*A \to Z$ be the mediating arrow with $m \circ \alpha = e \circ n$ (and $!_Z \circ \alpha = !_{f^*A}$). Obviously, α appears as pullback of e along m and, therefore, we write m^*e for α. By assumption there exists a unique extension $\overline{g}$ of g along m^*e. Let $\overline{f} : Y \to L\,A$ classify the partial map $(m, \overline{g})$. Then all squares in the following diagram are pullback squares.

$$
\begin{tikzcd}
f^*A \arrow[r, tail, "m^*e"] \arrow[dr, tail, "n"] \arrow[dd, tail, "g"'] & Z \arrow[dr, tail, "m"] \arrow[ddl, tail, "\overline{g}"] & \\
& X \arrow[r, tail, "e"'] \arrow[dd, tail, "f"] & Y \arrow[ddl, dashed, tail, "\overline{f}"] \\
A \arrow[dr, tail, "\eta_A"'] & & \\
& L\,A &
\end{tikzcd}
$$

As both $\overline{f} \circ e$ and f classify the partial map (n, g) it follows that $f = \overline{f} \circ e$.
For uniqueness of $\overline{f}$ assume that $f = h \circ e$. Let (m', g') be the partial map classified by h. As $L(!_A) \circ \overline{f} \circ e = L(!_A) \circ f = L(!_A) \circ h \circ e$, S is $\mathcal{L}$-complete and e is an S-iso, we get that $L(!_A) \circ \overline{f} = L(!_A) \circ h$ and, therefore, classify the same S-mono m. Thus, without loss

of generality we may assume that $m = m'$ and, therefore, $g = g' \circ m^*e$. But then $\overline{g} = g'$ as by assumption $m^*e \perp A$. Thus, we have $\overline{f} = h$ as both maps classify the partial map $(m, \overline{g})$. □

From this it follows immediately that lifting reflects $\mathcal{L}$-completeness provided $\mathcal{L}$ contains only S-isos.

Corollary 4.9. If $\mathcal{L}$ contains only S-isos then A is $\mathcal{L}$-complete if $L\,A$ is $\mathcal{L}$-complete.

Proof. If $L\,A$ is $\mathcal{L}$-complete then by Lemma 4.8 it follows that $m^*e \perp A$ for all $e : X \to Y$ in $\mathcal{L}$ and S-subobjects m of Y. As all identity maps are S-monos it follows that $e \perp A$ for all e in $\mathcal{L}$, *i.e.* that A is $\mathcal{L}$-complete. □

From Lemma 4.8 we now obtain the desired characterisation of those $\mathcal{L}$ for which $\mathcal{L}$-complete sets are closed under lifting.

Theorem 4.10. Let $\mathcal{L}$ be a class of maps in Set. Then $\mathcal{L}$-complete sets are closed under lifting iff $\mathcal{L}$ contains only S-isos and for all $\mathcal{L}$-complete A it holds that $m^*e \perp A$ whenever e is a map in $\mathcal{L}$ and m is an S-subobject of the codomain of e.

Proof. "$\Rightarrow$": From Lemma 4.6 it follows that all maps in $\mathcal{L}$ must be S-isos. The second condition is a consequence of Lemma 4.8.
"$\Leftarrow$": By Lemma 4.8. □

This characterisation gives rise to the following condition sufficient for $\mathcal{L}$-complete sets being closed under lifting.

Corollary 4.11. If $\mathcal{L}$ contains only S-isos and is closed under pullbacks along S-monos then $\mathcal{L}$-complete sets are closed under lifting.

Next we show a lemma that later will be essential for S-replete sets being closed under lifting.

Lemma 4.12. S-isos are stable under pullbacks along S-monos.

Proof. Let $e : X \to Y$ be an S-iso and $m : Z \rightarrowtail Y$ be an S-mono. Let $m^*e : P \to Z$ and $e^*m : P \rightarrowtail X$ be a pullback cone over m and e. Note that e^*m is an S-mono, too, and let $\chi : X \to S$ classify e^*m.
Let $f : P \to S$. We have to construct a unique extension $\overline{f}$ of f along m^*e.
Let $n : Q \rightarrowtail P$ be an S-mono classified by f and $g : X \to S$ be the classifier for the S-mono $e^*m \circ n$. As $g \circ e^*m$ classifies n and classifying maps are unique we have $f = g \circ e^*m$. As e is an S-iso there is a unique map $\overline{g} : Y \to S$ with $\overline{g} \circ e = g$. For the map $\overline{f} \triangleq \overline{g} \circ m$ we then have

$$\overline{f} \circ m^*e = \overline{g} \circ m \circ m^*e = \overline{g} \circ e \circ e^*m = g \circ e^*m = f$$

and, therefore, $\overline{f}$ is an extension of f along m^*e.
For uniqueness of $\overline{f}$ assume that $h_1, h_2 : Z \to S$ with $h_1 \circ m^*e = f = h_2 \circ m^*e$. For $i = 1, 2$ let $m_i : Z_i \rightarrowtail Z$ be an S-mono classified by h_i and $k_i : Y \to S$ classify the S-mono $m \circ m_i$. Then for $i = 1, 2$ the map $k_i \circ e$ classifies the S-mono $e^*m \circ n$ as n is classified by $f = h_i \circ m^*e$. Thus, $k_1 \circ e = k_2 \circ e$ and, therefore, $k_1 = k_2$ as e is an S-iso. As for $i = 1, 2$ both h_i and $k_i \circ m$ classify m_i it follows that $h_i = k_i \circ m$. Thus, we have $h_1 = h_2$ as $k_1 = k_2$. □

4.4.2. *Closure under Lifting for S-spaces*

Lemma 4.13. Let A be an S-space. Then $L\,A$ is isomorphic to

$$S_A \triangleq \{\phi \in S^2(A) \,|\, \phi(\lambda p{:}S(A).\top) = \top \Rightarrow \exists a{:}A.\, \phi = \varepsilon_A(a)\,\} \subseteq S^2(A)$$

and therefore an S-space.

Proof. Let A be an S-space. Then $L\,A$ and S_A are isomorphic via the maps $i : L\,A \to S_A$ and $j : S_A \to L\,A$ defined as

$$i(\langle s, a\rangle) \triangleq \lambda p{:}S(A).\, s \angle p \circ a$$

$$j(\phi) \triangleq \langle \phi(\lambda p{:}S(A).\top), \mathsf{get}(\phi)\rangle$$

where $\mathsf{get}(\phi)$ is the function such that $\mathsf{get}(\phi)(u)$ is the unique $a \in A$ with $\phi = \varepsilon_A(a)$ for every $u \in \mathsf{def}(\phi(\lambda h{:}S(A).\top))$. The existence of such an a is guaranteed by $\phi \in S_A$ and it is unique as A is an S-space.
Note that for the construction of get in type theory one needs the Axiom of Unique Choice. □

5. Predomains

In classical domain theory for the purpose of explaining the meaning of recursively defined elements of data types one uses so-called *cpos*, *i.e.* partially ordered sets where every ascending chain has a limit (*i.e.* a supremum). In this section we will give an "order-free" analogue of this completeness requirement which makes sense for arbitrary sets (and types).

We also investigate some stronger notions of completeness guaranteeing closure under lifting. The least restrictive notion of that kind is the notion of well-completeness introduced by Longley and Simpson in (LS96) whereas the most restrictive notion of that kind is S-repleteness as introduced by Hyland and Taylor in (Hyl91; Tay91). It will turn out that all S-replete sets will be S-spaces.

In the following *S-spaces that are well-complete or even S-replete will be called predomains.*

5.1. *Complete Sets*

Now we introduce an "order-free" analogue of the requirement that all ascending chains have limits. But, in the vein of staying "order-free" we consider so-called ω-chains instead of $\mathbb{N}$-indexed ascending chains where an ω-chain in a set A is a map $a : \omega \to A$.

Although any ω-chain $a : \omega \to A$ induces an ascending chain $a \circ \mathsf{step} : \mathbb{N} \to A$ it is not the case for arbitrary realisability models that every ascending chain $c : \mathbb{N} \to A$ can be obtained as $a \circ \mathsf{step}$ for some ω-chain $a : \omega \to A$ (see (LS96) for examples of S-spaces where all ω-chains have limits but some ascending $\mathbb{N}$-chains don't).

In the following we restrict attention to complete sets – though completeness could be defined for all types – simply because that is sufficient for our purposes.

Definition 5.1. A set A is called *complete* iff $\iota \perp A$, *i.e.* for all ω-chains $a : \omega \to A$ there is a unique map $\overline{a} : \overline{\omega} \to A$ with $\overline{a} \circ \iota = a$

$$\begin{tikzcd} \omega \arrow[r, tail, "\iota"] \arrow[d, tail, "a"'] & \overline{\omega} \arrow[dl, dashed, tail, "\overline{a}"] \\ A & \end{tikzcd}$$

Thus, the complete sets are the $\mathcal{L}$-complete sets for $\mathcal{L} = \{\iota\}$.

The following axiom states that S is complete.

Axiom 6. The map $\iota : \omega \to \overline{\omega}$ is an S-iso, *i.e.* $S(\iota)$ is an isomorphism.

Remark 5.1. Notice that our definition of "completeness" is an internal version of the external notion of completeness introduced by Longley and Simpson (LS96) for realisability models.
Notice also that the above definition of "completeness" is nothing but a weakening of Hyland and Taylor's notion of "replete object" (Hyl91; Tay91) (*cf.* also Section 5.3) where A is replete iff $e \perp A$ for *all* S-isos e. Thus, "complete" in the sense of the definition above is a minimal notion of closure whereas "replete" is the strongest possible one which still includes S.

5.1.1. *Closure Properties of Complete Sets* First we consider closure under dependent products.

Theorem 5.2. If $A \in \mathsf{Set}$ and $B : A \to \mathsf{Set}$ is a family of complete sets then $\Pi x{:}A.\, B(x)$ is a complete set, too. Moreover, if Set is an impredicative universe then complete S-spaces are closed under arbitrary dependent products.

Proof. Immediate from Theorems 3.2 and 3.12 by choosing $\mathcal{L} \triangleq \{\iota\}$. □

Similarly it follows that complete sets are closed under equalizers.

Theorem 5.3. Let $f_1, f_2 : A \to B$ be maps between complete sets. Then the equalizer $m : E \rightarrowtail A$ of f_1 and f_2 is complete, too.

Proof. Immediate by Lemma 3.5 by choosing $\mathcal{L} \triangleq \{\iota\}$. □

5.1.2. *A Representation Theorem for Complete S-spaces*

Theorem 5.4. A set is a complete S-space iff it is a complete subobject of some power of S by a set.

Proof. Immediate by Theorem 3.13 by putting $\mathcal{L} \triangleq \{\iota\}$. □

5.2. *Well-complete Sets*

The main problem with complete S-spaces is that they are not (known to be) closed under lifting. For this reason one has to look for a more restrictive notion which nevertheless satisfies all the closure properties of complete sets. Again, we just consider well-complete *sets*, though the notion would make sense for arbitrary types, too.

Definition 5.2. A set A is called *well-complete* iff its lifting $L\,A$ is orthogonal to ι, *i.e.* iff $L\,A$ is complete.

The following lemma identifies well-completeness as an instance of $\mathcal{L}$-completeness.

Lemma 5.5. A set A is well-complete iff $m^*\iota \perp A$ for all S-subobjects m of $\overline{\omega}$ (where $m^*\iota$ is the pullback of ι along m). Thus, a set is well-complete iff it is $\mathcal{L}$-complete for $\mathcal{L} \triangleq \{\, m^*\iota \mid m\ S\text{-subobject of } \overline{\omega}\,\}$.

Proof. Straightforward from Lemma 4.8. ☐

From this it follows that complete subsumes well-complete.

Lemma 5.6. Any well-complete set is complete.

Proof. If a set A is orthogonal to all $m^*\iota$ then A is orthogonal to ι by choosing for m the S-mono $id_{\overline{\omega}}$. ☐

Theorem 5.7. Well-complete sets are closed under equalizers, lifting, and dependent products of families indexed by a set. Moreover, if Set is impredicative then well-complete sets are closed even under arbitrary dependent products.

Proof. By Lemma 5.5 the well-complete sets are the $\mathcal{L}$-complete sets for $\mathcal{L} \triangleq \{\, m^*\iota \mid m\ S\text{-subobject of } \overline{\omega}\,\}$. Then, from Theorems 3.2 and 3.5 one gets closure under dependent products and equalizers, respectively. Closure under lifting is a consequence of Corollary 4.11 and Lemma 4.12. ☐

Corollary 5.8. Well-complete S-spaces are closed under equalizers, lifting, and dependent products of families indexed by sets. If Set is impredicative then well-complete S-spaces are closed even under arbitrary products.

Proof. Immediate from previous theorem and closure properties of S-spaces. ☐

Lemma 5.9. S is well-complete, *i.e.* $\iota \perp L\,S$.

Proof. S is complete by Axiom 6 and, therefore, $S \times S$ is complete, too. Furthermore, $L\,S$ appears as a retract of $S \times S$ via the maps $e : L\,S \to S \times S$ and $p : S \times S \to L\,S$ where $e(\langle u, f\rangle) \triangleq \langle u, u \angle f\rangle$ and $p(\langle u, v\rangle) \triangleq \langle u, \lambda p{:}\mathsf{def}(u).\, v\rangle$.
Thus, $L\,S$ appears as equaliser of $e \circ p$ and $id_{S\times S}$ from which it follows that $L\,S$ is complete as $S \times S$ is complete. ☐

Notice, that from this and Lemma 5.6 it follows that Axiom 6 is equivalent to S being well-complete (as the proof of Lemma 5.6 does not use Axiom 6).

5.2.1. *A Representation Theorem for Well-Complete S-Spaces*

Theorem 5.10. A set is a well-complete S-space iff it is a well-complete subobject of some power of S.

Proof. Immediate from Theorem 3.13 by taking for $\mathcal{L}$ the collection of all pullbacks of ι along S-subobjects of $\overline{\omega}$. ☐

5.3. *S-Replete Sets*

The notion of S-replete object or set was originally introduced by (Hyl91) and (Tay91) in the context of a more restrictive axiomatic setting taylored towards a domain theory *á la Scott*. But their definition makes sense in our more general setting, too.

In the previous subsection we have defined a set A to be "complete" iff $\iota \perp A$, *i.e.* iff A is closed under limits of ω-chains, whereas we now will call a set A to be *S-replete* or simply *replete* iff $e \perp A$ *for all* S-isos e. In (Str97) it has been explained in which sense S-isos can be considered as "generalised limit processes" and that, accordingly, S-repleteness of a set may be understood as being "closed under all generalised limit processes" (as given by S-isos).

Definition 5.3. A set A is called *S-replete* iff $e \perp A$ for all S-isos e.

Remark 5.11. Obviously, this is again an instance of $\mathcal{L}$-completeness with $\mathcal{L}$ being the collection of *all* S-isos. This observation is crucial for obtaining the subsequent closure properties.

Lemma 5.12. Dependent products and equalizers preserve S-repleteness. If Set is impredicative then S-replete sets are closed under arbitrary dependent products.

Proof. By Lemma 3.2 and Theorem 3.5. □

Lemma 5.13. The S-replete sets are closed under lifting.

Proof. By Corollary 4.11 and Lemma 4.12. □

Next we show that S-replete sets are already S-spaces. But first we need an auxiliary lemma (due to P. Taylor) characterising S-isos.

Lemma 5.14. A map $e : X \to Y$ is an S-iso iff e is an S-epi and $g \circ e = \varepsilon_X$ for some $g : Y \to S^2(X)$.

Proof. "$\Rightarrow$": If e is an S-iso, *i.e.* $S(e)$ is an iso, then $S(e)$ is monic, *i.e.* e is an S-epi. A (unique) $g : Y \to S^2(X)$ with $\varepsilon_X = g \circ e$ exists as $S^2(X)$ is S-replete.
"$\Leftarrow$": Suppose that $S(e)$ is monic and $\varepsilon_X = g \circ e$ for some $g : Y \to S^2(X)$. Let $p : X \to S$. We have to show that p can be extended uniquely to a map $\overline{p} : Y \to S$ with $\overline{p} \circ e = p$. Uniqueness follows from the assumption that e is an S-epi. For existence we define $\overline{p} : Y \to S$ as $\overline{p}(y) \triangleq (g\,y)\,p$ for which we have

$$\overline{p}(e(x)) = g(e(x))(p) = (g \circ e)(x)(p) = (\varepsilon_X(x))\,p = p(x)$$

as desired. □

Next we show that S-replete sets are automatically S-spaces.

Theorem 5.15. All S-replete sets are S-spaces.

Proof. Let A be S-replete. Let $\varepsilon_A = m_A \circ e_A$ be an epi-mono-factorisation of ε_A, *i.e.* $e_A : A \to R(A)$ is epic and $m_A : R(A) \rightarrowtail S^2(A)$ is monic, as exhibited in the diagram

$$\begin{tikzcd}
A \arrow[r, two heads, "e_A"] \arrow[dr, "\varepsilon_A"'] & R(A) \arrow[d, tail, "m_A"] \\
 & S^2(A)
\end{tikzcd}$$

As e_A is epic it is also an S-epi and therefore by Lemma 5.14 it follows that e_A is an S-iso. Thus, there is a unique map $i_A : R(A) \to A$ with $id_A = i_A \circ e_A$ as by assumption A is S-replete. So e_A is a split mono. As e_A is also epic it follows that e_A is actually an isomorphism.

Thus, the map ε_A is monic and, therefore, A appears as a subobject of $S^2(A)$ from which it follows by the representation theorem that A is an S-space. $\square$

This gives rise to the following observation.

Corollary 5.16. Any S-replete set is a well-complete (and thus complete) S-space.

Proof. By Lemma 5.15 A is a S-space. It is also well-complete as the S-isos are closed under pullbacks along S-subobjects (by Lemma 4.12). $\square$

6. Domains

The complete, well-complete and S-replete *S-spaces* studied so far are "synthetic" analogues of "classical" *predomains*, *i.e.* partial orders where every ascending chain has a supremum. As known from classical domain theory for the purposes of fixpoint theory one has to consider *domains*, *i.e.* predomains admitting a least element.

For S-spaces the synthetic analogue of the existence of a least element will be the existence of a so-called focal L-algebra structure. For arbitrary sets A there may be several different focal L-algebra structures on A. It turns out that for S-spaces focal L-algebra structures are unique provided they exist. Such S-spaces will be called *pointed S-spaces.*

Accordingly, *domains will be pointed S-spaces that are well-complete or even replete.*

6.1. *Focal L-Algebras and Pointed S-Spaces*

Definition 6.1. An L-algebra $\alpha : L\,A \to A$ is called *focal* iff $\alpha \circ \eta_A = id_A$. The element $\alpha(\perp_A)$ will be referred to as $\perp_\alpha$.
If $\alpha : L\,A \to A$ and $\beta : L\,B \to B$ are focal L-algebras then a *homomorphism from α to β* is a map $h : A \to B$ with

$$h \circ \alpha = \beta \circ L\,h$$

Notice that in general the focal L-algebra structure on a set A need not be unique.

We now formulate the last two axioms which guarantee the uniqueness of focal L-algebra structures for S-spaces and, moreover, are sufficient for proving all further results of this paper.

Axiom 7. For any set A the map $c_A = [\perp_A, \eta_A] : 1 + A \to L\,A$ is an S-epi, *i.e.*

$$p = q \Leftrightarrow p \circ c_A = q \circ c_A$$

for all $p, q : L\,A \to S$.

That means that S-predicates $p, q : L\,A \to S$ are equal if and only if $p \circ \eta_A = q \circ \eta_A$ and $p(\perp_A) = q(\perp_A)$, *i.e.* when p and q have the same behaviour on $\perp_A$ and elements of the form $\eta_A(a)$.

Axiom 8. For all $u \in S$ and $f : S \to S$ if $f(u) = \top$ then $f(\top) = \top$.

In other words, for all endo-maps f of S we have $f(\top) = \top$ if $f(u) = \top$ for some $u \in S$. That means that $\top$ is the greatest element w.r.t. the partial order $\sqsubseteq$ on S where $u \sqsubseteq_S v$ iff $f(u) = \top$ implies $f(v) = \top$ for all $f : S \to S$ (more about this after Definition 8.2).

From Axiom 7 we immediately get that any c_A is epic w.r.t. S-spaces.

Lemma 6.1. Let A be a set and B be an S-space. If $f_1, f_2 : A \to B$ with $f_1 \circ c_A = f_2 \circ c_A$ then $f_1 = f_2$.

Proof. As B is an S-space it suffices to prove that for all $p \in S(B)$ it holds that $p \circ f_1 = p \circ f_2$. Thus, by Axiom 7 it suffices to show that $p \circ f_1 \circ c_A = p \circ f_2 \circ c_A$ but this follows immediately from the assumption that $f_1 \circ c_A = f_2 \circ c_A$. □

Lemma 6.2. Let α be a focal L-algebra and $p \in S(A)$. Then for all $a \in A$ it holds that $p(a) = \top$ whenever $p(\alpha(\perp_A)) = \top$.

Proof. Let $a \in A$ and $h_a : S \to LA$ be the map defined as $h_a(u) \triangleq \langle u, \lambda p : \mathsf{def}(u).\, a\rangle$. Then $h_a(\perp) = \perp_A$ and $h_a(\top) = \eta_A(a)$. Thus, we have

$$(p \circ \alpha \circ h_a)(\perp) = p(\alpha(\perp_A)) = \top$$

$$(p \circ \alpha \circ h_a)(\top) = p(\alpha(\eta_A(a))) = p(a)$$

from which it follows by Axiom 8 that $p(a) = \top$. □

Theorem 6.3. An S-space A admits at most one focal L-algebra structure.

Proof. Suppose that $\alpha_1 : L\,A \to A$ and $\alpha_2 : L\,A \to A$ are focal L-algebra structures on an S-space A.
By the previous Lemma 6.1 for $\alpha_1 = \alpha_2$ it is sufficient to show that $\alpha_1 \circ c_A = \alpha_2 \circ c_A$, *i.e.* it suffices to show that $\alpha_1 \circ \eta_A = \alpha_2 \circ \eta_A$ and $\alpha_1(\perp_A) = \alpha_2(\perp_A)$. As α_1 and α_2 are focal L-algebras we have $\alpha_1 \circ \eta_A = id_A = \alpha_2 \circ \eta_A$. Thus, it remains to check that $\alpha_1(\perp_A) = \alpha_2(\perp_A)$.
As A is an S-space it suffices to show that $p(\alpha_1(\perp_A)) = \top$ iff $p(\alpha_2(\perp_A)) = \top$ but this follows immediately from Lemma 6.2 above. □

Lemma 6.4. Let A and B be S-spaces admitting (unique) focal L-algebra structures $\alpha : LA \to A$ and $\beta : LB \to B$, respectively. Then, for maps $f : A \to B$ the following two conditions are equivalent

(1) f is an L-algebra morphism from α to β
(2) f is strict, *i.e.* $f(\perp_\alpha) = \perp_\beta$.

Proof. The map $f : A \to B$ is an L-algebra morphism iff $f \circ \alpha = \beta \circ Lf$. By Lemma 6.1 this is equivalent to the following two requirements

$$f \circ \alpha \circ \eta_A = \beta \circ Lf \circ \eta_A$$

$$f(\alpha(\perp_A)) = \beta(Lf(\perp_A))$$

where the first requirement is valid anyway as we have

$$f \circ \alpha \circ \eta_A = f = \beta \circ \eta_B \circ f = \beta \circ Lf \circ \eta_A$$

and the second requirement is equivalent to $f(\alpha(\perp_A)) = \beta(\perp_B)$ as $Lf(\perp_A) = \perp_B$. □

Definition 6.2. A *pointed S-space* is an S-space A admitting a focal L-algebra structure denoted by α_A (as it is unique by Theorem 6.3 provided it exists). Furthermore, for a pointed S-space A we write $\perp^A$ for $\alpha_A(\perp_A)$ and call it “the bottom element of A”.

A function $f : A \to B$ between pointed S-spaces A and B will be called *strict* iff f preserves bottom elements, *i.e.* $f(\bot^A) = \bot^B$.

6.2. *Closure Properties of Focal L-Algebras and Pointed S-Spaces*

Theorem 6.5. Focal L-algebras are closed under dependent products.

Proof. Let $(\alpha_i : L\,A_i \to A_i \mid i \in I\,)$ be a family of focal L-algebras. Let $P \triangleq \Pi i{:}I.\,A_i$ be the (dependent) product of the underlying sets. Let $\varphi : L\,P \to P$ be the unique map with

$$\pi_i \circ \varphi \triangleq \alpha_i \circ L(\pi_i)$$

for all $i \in I$. We have $\varphi \circ \eta_P = id_P$ as for all $i \in I$ it holds that

$$\alpha_i \circ L(\pi_i) \circ \eta_P = \alpha_i \circ \eta_{A_i} \circ \pi_i = \pi_i$$

Thus φ endows P with the structure of a focal L-algebra.
It remains to show that for any focal L-algebra $\beta : L\,B \to B$ and any family $(h_i : \beta \to \alpha_i \mid i \in I\,)$ of homomorphisms the unique map $\overline{h} : B \to P$ with $h_i = \pi_i \circ \overline{h}$ for all $i \in I$ is actually a homomorphism, *i.e.* that $\varphi \circ L\,\overline{h} = \overline{h} \circ \beta$. But this is the case as we have

$$\pi_i \circ \varphi \circ L\,\overline{h} = \alpha_i \circ L(\pi_i) \circ L\,\overline{h} = \alpha_i \circ L(\pi_i \circ \overline{h}) = \alpha_i \circ L\,h_i = h_i \circ \beta = \pi \circ \overline{h} \circ \beta$$

for all $i \in I$. □

Corollary 6.6. Pointed S-spaces are closed under dependent products of families indexed by a set. Moreover, if Set is impredicative then pointed S-spaces are closed under all dependent products.

Proof. Immediate from the previous theorem and closure properties of S-spaces. □

Theorem 6.7. Focal L-algebras are closed under equalisers of L-algebra morphisms.

Proof. Let $\alpha : L\,A \to A$ and $\beta : L\,B$ be focal L-algebras and $f, g : A \to B$ homomorphisms. Assume that $e : E \rightarrowtail A$ is an equaliser of f and g. First notice that

$$f \circ \alpha \circ L\,e = \beta \circ L\,f \circ L\,e = \beta \circ L\,(f \circ e) = \beta \circ L\,(g \circ e) = \beta \circ L\,g \circ L\,e = g \circ \alpha \circ L\,e$$

i.e. $\alpha \circ L\,e$ equalisers f and g. Thus, there is a unique map $h : L\,E \to E$ with $e \circ h = \alpha \circ L\,e$ endowing E with a focal L-algebra structure as we have

$$e \circ h \circ \eta_E = \alpha \circ L\,e \circ \eta_E = \alpha \circ \eta_A \circ e = e$$

and since e is monic this entails $h \circ \eta_E = id_E$. As we have already shown that $e \circ h = \alpha \circ L\,e$ we also get that $e : h \to \alpha$, *i.e.* h is an L-algebra-morphism.
It remains to show that h has the universal property. Let $\gamma : L\,C \to C$ be a focal L-algebra and $k : \gamma \to \alpha$ a homomorphism with $f \circ k = g \circ k$. Then there exists a unique $k' : C \to E$ with $k = e \circ k'$. We have

$$e \circ k' \circ \gamma = k \circ \gamma = \alpha \circ L\,k = \alpha \circ L(e \circ k') = \alpha \circ L\,e \circ L\,k' = e \circ h \circ L\,k'$$

and therefore $k' \circ \gamma = h \circ L\,k'$, *i.e.* $k' : \gamma \to h$. □

Corollary 6.8. Pointed S-spaces are closed under equalisers of strict maps.

Proof. Immediate from the previous theorem and closure of S-spaces under equalisers. ☐

Next we show that there are enough L-algebras.

Lemma 6.9. Let A be a set and let $\mu_A : LLA \to L\,A$ be the classifier of the partial map $(\eta_{L\,A} \circ \eta_A, id_A)$

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, tail, "\eta_{L\,A} \circ \eta_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta_A"] \\ L^2 A \arrow[r, "\mu_A"'] & L\,A \end{tikzcd}$$

then μ_A is a focal L-algebra on LA.

Proof. As both $\mu_A \circ \eta_{L\,A}$ and $id_{L\,A}$ classify the partial map (η_A, id_A) it follows by uniqueness of classifying maps that $\mu_A \circ \eta_{L\,A} = id_{L\,A}$. ☐

Remark 6.10. Both η and μ are components of natural transformations endowing L with a *monad structure*. Therefore, L is often called the *lifting monad*.

Corollary 6.11. If A is an S-space then $L\,A$ is a pointed S-space.

Proof. Immediate from the previous Lemma 6.9 and closure of S-spaces under lifting. ☐

7. Fixpoints

We will next show that for any focal L-algebra structure $\alpha : L\,A \to A$ on a complete set A any map $f : A \to A$ admits a *canonical fixpoint* which can be constructed *à la Kleene* as in classical domain theory. Since well-complete and S-replete sets are all complete the results of this section are valid for all notions of predomains considered in this paper.

The exposition is essentially based on Rosolini's (Ros95).

Lemma 7.1. Let $\alpha : LA \to A$ be a focal L-algebra. For any $f : A \to A$ there is a canonical map $\mathsf{kl}_{\alpha,f} : \omega \to A$ called *Kleene chain of f (w.r.t. α)*, such that

$$\mathsf{kl}_{\alpha,f}(\mathsf{step}(n)) = f^n(\bot_\alpha)$$

for all $n \in \mathbb{N}$.

Proof. Let $f_\alpha \triangleq \alpha \circ L(f) : L\,A \to A$. Then by initiality of ϕ there exists a unique map $\mathsf{kl}_{\alpha,f} : \omega \to A$ such that

$$\begin{tikzcd} L\,\omega \arrow[r, "\phi"] \arrow[d, "L(\mathsf{kl}_{\alpha,f})"'] & \omega \arrow[d, "\mathsf{kl}_{\alpha,f}"] \\ L\,A \arrow[r, "f_\alpha"'] & A \end{tikzcd}$$

commutes. Moreover, we have

$$\begin{tikzcd}
1+\mathbb{N} \arrow[rr, "{[0,s]}", "\simeq"'] \arrow[d, "1+\mathsf{step}"'] & & \mathbb{N} \arrow[d, "\mathsf{step}"] \\
1+\omega \arrow[r, "c_\omega"] & L\,\omega \arrow[r, "\phi"] \arrow[d, "L(\mathsf{kl}_{\alpha,f})"'] & \omega \arrow[d, "\mathsf{kl}_{\alpha,f}"] \\
& L\,A \arrow[r, "f_\alpha"'] & A
\end{tikzcd}$$

which gives rise to the equations

$$\mathsf{kl}_{\alpha,f}(\mathsf{step}(0)) = \bot_\alpha$$

as $\mathsf{kl}_{\alpha,f}(\mathsf{step}(0)) = f_\alpha(\bot_A) = (\alpha \circ L\,f)(\bot_A) = \alpha(\bot_A) = \bot_\alpha$ and

$$\mathsf{kl}_{\alpha,f}(\mathsf{step}(n+1)) = f(\mathsf{kl}_{\alpha,f}(\mathsf{step}(n)))$$

as

$$\begin{aligned}
\mathsf{kl}_{\alpha,f}(\mathsf{step}(n+1)) &= (f_\alpha \circ L(\mathsf{kl}_{\alpha,f}) \circ \eta_\omega)(\mathsf{step}(n)) \\
&= (\alpha \circ L\,f \circ L(\mathsf{kl}_{\alpha,f}) \circ \eta_\omega)(\mathsf{step}(n)) \\
&= (\alpha \circ \eta_A \circ f \circ \mathsf{kl}_{\alpha,f})(\mathsf{step}(n)) = (f \circ \mathsf{kl}_{\alpha,f})(\mathsf{step}(n)) \\
&= f(\mathsf{kl}_{\alpha,f}(\mathsf{step}(n)))\ .
\end{aligned}$$

From these two equations it follows that $\mathsf{kl}_{\alpha,f}(\mathsf{step}(n)) = f^n(\bot_\alpha)$ by induction on n. $\square$

Corollary 7.2. Let A be a complete set and $\alpha : L\,A \to A$ be a focal L-algebra. For any $f : A \to A$ let $\overline{\mathsf{kl}_{\alpha,f}}$ be the unique extension of the Kleene-chain $\mathsf{kl}_{\alpha,f}$ along ι, *i.e.* $\mathsf{kl}_{\alpha,f} = \overline{\mathsf{kl}_{\alpha,f}} \circ \iota$. Then

$$\overline{\mathsf{kl}_{\alpha,f}}(\overline{\mathsf{step}}(n)) = f^n(\bot_\alpha)$$

for all $n \in \mathbb{N}$.

Proof. Immediate by the previous lemma as $\overline{\mathsf{kl}_{\alpha,f}} \circ \overline{\mathsf{step}} = \overline{\mathsf{kl}_{\alpha,f}} \circ \iota \circ \mathsf{step} = \mathsf{kl}_{\alpha,f} \circ \mathsf{step}$. $\square$

Theorem 7.3. (G. Rosolini (Ros95))
Let A be a complete set and $\alpha : LA \to A$ be a focal L-algebra structure on A. Then for any endomap $f : A \to A$ the object

$$\mathsf{fix}_\alpha(f) \triangleq \overline{\mathsf{kl}_{\alpha,f}}(\infty) \in A$$

is a fixpoint of f, *i.e.*

$$f(\mathsf{fix}_\alpha(f)) = \mathsf{fix}_\alpha(f)$$

Proof. Recall that $\overline{\sigma} : \overline{\omega} \to \overline{\omega}$ has been defined as $\upsilon^{-1} \circ \eta_{\overline{\omega}} : \overline{\omega} \to \overline{\omega}$ and that $\overline{\sigma}(\infty) = \infty$ by Lemma 4.5.
Thus, for proving $f(\mathsf{fix}_\alpha(f)) = \mathsf{fix}_\alpha(f)$ it is sufficient to prove

$$\overline{\mathsf{kl}_{\alpha,f}} \circ \overline{\sigma} = f \circ \overline{\mathsf{kl}_{\alpha,f}}$$

as this entails

$$f(\mathsf{fix}_\alpha(f)) = (f \circ \overline{\mathsf{kl}_{\alpha,f}})(\infty) = (\overline{\mathsf{kl}_{\alpha,f}} \circ \overline{\sigma})(\infty) = \overline{\mathsf{kl}_{\alpha,f}}(\overline{\sigma}(\infty)) = \overline{\mathsf{kl}_{\alpha,f}}(\infty)) = \mathsf{fix}_\alpha(f)\ .$$

As A is complete it is sufficient to show

$$\overline{\mathsf{kl}_{\alpha,f}} \circ \overline{\sigma} \circ \iota = f \circ \overline{\mathsf{kl}_{\alpha,f}} \circ \iota$$

which follows from the following commuting diagram

$$\begin{tikzcd}
 & & & L\,A \arrow[dd, "f_\alpha"] \\
\omega \arrow[r, "\eta_\omega"] \arrow[d, "\iota"'] & L\,\omega \arrow[r, "\phi"'] \arrow[d, "L\,\iota"'] \arrow[urr, "L(\mathsf{kl}_{\alpha,f})"] & \omega \arrow[d, "\iota"'] \arrow[dr, "\mathsf{kl}_{\alpha,f}"] & \\
\overline{\omega} \arrow[r, "\eta_{\overline{\omega}}"'] \arrow[rr, bend right, "\overline{\sigma}"'] & L\,\overline{\omega} \arrow[r, "\upsilon^{-1}"'] & \overline{\omega} \arrow[r, "\overline{\mathsf{kl}_{\alpha,f}}"'] & A
\end{tikzcd}$$

as

$$\overline{\mathsf{kl}_{\alpha,f}} \circ \overline{\sigma} \circ \iota = f_\alpha \circ L(\mathsf{kl}_{\alpha,f}) \circ \eta_\omega =$$

$$\alpha \circ L\,f \circ L(\mathsf{kl}_{\alpha,f}) \circ \eta_\omega = \alpha \circ L\,(f \circ \mathsf{kl}_{\alpha,f}) \circ \eta_\omega =$$

$$\alpha \circ \eta_A \circ (f \circ \mathsf{kl}_{\alpha,f}) = f \circ \overline{\mathsf{kl}_{\alpha,f}} \circ \iota$$

by definition of f_α, naturality of η and focality of α. $\square$

8. Domain-Theoretic Induction Principles

It is not sufficient to explain the meaning of recursive definitions of objects (typically recursively defined functions) by means of a canonical fixpoint construction for endofunctions on focal L-algebras. For the purposes of *program verification* one also needs proof principles for these canonical fixpoints.

For example if one considers the identity map id_A on a focal L-algebra $\alpha : L\,A \to A$ then all elements of A appear as fixpoints of id_A but the canonical fixpoint construction *à la Kleene* picks out precisely one element of A. Of course, the Kleene chain $\mathsf{kl}_{\alpha,id_A}$ is constant with value $\bot_\alpha = \alpha(\bot_A)$ and so is $\overline{\mathsf{kl}_{\alpha,id_A}}$ from which we may conclude that $\mathsf{fix}_\alpha(id_A) = \bot_\alpha$. But usually, the Kleene chain will not be constant and, therefore, for arbitrary $f : A \to A$ it is more difficult to predict the behaviour $\overline{\mathsf{kl}_{\alpha,f}}$ at ∞.

The traditional methods for verifying properties of recursively defined objects were called "induction principles" because *formally* they resemble the scheme of induction over $\mathbb{N}$. The most prominent induction principles known from classical domain theory are *Fixpoint Induction*, *Computational Induction* and *Park Induction*.

We now will derive "synthetic" analogues of these induction principles. This is a somewhat delicate question as we deliberately want to avoid reference to order-theoretic notions and, accordingly, cannot employ the "classical" characterisations of canonical fixpoints as *least fixpoints* or *order-theoretic suprema of the Kleene chain*.

It is well known from classical domain theory that most of the induction principles are *admissible* only for predicates that are well-behaved w.r.t. limits, namely the so-called *admissible predicates.*

Classically, a predicate P on a cpo A is called *admissible* iff P is closed under order-theoretic suprema of ascending chains, *i.e.* $P(\bigsqcup a)$ whenever $a : \mathbb{N} \to A$ is an ascending chain that is contained in P, *i.e.* $P(a(n))$ for all $n \in \mathbb{N}$. The straightforward "synthetic" analogue of the "classical" notion of admissibility would be the requirement that P is complete, *i.e.* orthogonal to ι, because the key idea of the synthetic approach is to replace ascending $\mathbb{N}$-indexed chains by ω-chains.

Alas, this straightforward choice would enforce a *reformulation* already of fixpoint induction as illustrated by the following theorem and subsequent discussion.

Theorem 8.1. *("Synthetic" Fixpoint Induction)*
Let A be a complete set, $f : A \to A$ and $\alpha : LA \to A$ be a focal L-algebra. If $m : P \rightarrowtail A$ is complete and also a subalgebra of f_α, *i.e.* $f_\alpha \circ L(m)$ factors through m via some $\pi : LP \to P$ as illustrated in the following diagram

$$\begin{tikzcd} LP \arrow[r, dashed, "\pi"] \arrow[d, "Lm"'] & P \arrow[d, tail, "m"] \\ LA \arrow[r, "f_\alpha"'] & A \end{tikzcd}$$

then $\overline{\mathsf{kl}_{\alpha,f}}$ factors through m and, therefore, $\mathsf{fix}_\alpha(f) \in P$.

Proof. By initiality of $\phi : L\omega \to \omega$ there exists a unique map $h : \omega \to P$ with $h \circ \phi = \pi \circ Lh$. Again by initiality of ϕ we get that $\mathsf{kl}_{\alpha,f} = m \circ h$.
As by assumption P is complete there is a unique $\overline{h} : \overline{\omega} \to P$ with $h = \overline{h} \circ \iota$. It follows that

$$\overline{\mathsf{kl}_{\alpha,f}} \circ \iota = \mathsf{kl}_{\alpha,f} = m \circ h = m \circ \overline{h} \circ \iota$$

and, therefore, by completeness of A we get

$$\overline{\mathsf{kl}_{\alpha,f}} = m \circ \overline{h}$$

which entails

$$\mathsf{fix}_\alpha(f) = \overline{\mathsf{kl}_{\alpha,f}}(\infty) = m(\overline{h}(\infty))$$

from which it follows that $P(\mathsf{fix}_\alpha(f))$. □

Notice that the above theorem does not provide us with "classical" fixpoint induction as the premiss of "synthetic" fixpoint induction is that the complete subobject P is a sub-L-algebra of f_α whereas the premiss of the "classical" fixpoint induction principle is that $P(\bot_\alpha)$ and $\forall x{:}A.\, P(x) \Rightarrow P(f(x))$. That these are not necessarily equivalent can be seen by unfolding the requirement "π is a sub-L-algebra of f_α" as

$$\forall u{:}S.\ \forall a{:}A^{\mathsf{def}(u)}.\ (\forall p{:}\mathsf{def}(u).\, P(a(p))) \Rightarrow P(\alpha(\langle u, f \circ a\rangle))$$

which would be equivalent to the premiss of classical fixpoint induction if it were sufficient to analyse the cases $u = \top$ and $u = \bot$. But such case analysis is not a valid proof principle in general. Actually, the proposition

$$\forall u{:}S.\, u = \top \vee u = \bot$$

is wrong in all realisability models. However, the weaker statement

$$\forall u{:}S.\, \neg\neg(u = \top \vee u = \bot)$$

can be derived form our axioms and, therefore, we have

$$P(\bot) \wedge P(\top) \Rightarrow \forall u{:}S.\, P(u)$$

provided the predicate P is *$\neg\neg$-closed*, *i.e.* $\forall u{:}S.\, \neg\neg P(u) \Rightarrow P(u)$. Of course, this is only useful if sufficiently many predicates P on S are $\neg\neg$-closed. But this is the case in realisability models and has been exploited axiomatically in the first author's Thesis (Reu95; Reu96) by postulating a generalised Markov's Principle which is further discussed in Section 9.

The rest of this section, however, is devoted to the derivation of induction principles without such an assumption. There are two ways how to proceed. One is to content oneself with the "synthetic" fixpoint induction principle of Theorem 8.1. The other one is to employ a more suitable notion of "admissible predicate" validating the traditional form of fixpoint induction. That is precisely what we do next.

8.1. *Admissible Predicates*

Recall that in classical domain theory a predicate P on a cpo A is called "admissible" iff for every ascending chain $a : \mathbb{N} \to A$ its limit $\bigsqcup A$ is in P provided all $a(n)$ are in P.

Replacing "ascending chain" by "ω-chain", "$a(n)$" by "$a(\mathsf{step}(n))$" and "$\bigsqcup A$ is in P" by "$\overline{a}$ factors through P" we obtain the following notion of "admissibility".

Definition 8.1. Let A be a complete set. Then a subobject $m_P : P \rightarrowtail A$ is called *admissible* iff $\overline{\mathsf{step}} \perp m_P$.

Thus $m_P : P \rightarrowtail A$ is admissible iff for all $a : \omega \to A$ the unique extension $\overline{a} : \overline{\omega} \to A$ factors through m_P provided $a \circ \mathsf{step} = \overline{a} \circ \overline{\mathsf{step}}$ factors through m_P, *i.e.* there exists a (necessarily unique) map $\overline{p} : \overline{\omega} \to P$ such that

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\mathsf{step}"] \arrow[d] & \omega \arrow[r, tail, "\iota"] & \overline{\omega} \arrow[d, "\overline{a}"] \arrow[dll, dashed, "\overline{p}"'] \\
P \arrow[rr, tail, "m_P"'] & & A
\end{tikzcd}$$

commutes. This is more akin to the "classical" definition of "admissible predicate" as it says that for an ω-chain $a : \omega \to A$ the image of $\overline{a} : \overline{\omega} \to A$ is contained in P iff the image of $a \circ \mathsf{step} = \overline{a} \circ \overline{\mathsf{step}}$ is contained in P, *i.e.* iff all "finite approximations" $a(\mathsf{step}(n))$ satisfy predicate P.

Next we show that admissible predicates on complete sets are always complete.

Lemma 8.2. Let A be a complete set and $m_P : P \rightarrowtail A$ be an admissible subobject of A then P is complete, *i.e.* $\iota \perp m_P$.

Proof. Let $p : \omega \to P$. Define $a \triangleq m_P \circ p$. As A is complete there exists a unique $\overline{a} : \overline{\omega} \to A$ with $\overline{a} \circ \iota = a$. It holds that

$$\overline{a} \circ \overline{\mathsf{step}} = m_P \circ p \circ \mathsf{step}$$

as $\overline{a} \circ \overline{\mathsf{step}} = \overline{a} \circ \iota \circ \mathsf{step} = a \circ \mathsf{step} = m_P \circ p \circ \mathsf{step}$. Thus, by admissibility of P, there exists a (necessarily unique) map $\overline{p} : \overline{\omega} \to P$ with

$$m_P \circ \overline{p} = \overline{a} \quad \text{and} \quad \overline{p} \circ \overline{\mathsf{step}} = p \circ \mathsf{step}$$

Therefore, we get

$$m_P \circ \overline{p} \circ \iota = \overline{a} \circ \iota = a = m_P \circ p$$

which entails $\overline{p} \circ \iota = p$ as m_P is monic.
For uniqueness of $\overline{p}$ assume that $\tilde{p} : \overline{\omega} \to P$ is a map with $\tilde{p} \circ \iota = p$. Then

$$m_P \circ \tilde{p} \circ \iota = m_P \circ p = a = \overline{a} \circ \iota$$

which, by completeness of A, entails $m_P \circ \tilde{p} = \overline{a}$. Thus, we get $\tilde{p} = \overline{p}$ as m_P is monic and $m_P \circ \overline{p} = \overline{a}$. □

Furthermore, we can characterise admissible predicates on a complete set as the complete predicates which are orthogonal to step.

Lemma 8.3. A subobject $m_P : P \rightarrowtail A$ of a complete set A is admissible iff P is complete and $\mathsf{step} \perp m_P$.

Proof. Let $m_P : P \rightarrowtail A$ be a subobject of a complete set A.
"$\Leftarrow$": If m_P is orthogonal to both ι and step then m_P is also orthogonal to $\overline{\mathsf{step}} = \iota \circ \mathsf{step}$ (by a well-known straightforward argument).
"$\Rightarrow$": On the other hand, if m_P is admissible then $\iota \perp m_P$ by Lemma 8.2. For $\mathsf{step} \perp m_P$ assume that $a : \omega \to A$ is a map with $a \circ \mathsf{step} = m_P \circ h$ for some (necessarily unique) $h : \mathbb{N} \to P$. As A is complete there exists a (necessarily unique) extension $\overline{a} : \overline{\omega} \to A$ with $\overline{a} \circ \iota = a$. Therefore, we have

$$\overline{a} \circ \overline{\mathsf{step}} = m_P \circ h$$

and, therefore, by admissibility of m_P, there exists a (necessarily unique) map $\overline{h} : \overline{\omega} \to P$ with

$$m_P \circ \overline{h} = \overline{a} \quad \text{and} \quad \overline{h} \circ \overline{\mathsf{step}} = h \quad .$$

Putting $k \triangleq \overline{h} \circ \iota$ we then get

$$m_P \circ k = m_P \circ \overline{h} \circ \iota = \overline{a} \circ \iota = a$$

and

$$k \circ \mathsf{step} = \overline{h} \circ \iota \circ \mathsf{step} = \overline{h} \circ \overline{\mathsf{step}} = h$$

establishing the existence of the desired mediating arrow k. This k is unique as m_P is monic. □

For admissibility of equality predicates on complete S-spaces we need that $\overline{\mathsf{step}}$ is an S-epi. As a preparation for this we prove the following auxiliary lemma.

Lemma 8.4. S-epis are preserved by the lifting functor L.

Proof. Let $e : X \to Y$ be an S-epi. For showing that $L\,e$ is an S-epi assume that $p_1, p_2 : LY \to S$ with $p_1 \circ L\,e = p_2 \circ L\,e$. We have to show $p_1 = p_2$. As c_Y is an S-epi by Axiom 7 it suffices to show that $p_1 \circ c_Y = p_2 \circ c_Y$, *i.e.* that $p_1 \circ \bot_Y = p_2 \circ \bot_Y$ and $p_1 \circ \eta_Y = p_2 \circ \eta_Y$.
Notice that for $i = 1, 2$ we have

$$p_i \circ \bot_Y = p_i \circ L\,?_Y = p_i \circ L(e \circ ?_X) = p_i \circ L\,e \circ L\,?_X = p_i \circ L\,e \circ \bot_X$$

and, therefore,

$$p_1 \circ \bot_Y = p_1 \circ L\,e \circ \bot_X = p_2 \circ L\,e \circ \bot_X = p_2 \circ \bot_Y \quad .$$

As e is an S-epi for the second requirement it suffices to show that $p_1 \circ \eta_Y \circ e = p_2 \circ \eta_Y \circ e$. By naturality of η we have $\eta_Y \circ e = L\,e \circ \eta_X$ and, therefore,

$$p_1 \circ \eta_Y \circ e = p_1 \circ L\,e \circ \eta_X = p_2 \circ L\,e \circ \eta_X = p_2 \circ \eta_Y \circ e$$

as required. □

The idea behind the proof of the next theorem is due to Simpson, *cf.* (Sim96), where, however, it has been used for quite a different purpose.

Theorem 8.5. The maps $\mathsf{step} : \mathbb{N} \to \omega$ and $\overline{\mathsf{step}} : \mathbb{N} \to \overline{\omega}$ are S-epic.

Proof. As ι is an S-iso and $\overline{\mathsf{step}} = \iota \circ \mathsf{step}$ it suffices to show that step is an S-epi. For this purpose let $m : P \rightarrowtail \omega$ be the *greatest* subobject of ω through which step factors by an S-epi. Let $e : \mathbb{N} \to P$ be the unique map with $\mathsf{step} = m \circ e$. Of course, e is an S-epi. Notice that

$$\phi \circ L(\mathsf{step}) \circ c_{\mathbb{N}} = \phi \circ c_\omega \circ (1 + \mathsf{step}) = \mathsf{step} \circ [0, s]$$

as

$$L(\mathsf{step}) \circ c_{\mathbb{N}} = c_\omega \circ (1 + \mathsf{step})$$

by naturality of c and

$$\phi \circ c_\omega \circ (1 + \mathsf{step}) = \mathsf{step} \circ [0, s]$$

by definition of step.
Now consider the following diagram

$$\begin{tikzcd}
 & L(\mathbb{N}) \arrow[rr, two heads, "L(\mathsf{step})"] \arrow[d, equal] & & L(\omega) \arrow[d, equal] \\
1 + \mathbb{N} \arrow[r, two heads, "c_{\mathbb{N}}"] & L(\mathbb{N}) \arrow[r, two heads, "L(e)"] & L(P) \arrow[r, tail, "L(m)"] \arrow[d, two heads] & L(\omega) \arrow[d, two heads, "\phi"] \\
\mathbb{N} \arrow[u, "{[0,s]^{-1}}"] \arrow[rr, two heads, "e'"'] \arrow[d, equal] & & \phi(L(P)) \arrow[r, tail, "m'"'] & \omega \arrow[d, equal] \\
\mathbb{N} \arrow[rrr, two heads, "\mathsf{step}"'] & & & \omega
\end{tikzcd}$$

where m' is the image of $\phi \circ L(m)$ and e' is the unique map with $\mathsf{step} = m' \circ e'$.
As both $c_{\mathbb{N}}$ and $L(e)$ are S-epis (by Axiom 7 and Lemma 8.4) the map e' is an S-epi, too, as it arises as composition of S-epis. Therefore, m' is contained in m from which it follows that P is a sub-L-algebra of ϕ via m. As ϕ is the initial L-algebra the L-algebra morphism m is an isomorphism and, therefore, step is an S-epi. □

Now we are ready to establish the usual closure properties for admissible predicates.

Theorem 8.6. Equality on complete S-spaces is admissible.

Proof. Let A be a complete S-space. We have to show that the equality predicate on A given by the subobject $\delta_A \triangleq \langle id_A, id_A \rangle$ is orthogonal to $\overline{\mathsf{step}}$.
For this purpose let $f = \langle f_1, f_2 \rangle : \overline{\omega} \to A \times A$ and $g : \mathbb{N} \to A$ be maps such that the following diagram

$$\begin{tikzcd}
\mathbb{N} \arrow[r, tail, "\overline{\mathsf{step}}"] \arrow[d, tail, "g"'] & \overline{\omega} \arrow[d, tail, "f = \langle f_1, f_2 \rangle"] \\
A \arrow[r, tail, "\delta_A"'] & A \times A
\end{tikzcd}$$

commutes. Then $f_1 \circ \overline{\mathsf{step}} = g = f_2 \circ \overline{\mathsf{step}}$ entailing $f_1 = f_2$ as A is an S-space and $\overline{\mathsf{step}}$ is an S-epi by Theorem 8.5. □

Lemma 8.7. Admissible predicates on complete set are closed under arbitrary intersections and, therefore, in particular under conjunction and universal quantification.

Proof. Let $(P_i)_{i \in I}$ be a family of admissible predicates on a complete set A. Let $\overline{a} : \overline{\omega} \to A$ be a map such that $\forall n{:}\mathbb{N}.\, \forall i{:}I.\, P_i(\overline{a}(\overline{\mathsf{step}}(n)))$, *i.e.* $\forall i{:}I.\, \forall n{:}\mathbb{N}.\, P_i(\overline{a}(\overline{\mathsf{step}}(n)))$. As all P_i are admissible it follows that $\forall i{:}I.\, \forall u{:}\overline{\omega}.\, P_i(\overline{a}(u))$, *i.e.* $\forall u{:}\overline{\omega}.\, \forall i{:}I.\, P_i(\overline{a}(u))$.
Thus, the predicate $\lambda x{:}A.\, \forall i{:}I.\, P_i(x)$ is admissible. □

As in type theory implication is a special case of universal quantification we have the following corollary.

Corollary 8.8. Let P be an admissible predicate on a complete set A and Q be an arbitrary proposition then $\lambda x{:}A.\, Q \Rightarrow P(x)$ is an admissible predicate on A.

Proof. Immediate from Theorem 8.7 by the observation that

$$(Q \Rightarrow P(x)) \iff (\forall q{:}Q.\, P(x))$$

for all $x \in A$. □

Moreover, admissible predicates on complete sets are stable under substitution.

Lemma 8.9. Let $f : B \to A$ be a map between complete sets and P be an admissible predicate on A. Then $\lambda y{:}B.\, (P \circ f)(y)$ is an admissible predicate on B.

Proof. Let $\overline{b} : \overline{\omega} \to B$ with $(P \circ f)(\overline{b}(\overline{\mathsf{step}}(n)))$ for all $n \in \mathbb{N}$. Then for $\overline{a} \triangleq f \circ \overline{b}$ we have $P(\overline{a}(\overline{\mathsf{step}}(n)))$ for all $n \in \mathbb{N}$. By admissibility of P it follows that $P(\overline{a}(u))$ for all $u \in \overline{\omega}$, *i.e.* $(P \circ f)(\overline{b}(u))$ for all $u \in \overline{\omega}$. □

An important special case is the following.

Corollary 8.10. The binary predicate on S given by $(u \wedge v) = u$ is admissible.

Proof. The predicate on S given by $(x \wedge y) = x$ can be written more formally as $\lambda z : S \times S.\, (\pi_0(z) \wedge \pi_1(z)) = \pi_0(z)$ which, by the previous Theorem 8.9, is admissible as it is obtained by substitution along $\lambda z : S \times S.\, \langle \pi_0(z) \wedge \pi_1(z), \pi_0(z) \rangle$ from the equality predicate on S which is admissible by Theorem 8.6. $\square$

Definition 8.2. For any set A, the *information (pre)order* is a binary predicate, denoted $\sqsubseteq_A$, such that

$$x \sqsubseteq_A y \;:\Leftrightarrow\; \forall p{:}S(A).\, (p(x) = \top) \Rightarrow (p(y) = \top)$$

holds.

Finally, the "information (pre)order" is an admissible predicate on complete sets.

Theorem 8.11. Let A be a complete set. Then the information preorder $\sqsubseteq_A$ is admissible.

Proof. First notice that for all $u, v \in S$ we have that $u = \top \Rightarrow v = \top$ is equivalent to $((u = \top) \wedge (v = \top)) \Leftrightarrow u = \top$ which in turn is equivalent to $(u \wedge v) = \top \Leftrightarrow u = \top$ as $(u \wedge v = \top) \Leftrightarrow ((u = \top) \wedge (v = \top))$. As by Axiom 5 $(u \wedge v = \top) \Leftrightarrow u = \top$ is equivalent to $u \wedge v = u$ and the latter is an admissible binary predicate on S by Corollary 8.10 we get that the binary predicate on S given by

$$(u = \top) \Rightarrow (v = \top)$$

is admissible.
As admissible predicates are closed under substitution (by Lemma 8.9) and universal quantification (by Lemma 8.7) we get that the binary predicate $\sqsubseteq_A$ on A is admissible. $\square$

8.2. *Computational Induction and Fixpoint Induction*

We now prove the validity of computational induction for admissible predicates on complete sets.

Theorem 8.12. *(Computational Induction)*
Let $\alpha : LA \to A$ be a focal L-algebra structure on a complete set A. Then for any map $f : A \to A$ and admissible predicate P on A it holds that $P(\mathsf{fix}_\alpha(f))$ if $P(f^n(\perp_\alpha))$ for all $n \in \mathbb{N}$.

Proof. By Lemma 7.1 we know that $\mathsf{kl}_{\alpha,f}(\mathsf{step}(n)) = f^n(\perp_\alpha)$. The assumption that $P(f^n(\perp_\alpha))$ for all $\mathbb{N}$ expresses that $\mathsf{kl}_{\alpha,f} \circ \mathsf{step} = \overline{\mathsf{kl}_{\alpha,f}} \circ \overline{\mathsf{step}}$ factors through $m_P : P \rightarrowtail A$ (*cf.* Corollary 7.2). Thus, by admissibility of P the map $\overline{\mathsf{kl}_{\alpha,f}}$ factors through m_P and, therefore, $\mathsf{fix}_\alpha(f) = \overline{\mathsf{kl}_{\alpha,f}}(\infty) \in P$. $\square$

As usual one may derive Fixpoint Induction from Computational Induction.

Theorem 8.13. *(Fixpoint Induction for Admissible Predicates)*
Let $\alpha : LA \to A$ be a focal L-algebra structure on a complete set A. Then for any map $f : A \to A$ and admissible predicate P on A it holds that $P(\mathsf{fix}_\alpha(f))$ provided $P(\perp_\alpha)$ and $\forall x{:}A.\, P(x) \Rightarrow P(f(x))$.

Proof. By Theorem 8.12 it is sufficient to prove $\forall n{:}\mathbb{N}.\, P(f^n(\bot_\alpha))$ which follows by induction on $n \in \mathbb{N}$ from the assumptions $P(\bot_\alpha)$ and $\forall x{:}A.\, P(x) \Rightarrow P(f(x))$. $\square$

8.3. *Park Induction*

What is called "Park induction" in classical domain theory is simply the statement of the fact that the least fixpoint of a (continuous) endofunction $f : A \to A$ on a domain A is also the *least prefixpoint* of f. Of course, "least" and "prefixpoint" refer to the *information ordering on* A $\sqsubseteq_A$ (or simply $\sqsubseteq$ when A is clear from the context).

Notice that Park induction is needed for proving that a recursively defined function $f \triangleq \mathsf{fix}(\Phi) : A \to B$ *diverges* for some argument $a \in A$. Usually, this is achieved by exhibiting a function $g \in B^A$ with $\Phi(g) \sqsubseteq g$ and $g(a) = \bot_B$.

But before we can prove the validity of Park induction we need the following lemma.

Remark 8.14. Let $\alpha : LA \to A$ be a focal algebra structure on a set A. Then by Lemma 6.2 $\bot_\alpha \sqsubseteq a$ for all $a \in A$.

Thus, $\bot_\alpha$ is a least element of A which is unique if A is an S-space.

Theorem 8.15. *(Park Induction)*
Let A be a complete set and $\alpha : LA \to A$ be a focal L-algebra. Then for all $f : A \to A$ and $a \in A$ with $f(a) \sqsubseteq_A a$ it holds that $\mathsf{fix}_\alpha(f) \sqsubseteq_A a$.

Proof. Suppose $f(a) \sqsubseteq_A a$. As the information ordering $\sqsubseteq_A$ is admissible by Theorem 8.11 the predicate $\lambda x{:}A.\, x \sqsubseteq_A a$ is admissible and, therefore, we may use fixpoint induction for proving $\mathsf{fix}_\alpha(f) \sqsubseteq_A a$. By Remark 8.14 we have $\bot_\alpha \sqsubseteq a$.
If $x \sqsubseteq_A a$ then $f(x) \sqsubseteq_A f(a)$ as every map preserves the information ordering. From assumption $f(a) \sqsubseteq_A a$ it follows that $f(x) \sqsubseteq_A a$. $\square$

Thus, for a complete set A we have that the canonical fixpoint $\mathsf{fix}_\alpha(f)$ is actually a least fixpoint (as it is even a least prefixpoint). Moreover, if A is an S-space then $\mathsf{fix}_\alpha(f)$ is *the* least fixpoint of f which does not depend on choice of α as this is unique for an S-spaceprovided it exists.

Thus, summarising we have the following theorem.

Theorem 8.16. Every map $f : A \to A$ on a complete pointed S-space A has a least fixpoint $\mathsf{fix}(f)$, namely $\mathsf{fix}_\alpha(f)$ where α is the unique focal L-algebra structure on A.

9. Markov's Principle and its Consequences

In the intended models of our theory, namely the *realisability models*, a further principle is valid which we have not exploited yet, namely that equality on all types is $\neg\neg$-closed, *i.e.*

$$\forall x, y : X.\, \neg\neg\ x = y \ \Rightarrow\ x = y$$

for all types X.

We will show that equality on S-spaces is $\neg\neg$-closed already if equality on S is $\neg\neg$-closed. But, first consider the following reformulation of the latter requirement.

Lemma 9.1. Equality on S is $\neg\neg$-closed iff def is $\neg\neg$-closed.

Proof. Due to Axiom 5 for all $u, v \in \mathsf{Set}$ we have $u = v$ iff $\mathsf{def}(u) \Leftrightarrow \mathsf{def}(v)$. So equality on S is $\neg\neg$-closed if def is $\neg\neg$-closed. □

Thus, equality on S is $\neg\neg$-closed iff def embeds S into the $\neg\neg$-closed propositions of Prop which is known as *Markov's Principle* for the case when *S is identified with the collection of all Σ^0_1-propositions asserting termination of computations.*

As even in our more general setting S corresponds to the class of propositions asserting termination of computations[§] it makes sense to refer to $\neg\neg$-closedness of def as *(generalised) Markov's Principle.*

Theorem 9.2. In presence of Markov's Principle equality on S-spaces is $\neg\neg$-closed. Actually, this is equivalent to Markov's Principle.

Proof. For an S-space A we have $x =_A y$ iff $\forall p{:}S(A).\, p(x) = p(y)$. But the latter proposition is $\neg\neg$-closed as equality on S is $\neg\neg$-closed and $\neg\neg$-closed propositions are closed under universal quantification.
The other direction holds as S is an S-space. □

Thus, in presence of generalised Markov's Principle many interesting predicates on domains are $\neg\neg$-closed. This turns out to be useful as for $\neg\neg$-closed predicates some forms of case analysis are valid even constructively. These proof principles are needed for developing a more special synthetic domain theory à la Scott as can be found in the first author's Thesis (Reu95; Reu96).

Lemma 9.3. In presence of Markov's Principle for all $\neg\neg$-closed predicates P on S we have that $\forall u{:}S.\, P(u)$ iff $P(\top)$ and $P(\bot)$.

Proof. Let P be a $\neg\neg$-closed predicate on S and $u \in S$. If $P(\top)$ and $P(\bot)$ then $(u = \top \vee u = \bot) \Rightarrow P(u)$. By applying contraposition twice we get that

$$\neg\neg(u = \top \vee u = \bot) \Rightarrow \neg\neg P(u) ,$$

i.e. $\neg\neg(u = \top \vee u = \bot) \Rightarrow P(u)$ by $\neg\neg$-closedness of P. But by Lemma 4.2 the premiss is true and, thus, we may conclude $P(u)$. □

A similar result follows for lifted types.

Lemma 9.4. In presence of Markov's Principle for any type X and $\neg\neg$-closed predicate P on $L\,X$ we have that $\forall z : L\,X.\, P(z)$ iff $P(\bot_X)$ and $\forall x{:}X.\, P(\eta_X(x))$.

Proof. Let P be a $\neg\neg$-closed predicate on $L\,X$. We define a predicate Q on S by

$$Q(u) :\Longleftrightarrow \forall f : X^{\mathsf{def}(u)}.\, P(\langle u, f\rangle)$$

which is $\neg\neg$-closed as $\neg\neg$-closed predicates are closed under substitution and universal quantification.
By definition of lifting $\forall z : L\,X.\, P(z)$ is equivalent to

$$\forall u{:}S.\ \forall f : X^{\mathsf{def}(u)}.\ P\left(\langle u, f\rangle\right) \quad ,$$

i.e. iff $\forall u{:}S.\, Q(u)$. By Lemma 9.3 it suffices to prove $Q(\bot)$ and $Q(\top)$. The first condition

[§] But notice that in general S will not be closed under disjunction and existential quantification over $\mathbb{N}$. So in general not all Σ^0_1-propositions need to be equivalent to some $\mathsf{def}(u)$.

$Q(\bot)$ follows from the assumption $P(\bot_A)$ because there is only one map from $\mathsf{def}(\bot)$ to X (as $\mathsf{def}(\bot)$ is empty). The second condition $Q(\top)$ follows from the assumption that $P(\eta_X(x))$ for all $x \in X$ as all maps from $\mathsf{def}(\top)$ to X are constant (as $\mathsf{def}(\top)$ contains precisely one element due to Axiom 1). $\square$

From this lemma Axiom 7 follows as an immediate corollary.

Corollary 9.5. In presence of Markov's Principle it holds for all types X and maps $p, q : LX \to S$ that $p = p$ iff $p \circ \eta_X = q \circ \eta_X$ and $p(\bot_X) = q(\bot_X)$.

Proof. From Lemma 9.4 applied to the $\neg\neg$-closed predicate $\lambda z : L\,X.\, p(x) = q(x)$ the corrolary follows immediately. $\square$

This nicely illustrates the power of (generalised) Markov's principle as in its presence for developing basic synthetic domain theory it suffices to require – beyond the purely logical Axioms 1-4 – simply that

— there is a dominance
— ι is an S-iso
— $u \sqsubseteq_S \top$ for all $u \in S$.

10. Domain Equations

So far we have shown that well-complete (or replete) pointed S-spaces provide a suitable notion of domain as far as recursive definitions of objects and induction principles for them are concerned. However, in functional programming one is interested also in recursively defined types.

Recursive types are given by domain equations $D \cong F(D, D)$ where F is an internal *mixed-variance bifunctor*, *i.e.* $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ where $\mathcal{C}$ is the category of well-complete (or replete) pointed S-spaces and strict maps.

Following a suggestion of Simpson (*cf.* (Sim96)) – under the assumption that the *universe* Set *is impredicative* – one may show the existence of solutions of domain equations as follows.

First one shows that $\mathcal{C}$ is *algebraically compact* which means that for any internal functor $F : \mathcal{C} \to \mathcal{C}$ there exists an initial/terminal F-algebra, *i.e.* an isomorphism $\alpha : FA \to A$ such that α is an initial F-algebra and $\alpha^{-1} : A \to FA$ is a terminal F-coalgebra. As, by (Fre92), algebraically compact categories are closed under $(\_)^{\mathrm{op}}$ and $\times$ the category $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is algebraically compact, too. Therefore, any internal mixed-variant functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ admits a canonical solution for $D \cong F(D, D)$ as given by the initial/terminal algebra for the functor $F^{\S} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ where $F^{\S}(\_1, \_2) = \langle F(\_2, \_1), F(\_1, \_2)\rangle$.

Theorem 10.1. The category $\mathcal{C}$ of well-complete (or replete) pointed S-spaces is algebraically compact.

Proof. Let $F : \mathcal{C} \to \mathcal{C}$ be an internal functor. By Theorems 6.5 and 6.7 the category $\mathcal{C}$ is small complete. Accordingly, the category of F-algebras is small complete, too.
The initial F-algebra $\alpha : FA \to A$ is constructed as the least sub-F-algebra of the weakly

initial F-algebra $\phi : FP \to P$ where $P := \Pi_{\alpha:FA\to A}A$ and

$$\pi_\alpha \circ \phi = \alpha \circ F(\pi_\alpha)$$

for every F-algebra $\alpha : FA \to A$.
Let $\beta : B \to FB$ be some F-coalgebra. Then let $h : B \to A$ be the canonical solution of the fixpoint equation

$$h = \alpha \circ Fh \circ \beta$$

which exists as the set $[B \to_{st} A]$ of strict maps from B to A is a domain. It remains to show that this $h : \beta \to \alpha^{-1}$ is unique. Suppose that $k : \beta \to \alpha^{-1}$. Let $e : A \to A$ be the canonical solution of $e = \alpha \circ Fe \circ \alpha^{-1}$. By initiality of α any of its endomorphisms is equal to id_A and therefore $e = id_A$. As $\bot_{B,A} = \bot_{A,A} \circ k$ and $f = g \circ k$ implies $\alpha \circ Ff \circ \beta = \alpha \circ Fg \circ Fk \circ \beta = \alpha \circ Fg \circ \alpha^{-1} \circ k$ it follows by fixpoint induction that $h = e \circ k$. Thus, we have $h = k$ as $e = id_A$. $\square$

The above construction of an initial F-algebra uses impredicativity of the universe Set in an essential way as the initial F-algebra is identified as the least sub-F-algebra of $\phi : FP \to P$ where P is the product of all carrier sets of F-algebras including ϕ itself.

This sort of "cheating" is impossible in classical domain theory due to the absence of a non-trivial impredicative universe Set. Instead, canonical solutions of domain equations are constructed explicitely as inverse limits. This method has originally been used by D. Scott for obtaining models of the untyped λ-calculus and has later been made more systematic by Plotkin and Smyth (SP82). This classical construction can be shown to work in a special version of SDT suitable for Domain Theory á la Scott and essentially based on the axioms of (Hyl91; Tay91). See (Reu95) for a type-theoretic formalisation which has been machine-checked in an extension of the LEGO system.

In general SDT, however, there arises the following problem with the usual inverse limit construction. Let $i_n : D_n \to D$, $q_n : D \to D_n$ be the embeddings and projections for the n-th approximation D_n to the inverse limit D. Then $\lambda n : \mathbb{N}.\, i_n \circ q_n$ is an ascending $\mathbb{N}$-indexed chain in $D \to D$ but its limit is not guaranteed to exist as it is not clear how to extend this chain to an $\overline{\omega}$-chain (along $\overline{\mathsf{step}}$).

In (Sim96), however, Simpson has suggested to avoid this problem by considering inverse limits of ω-diagrams instead of $\mathbb{N}$-diagrams. The underlying logic of his approach is some version of Intuitionistic Set Theory (IST) where he can endow the class of all domains with an L-algebra structure allowing him to construct the ω-diagram whose inverse limit provides a solution for $D \cong F(D, D)$. It remains to be checked whether his set-theoretic construction together with its verification can be expressed in type theory, too, but it seems as if there were no severe problems.

11. Conclusion

We have presented a logical approach to general SDT avoiding any reference to external notions. Any realisability structure as considered in (LS96) provides a model of the logic and axioms employed in this paper. It is, however, not clear how the sheaf models of Fiore and Rosolini (FR96) can be extended to models of SDT in our sense. The main obstacle is to identify an appropriate type-theoretic universe Set in their sheaf models. (It is only known that such a universe cannot be impredicative.)

In (Reu95) extensional S-spaces have been studied as an axiomatic analogue of the *extensional pers* of Freyd et. al. (FMRS92) but under the stronger assumptions of (Hyl91; Tay91) and Markov's Principle. It is planned to investigate in a subsequent paper how the theory of extensional S-spaces looks like under the weaker assumptions of general SDT.

Furthermore, there is need for a more detailed account of domain equations.

Finally, it should be investigated to which extent our logical approach to general SDT may have an impact on the methods for the verification of functional programs.

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