General Synthetic Domain Theory – A Logical Approach (Extended abstract)

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Abstract. Synthetic Domain Theory (SDT) is a version of Domain Theory where "all functions are continuous". In [14, 12] there has been developed a logical and axiomatic version of SDT which is special in the sense that it captures the essence of Domain Theory *à la Scott* but rules out other important notions of domain.

In this article we will give a *logical* and *axiomatic* account of *General Synthetic Domain Theory* (GSDT) aiming to grasp the structure common to all notions of domain as advocated by various authors. As in [14, 12] the underlying logic is a sufficiently expressive version of *constructive type theory*. We start with a few basic axioms giving rise to a core theory on top of which we study various notions of predomains as well-complete and replete S-spaces [9], define the appropriate notion of domain and verify the usual *induction principles*.

1 Introduction

In various lectures starting from end of the 70's Dana Scott has promoted the idea of using intuitionistic higher order logic or set theory as an adequate logical framework for axiomatising Domain Theory as an *Extensional Theory of Computation* where *all functions between domains are continuous* and even *computable*. Moreover, there are *intended models* for this situation, namely the various *realisability models*.

In [15] the basic notion of Synthetic Domain Theory was introduced, based on previous work of Scott and Mulry: the "r.e. subobject classifier" Σ providing the collection of propositions equivalent to some Σ^0_1-sentence. Using Σ one may *define* for every intuitionistic set a relation of "observational inequality" corresponding to the "information ordering" of Classical Domain Theory (CDT). This line of research was taken up again by W. Phoa in his Ph.D. Thesis [11] where he gave a detailed account of Domain Theory in the Effective Topos. Whereas these approaches were model-based, in 1989 J.M.E. Hyland [6] and P. Taylor [21] independently introduced two (essentially) equivalent formulations of a synthetic notion of domain, namely the so-called *Σ-replete objects*. This property of Σ-repleteness can be shown to be equivalent to closure under "all generalised limit processes", cf. [20]. In "traditional" accounts of SDT it often is not clear whether definitions and theorems have to be understood internally or externally[1].

In contrast the starting point of our *Logical Approach to SDT* is that *an axiomatic theory of domains has to be developed in a purely logical way* if one wants it to provide a *logical framework for the verification of functional programs.* Of course, this does not at all mean to avoid categorical notions and diagrams but, rather, to understand them as *formulated in the internal language.* For this purpose one needs a *full internal subcategory* Set of the ambient category which is a model for the type theory in use. Such a full internal subcategory provides an appropriate *type-theoretic universe* Set. Although we will often use diagrammatic language we emphasize that *all diagrams have to be understood internally as living in the universe Set.*

Using such a universe Set in this paper we develop elementary domain theory in an appropriate type theory concentrating on the verification of the usual induction principles for canonical fixpoints, for which purpose we introduce a new notion of admissible predicate. We also show how to obtain canonical solutions of domain equations if the universe Set is impredicative thereby following a suggestion of A. Simpson. Most proofs are omitted due to space restrictions, a full version, however, is in preparation.

2 General Synthetic Domain Theory

The basic idea of synthetic domain theory à la Scott is that the set Σ^X of Σ-predicates on a set X provides sort of a topology on X. But already the requirement that Σ-predicates are closed under finite disjunction gives rise to the function $\lambda u, v{:}\Sigma.\, u \vee v : \Sigma \times \Sigma \longrightarrow \Sigma$ which *cannot be computed sequentially* (as it is not even stable). Thus, there is need for a *more general theory* which does not rule out interesting models but, nevertheless, allows one to develop basic domain theory as needed for the verification of functional programs.

A fairly general categorical axiomatisation has been developed by G. Rosolini in [16]. His presentation, however, is not free from external arguments and his axioms do not seem to be strong enough for deriving the usual induction principles for canonical fixpoints.

A detailed account of domain theory in *general realisability models* has been given by John Longley and Alex Simpson in [9]. Their exposition is *purely model based* and refrains from developing their theory axiomatically. In [17] Simpson gives a logical treatment of well-complete sets based on intuitionistic set theory instead of type theory. (This may be more in accordance with mainstream mathematics but makes rather strong ontological assumptions.)

Our logical approach to *General Synthetic Domain Theory* (GSDT) is, of course, based on and inspired by the above mentioned work. The main purpose of our logical approach is to demonstrate that the *consequent use of type-theoretic language* facilitates a *simpler* presentation avoiding complicated external category-theoretic arguments as in [16] as well as complicated explicit

[1] see also the remark after Definition 4.2

constructions in realisability models as in [9]. This is also the case for [17] but we think that our type theoretic formulation is even simpler and more suitable for formalisation, see [12].

3 The Logic

The logic underlying our formulation of SDT is the Extended Calculus of Constructions (ECC) as described in [10] and implemented within the LEGO system. The Extended Calculus of Constructions is a constructive type theory with an *impredicative* universe Prop of propositions together with an infinite hierarchy of *predicative* universes $\mathsf{Type}_0 \in \mathsf{Type}_1 \in \ldots \in \mathsf{Type}_n \in \ldots$ where $\mathsf{Prop} \in \mathsf{Type} := \mathsf{Type}_0$. This hierarchy of universes is *cumulative* in the sense that we also have $\mathsf{Prop} \subseteq \mathsf{Type}_0 \subseteq \mathsf{Type}_1 \subseteq \ldots \subseteq \mathsf{Type}_n \subseteq \ldots$.

For our purposes we need the following extension of ECC.

Axiom 1 *The universe* Prop *of propositions is "proof-irrelevant", i.e.*

$$\forall P{:}\mathsf{Prop}.\, \forall p, q{:}P.\ p = q$$

Axiom 2 *There is a universe* $\mathsf{Set} \in \mathsf{Type}$ *with* $\mathsf{Prop} \subseteq \mathsf{Set}$ *but* **not** $\mathsf{Prop} \in \mathsf{Set}$. *Furthermore,* Set *contains the inductive type* $\mathbb{N}$ *of natural numbers.*

In the following the word "set" refers to objects of the universe Set.

Axiom 3 *Functions between sets are extensional, i.e.*

$$\forall A{:}\mathsf{Set}.\, \forall B{:}A \to \mathsf{Set}.\, \forall f, g : \Pi x{:}A.\, B(x).\ (\forall a{:}A. f\, a = g\, a) \Rightarrow f = g$$

Axiom 4 *Functional relations are required to be traced by functions, i.e. we require the Axiom of Unique Choice (AUC)*

$$\begin{aligned}&\Pi A : \mathsf{Set}.\, \Pi B : A \to \mathsf{Set}.\, \Pi P : \Pi x{:}A.\, B(x) \to \mathsf{Prop}.\\ &\qquad (\forall x{:}A.\, \exists!\, y{:}B(x).\, P\, x\, y) \to \Sigma f : \Pi x{:}A.\, B(x).\, \forall a{:}A.\, P\, a\, (f\, a)\end{aligned}$$

The universe Set may be impredicative but need not.

It has been shown in [19, 10] that ECC can be interpreted in the category of realisability sets over an arbitrary partial combinatory algebra A. For interpreting the extension above in arbitrary realisability models one has to interpret the universe Set as the collection of partial equivalence relations on A and Prop as the sub-collection of those partial equivalence relations R with $A_{/R}$ containing at most one element. See [12] for detailed description of these models and the verifications of Axioms 3 and 4.

Notice that our extension of type theory is closely related to the internal language of toposes, see e.g. [11]. But it is different from topos logic in the following two aspects. On the one hand in a topos there need not exist a universe Set on the other hand our type theory does not assume that equivalent propositions are always equal. It turns out that subsequently this principle will never be needed.

This is good so since for the case of impredicative universe Set it is unknown whether it is consistent with our other assumptions.

Notice that due to the assumption that the universe Prop is proof-irrelevant we have subset types available as in topos logic. If A is a set and $P : A \to \mathsf{Prop}$ is a predicate then the corresponding subset is given by

$$\{ x \in A \mid P(x) \} \equiv \Sigma x{:}A.\, P(x)$$

whose inclusion into A is given by projection on the first component, i.e. the term $\lambda z : \{ x \in A \mid P(x) \}.\, \pi_0(z)$, which is one-to-one as for $z, z' \in \{ x \in A \mid P(x) \}$ we have $\pi_1(z) = \pi_1(z')$ provided $\pi_0(z) = \pi_0(z')$ as $P(\pi_0(z)) = P(\pi_0(z'))$ contains at most one element due to Axiom 1.

Following the practice of topos theory we will use informal set-theoretic notation instead of the formal type theoretic language which makes sense as we have subset types and set comprehension available. Thus by *abus de langage* we (almost always) omit the inclusion maps $\pi_0 : \{ x \in A \mid P(x) \} \longrightarrow A$ for subset types. The only restriction compared to the internal language of a topos is that one must not consider equivalent predicates as equal.

Furthermore, all diagrams below have to be understood as living in some of the universes $\mathsf{Set}, \mathsf{Type}_0, \ldots, \mathsf{Type}_n, \ldots$ considered as *full internal subcategories* of the ambient category establishing a model of the type theory in use.

4 $\mathcal{L}$-completeness and Separatedness

4.1 $\mathcal{L}$-completeness

Many notions of completeness can be formulated quite elegantly (cf. Section 6) via the following notion of *orthogonality*, originally introduced in [4].

Definition 4.1 *A map $f : X \longrightarrow Y$ is called* orthogonal *to a map $g : Z \longrightarrow U$, abbreviated as $f \perp g$, iff for all $h : X \longrightarrow Z$ and $k : Y \longrightarrow U$ with $k \circ f = g \circ h$ there is a unique map $\alpha : Y \longrightarrow Z$ (called "fill-in") with $\alpha \circ f = h$ and $g \circ \alpha = k$, i.e. the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "k"] \arrow[dl, dotted, "\alpha"] \\ Z \arrow[r, "g"'] & U \end{tikzcd}$$

commutes.

If U is terminal then we write $f \perp Z$ for $f \perp !_Z$. That means that any $h : X \longrightarrow Z$ uniquely extends along f to a map $\overline{h} : Y \longrightarrow Z$, i.e. $h = \overline{h} \circ f$.

Using the notion of orthogonality we define an abstract notion of completeness relative to a class $\mathcal{L}$ of maps in some universe.

Definition 4.2 *Let $\mathcal{L}$ be a class of maps, i.e. a predicate on*

$$\Sigma X, Y : U.\, Y^X$$

where U is one of the universes $\mathsf{Set} \subseteq \mathsf{Type}_0 \subseteq \mathsf{Type}_1 \ldots$. A type A is called $\mathcal{L}$-complete iff $e \perp A$ for all $e \in \mathcal{L}$ (i.e. $(X, Y, e) \in \mathcal{L}$ where $e : X \to Y$).
A map m is called $\mathcal{L}$-closed *iff $e \perp m$ for all $e \in \mathcal{L}$.*

This general pattern will later be instantiated by specific choices of $\mathcal{L}$ in order to obtain particular notions of predomain as e.g. complete or replete sets.

Remark: Note that in the rest of this paper we will mostly consider the case $U = \mathsf{Set}$ and thus speak about $\mathcal{L}$-complete sets. In principle, however, we could also use any other of the universes Type_n instead of Set.

In any case, the important point is that *our type theory allows us to quantify over* $\mathcal{L}$ as $\mathcal{L}$ is contained in some universe[2].

Theorem 4.1 *$\mathcal{L}$-complete sets enjoy the following closure properties:*

(i) Dependent products preserve $\mathcal{L}$-completeness.

(ii) Let A be $\mathcal{L}$-complete and $m : P \rightarrowtail A$ be a subobject of A. Then P is $\mathcal{L}$-complete iff m is $\mathcal{L}$-closed.

(iii) Let $f_1, f_2 : A \longrightarrow B$ be maps between $\mathcal{L}$-complete sets. If $m : E \rightarrowtail A$ is an equaliser of f_1 and f_2 then E is also $\mathcal{L}$-complete.

4.2 S-spaces

Now we define the notion of *S-space* as generalisation of T_0-spaces where points are determined uniquely by its open neighbourhoods represented by continuous maps to *Sierpinski space* S. This notion has been considered by Rosolini and later by W. Phoa [11] in the particular context of the effective topos choosing for S the "r.e.-subobject-classifier" Σ.

In this subsection we just assume S to be an arbitrary set, i.e. $S \in \mathsf{Set}$.

Notation. We write $S(\_)$ for $S^{\_}$ and $S^n(\_)$ for the n-times application of the functor $S(\_)$.

Definition 4.3 *For any type X we define $\varepsilon_X : X \longrightarrow S^2(X)$ as $\varepsilon_X(x) = \lambda p{:}S(X).\, p(x)$.*

[2] Notice that the collection of *all* S-isos does not form an internal collection unless one restricts to S-isos in some universe U. Typically, one takes Set for U, *c.f.* Section 6.3. However, if the ambient category is a topos there need not exist such a universe and, therefore, repleteness cannot be expressed by a predicate expressible in the internal language.

Remark Notice that ε is the unit of the so-called "continuation monad" $S^2(\_)$ and therefore a natural transformation from the identity functor to $S^2(\_)$ which restricts to any type universe Type_i or Set.

Definition 4.4 *A set $X \in \mathsf{Set}$ is called an S-*space *or S-*separable *iff ε_X is monic.*

The following representation theorem is useful in many places, e.g. for showing closure properties of S-spaces.

Theorem 4.2 *A set X is an S-space iff it is a subobject of $S(I)$ for some $I \in \mathsf{Set}$.*

Proof. "$\Rightarrow$": If $\varepsilon_X : X \longrightarrow S^2(X)$ is monic then X is a subobject of $S^2(X)$.
"$\Leftarrow$": Suppose $m : X \rightarrowtail S(I)$ is monic. By naturality of ε we get that

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, tail, "m"'] & S^2(X) \arrow[d, "S^2(m)"] \\ S(I) \arrow[r, tail, "\varepsilon_{S(I)}"'] & S^3(I) \end{tikzcd}$$

where $\varepsilon_{S(I)}$ is split monic since $S(\varepsilon_X) \circ \varepsilon_{S(X)} = \mathsf{id}_{S(X)}$ (as can be verified by straightforward computation). Thus, ε_X is monic and X is is an S-space.

A representation theorem for $\mathcal{L}$-complete S-spaces follows from the representation theorem for S-spaces.

Corollary 4.3 *A set is an $\mathcal{L}$-complete S-space iff it is an $\mathcal{L}$-closed subobject of some power of S by a set.*

S-spaces have the following closure properties.

Theorem 4.4 *If $A \in \mathsf{Set}$ and $B : A \longrightarrow \mathsf{Set}$ with $B(x)$ an S-space for all $x \in A$ then $\Pi x{:}A.\, B(x) \in \mathsf{Set}$ is an S-space, i.e. S-spaces are closed under products of families indexed over a set.*

Moreover, if Set is an impredicative universe then S-spaces are closed under products of arbitrary *families.*

Furthermore, S-spaces are closed under equalisers.

5 Initial and Terminal Lift-algebra

5.1 The Dominance S

Up to now we have made no assumptions about S. Now we will assume that S is a set of propositions providing a notion of "well-behaved" subobject as introduced originally by G. Rosolini in [15] for the case of toposes under the name *dominance*.

As opposed to toposes in our case equivalent propositions need not be equal. Accordingly, we cannot consider S simply as a subset of Prop. Instead, we postulate an element $\top \in S$ giving rise to a map $\mathsf{def} := \lambda u{:}S.\,(u = \top) : S \to \mathsf{Prop}$ required to reflect equivalence to equality, i.e. that $u = v \in S$ iff the propositions $u = \top$ and $v = \top$ are equivalent. Thus, via def the set S may be considered "as a subobject of Prop *modulo equivalence*".

Axiom 5 *There is a distinguished $S \in \mathsf{Set}$ together with distinguished elements $\bot, \top \in S$ such that*

$$\neg(\bot = \top)$$

and

$$\forall u, v{:}S.\,[u = \top \Leftrightarrow v = \top] \Rightarrow u = v$$

i.e. the map $\mathsf{def} := \lambda u{:}S.\,(u = \top) : S \to \mathsf{Prop}$ reflects logical equivalence in Prop to equality in S.

Moreover, S is required to be a dominance, *i.e. for any $u \in S$ and $v : \mathsf{def}(u) \to S$ there exists a (necessarily unique) $u \angle v \in S$ with*

$$\mathsf{def}(u \angle v) \Leftrightarrow \exists p{:}\mathsf{def}(u).\,\mathsf{def}(v(p))$$

providing a "dependent conjunction" on S.

Remark: Notice that $\angle$ provides us with an operation $\wedge_S : S \times S \longrightarrow S$ given by $u \wedge_S v := u \angle \lambda p{:}\mathsf{def}(u).\, v$ satisfying $\mathsf{def}(u \wedge_S v) \Leftrightarrow \mathsf{def}(u) \wedge \mathsf{def}(v)$. Therefore we write $u \wedge v$ for $u \wedge_S v$.

Some useful notions are gathered in the following definition.

Definition 5.1 *A map $e : X \longrightarrow Y$ in Set is called S-*iso *iff $e \perp S$. Accordingly, a map e in Set is called S-*epi *iff $S(e)$ is monic, i.e.*

$$p \circ e = q \circ e \Rightarrow p = q$$

for all $p, q \in Y \longrightarrow S$.

*A mono $m : X \rightarrowtail Y$ is called S-*mono *or S-*subobject of Y *iff there exists a (necessarily unique) map $\chi : Y \longrightarrow S$ with $\mathsf{def}(\chi(y)) \Leftrightarrow \exists x{:}X.\, y = m(x)$ for all $y \in Y$ called* classifying map for m.

Remark: Notice that from the assumption that $\top \in S$ is a dominance it follows that S-monos are closed under composition and pullbacks along arbitrary maps in Set.

5.2 Lifting

Now we will construct a *lifting operation* from the dominance allowing one to classify partial maps with domain of definition given by an S-valued predicate. Note that – just as for the notion of $\mathcal{L}$-completeness – we could consider lifting on different universes, but for the purposes of this paper it suffices to have a lifting operation on the universe Set.

Definition 5.2 *The lifting functor $L : \mathsf{Set} \to \mathsf{Set}$ is defined as follows. For any $A \in \mathsf{Set}$*

$$LA \triangleq \Sigma u{:}S.\, (\mathit{def}(u) \longrightarrow A)$$

and for any $f : A \longrightarrow B$ in Set

$$L(f) \triangleq \lambda z{:}LA.\, \langle \pi_0(z), \lambda p{:}\mathit{def}(\pi_0(z)).\, f(\pi_1(z)(u))\rangle$$

Moreover, let η be the natural transformation from the identity functor on Set to L given by

$$\eta_A \triangleq \lambda x{:}A.\, \langle \top, \lambda p{:}\mathit{def}(\top).\, x\rangle$$

for all $A \in \mathsf{Set}$.

Notice that $L0 \cong 1$ and $L1 \cong S$ when 0 is an empty set and 1 is a "terminal" set containing precisely one element, say $*$.

Convention : In the sequel we adopt the convention of not distinguishing between $a \in A \in \mathsf{Set}$ and the function $1 \longrightarrow A$ sending $*$ to a.

Definition 5.3 *For any set A let $\perp_A \in LA$ be defined as*

$$\perp_A \triangleq \langle \perp, \lambda p{:}\mathit{def}(\perp).\, ?_A(p)\rangle$$

where $?_A$ is the unique map $\mathit{def}(\perp) \cong 0 \longrightarrow A$ and let

$$c_A \triangleq [\perp_A, \eta_A] : 1 + A \longrightarrow LA$$

called covering *map of LA.*

Remark: Notice that for any set A the map $\eta_A : A \longrightarrow LA$ *classifies* partial maps into A whose domain of definition is given by an S-subobject. More precisely, for any $B \in \mathsf{Set}$, $p : B \longrightarrow S$, $B' \triangleq \{\, x \in B \mid \mathsf{def}(p(x)) \,\}$ and $f : B' \longrightarrow A$ there exists a *unique classifying* map $\chi_f : B \longrightarrow LA$ such that the following diagram is a pullback

$$\begin{tikzcd}
B' \arrow[r, "f"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "\eta_A"] \\
B \arrow[r, "\chi_f"'] & LA
\end{tikzcd}$$

where χ_f is given explicitly as $\chi_f(b) = \langle\, p(b), \lambda u{:}\mathsf{def}(p(b)).\, f(\langle b, u\rangle)\,\rangle$.

5.3 Initial L-Algebra and Terminal L-Coalgebra

Following [21, 7] one may show the existence of initial and terminal L-algebras in Set.

Theorem 5.1 *In Set there exists an initial L-algebra $\phi : L\omega \longrightarrow \omega$ and a terminal L-coalgebra $\upsilon : \overline{\omega} \longrightarrow L\overline{\omega}$ where $\overline{\omega}$ is a subobject of $S(\mathbb{N})$ and ω is the least sub-L-algebra of υ^{-1}.*

As ϕ and υ are known to be isomorphisms, c.f. [3], the following definitions make sense.

Definition 5.4 *The maps*

$$\sigma \triangleq \phi \circ \eta_\omega : \omega \longrightarrow \omega$$

$$\overline{\sigma} \triangleq \upsilon^{-1} \circ \eta_{\overline{\omega}} : \overline{\omega} \longrightarrow \overline{\omega}$$

are called successor maps *on ω and $\overline{\omega}$, respectively.*

The unique L-algebra morphism from ϕ to υ^{-1} is called $\iota : \omega \longrightarrow \overline{\omega}$ satisfying $\upsilon \circ \iota \circ \phi = L\iota$.

From these definitions and naturality of η it is obvious that $\overline{\sigma} \circ \iota = \iota \circ \sigma$.

Definition 5.5 *Whenever $\alpha : LA \longrightarrow A$ is an L-algebra let $\mathsf{step}_\alpha : \mathbb{N} \longrightarrow A$ be the unique map satisfying*

$$\begin{tikzcd}
1+\mathbb{N} \arrow[rr, "{[0,s]}"] \arrow[d, "1+\mathsf{step}_\alpha"'] & & \mathbb{N} \arrow[d, "\mathsf{step}_\alpha"] \\
1+A \arrow[r, "c_A"'] & LA \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

where $s(n) = n+1$. The uniqueness of step_α follows from the fact that $[0,s]$ by definition is the initial algebra for the functor $1 + \_$.

We write step for $\mathsf{step}_\phi : \mathbb{N} \longrightarrow \omega$ and $\overline{\mathsf{step}}$ for $\mathsf{step}_{\upsilon^{-1}} : \mathbb{N} \longrightarrow \overline{\omega}$ and, therefore, $\overline{\mathsf{step}} = \iota \circ \mathsf{step}$.

Next we define an object $\infty \in \overline{\omega}$ corresponding intuitively to the "limit" of the sequence $(\mathsf{step}(n))_{n\in\mathbb{N}}$ in $\overline{\omega}$. It will play a key role in the construction of canonical fixpoints for endofunctions on domains (cf. [16]).

Definition 5.6 *Let $\infty : 1 \longrightarrow \overline{\omega}$ be the unique L-coalgebra morphism from η_1 to υ.*

Lemma 5.2 *The object ∞ is a fixpoint of $\overline{\sigma}$, i.e. $\overline{\sigma} \circ \infty = \infty$, and, moreover, ∞ is an equaliser of $\mathsf{id}_{\overline{\omega}}$ and $\overline{\sigma}$.*

5.4 Closure under Lifting

We first give a necessary condition for a class $\mathcal{L}$ of maps in Set such that $\mathcal{L}$-complete sets are closed under lifting.

Lemma 5.3 *If $\mathcal{L}$ is a class of maps in Set such that $\mathcal{L}$-complete sets are closed under lifting then $e \perp S$ for all e in $\mathcal{L}$.*

Proof. As 1 is $\mathcal{L}$-complete and $\mathcal{L}$-complete sets are closed under lifting it follows that $S \cong L1$ is $\mathcal{L}$-complete i.e. $e \perp S$ for all e in $\mathcal{L}$.

Maps e with $e \perp S$ were originally introduced by Martin Hyland in [6] and called S-isos. As closure under lifting is sort of a minimal requirement for a class of predomains given as $\mathcal{L}$-complete sets for some $\mathcal{L}$ the previous lemma tells us that one may restrict attention to classes $\mathcal{L}$ of S-isos. A detailed explanation of S-isos as "generalised limit process" has been given in [20]. Thus one may consider a class $\mathcal{L}$ of S-isos as a *class $\mathcal{L}$ of generalised limit processes.*

The following lemma will be crucial for giving a necessary and sufficient characterisation of those $\mathcal{L}$ for which $\mathcal{L}$-complete sets are closed under lifting.

Lemma 5.4 *If $\mathcal{L}$ is a class of S-isos and $A \in \mathsf{Set}$ then $L\,A$ is $\mathcal{L}$-complete iff $m^*e \perp A$ for all $e \in \mathcal{L}$ and S-monos m.*

Proof. The proof idea is illustrated by the following diagram

$$\begin{tikzcd}
m^*X \arrow[rr, "m^*e"] \arrow[dd, "g"'] \arrow[dr, "n"] & & Z \arrow[ddll, "\bar{g}"] \arrow[dr, "m"] & \\
& X \arrow[rr, "e"'] \arrow[dd, "f"] & & Y \arrow[ddll, "\bar{f}"] \\
A \arrow[dr, "\eta_A"'] & & & \\
& L\,A & &
\end{tikzcd}$$

where all squares are pullback squares.

Corollary 5.5 *If $\mathcal{L}$ contains only S-isos then A is $\mathcal{L}$-complete if $L\,A$ is $\mathcal{L}$-complete.*

Proof. If $L\,A$ is $\mathcal{L}$-complete then by Lemma 5.4 it follows that $m^*e \perp A$ for all $e : X \longrightarrow Y$ in $\mathcal{L}$ and S-subobjects m of Y. As all identity maps are S-monos we get that $e \perp A$ for all e in $\mathcal{L}$, i.e. that A is $\mathcal{L}$-complete.

Furthermore, we have the following lemma.

Lemma 5.6 *S-isos are stable under pullbacks along S-monos.*

From this we get a sufficient condition for $\mathcal{L}$-complete sets being closed under lifting.

Theorem 5.7 *If $\mathcal{L}$ is a class of S-isos stable under pullbacks along S-monos then the $\mathcal{L}$-complete sets are closed under lifting.*

Finally, we observe that S-spaces are closed under lifting.

Lemma 5.8 *The S-spaces are closed under lifting.*

6 Predomains

In this section we study several notions of predomain arising from different choices of the set $\mathcal{L}$ of generalised limit processes.

6.1 Complete Sets

As known from classical domain theory it is not sufficient to consider partial orders for interpreting recursive definitions of objects. Instead, one has to require that any ω-chain has a supremum, i.e. a *limit.*

In the vein of being "order-free" we have replaced partially ordered sets by S-spaces. Next we will give an "order-free" analogue of the requirement that ascending chains have limits.

Instead of $\mathbb{N}$-indexed ascending chains in an S-space A we consider ω-chains in A, i.e. maps $a : \omega \longrightarrow A$ where ω is the initial L-algebra. Although for any such $a : \omega \longrightarrow A$ one may obtain an $\mathbb{N}$-indexed ascending chain $a \circ \mathsf{step} : \mathbb{N} \longrightarrow A$ there exist even realisability models where some ascending chains $c : \mathbb{N} \longrightarrow A$ cannot be obtained as $a \circ \mathsf{step}$ from some $a : \omega \longrightarrow A$ (cf. [9]).

Definition 6.1 *A set A is called* complete *iff $\iota \perp A$, i.e. for any $a : \omega \longrightarrow A$ there is a unique $\overline{f} : \overline{\omega} \longrightarrow A$ with $\overline{f} \circ \iota = f$.*

Thus, the complete sets are the $\mathcal{L}$-complete sets for $\mathcal{L}$ containing precisely the map $\iota : \omega \longrightarrow \overline{\omega}$ and therefore satisfy the following closure properties.

Theorem 6.1 *If A is a set and $B : A \longrightarrow \mathsf{Set}$ is a family of complete sets (S-spaces) then $\Pi x{:}A.\, B(x)$ is a complete set (S-space), too.*

Moreover, if Set is an impredicative universe then complete sets (S-spaces) are closed under arbitrary dependent products.

Complete sets (S-spaces) are closed under equalisers.

In order to ensure that S itself is complete, however, we have to postulate

Axiom 6 $\iota \perp S$.

Remark: Our definition of "completeness" is an internal version of the external notion of completeness introduced by Longley and Simpson [9] for realisability models. Notice also that the above definition of "completeness" is nothing but a weakening of Hyland and Taylor's notion of "replete object" [6, 21] (cf. also Section 6.3). Thus completeness in the above sense is a minimal closure property whereas repleteness is a maximal closure property.

6.2 Well-complete Sets

The main problem with complete S-spaces is that they are not (known to be) closed under lifting. Therefore one has to look for a more restrictive notion which nevertheless satisfies all the closure properties of complete sets and S-spaces.

A minimal restriction in order to obtain closure under lifting is the notion of *well-completeness* as suggested by [8, 9].

Definition 6.2 *A set A is called* well-complete *iff its lifting $L\,A$ is orthogonal to ι, i.e. iff $L\,A$ is complete.*

From Corollary 5.5 we immediately get that

Lemma 6.2 *Any well-complete set A is complete.*

The following Lemma identifies well-complete as an instance of the notion of $\mathcal{L}$-completeness.

Lemma 6.3 *A set A is well-complete iff $m^*\iota \perp A$ for all S-subobjects m of $\overline{\omega}$, i.e. iff A is $\mathcal{L}$-complete for $\mathcal{L} \triangleq \{\, m^*\iota \mid m \text{ } S\text{-subobject of } \overline{\omega}\,\}$.*

Proof. Immediate by Lemma 5.4.

Therefore, we get the usual desired closure properties.

Theorem 6.4 *Well-complete sets (S-spaces) are closed under equalisers, lifting, and dependent products of families indexed over a set. Moreover, if Set is impredicative then well-complete sets (S-spaces) are closed under arbitrary dependent products.*

Lemma 6.5 *S is well-complete, i.e. $\iota \perp L\,S$.*

Proof. $L\,S$ is a retract of $S \times S$ which can be seen by establishing the retraction maps $e : L\,S \longrightarrow S \times S$ such that $e : \langle u, f\rangle \mapsto \langle u, u \angle f\rangle$ and $p : S \times S \longrightarrow L\,S$ such that $p : \langle u, v\rangle \mapsto \langle u, \lambda p{:}\mathsf{def}(u).\, v\rangle$.
By Axiom 6 and the closure properties of complete sets it follows that $L\,S$ must be complete too.

Note that by Lemma 6.2 the above Lemma is even equivalent to Axiom 6. Accordingly, we get the following representation theorem for well-complete S-spaces.

Theorem 6.6 *A set is a well-complete S-space iff it is a well-complete subobject of some power of S.*

6.3 S-replete Sets

The notion of S-replete set has been introduced originally in [6] and [21] in the context of a more restrictive axiomatic setting tailored towards a domain theory *á la Scott.* Nevertheless, their definition makes sense also in our more general framework.

Definition 6.3 *A set A is called (S-)*replete *iff $e \perp A$ for all S-isos e.*

Obviously, this is again an instance of $\mathcal{L}$-completeness with $\mathcal{L}$ being the collection of *all* S-isos. This observation is crucial for obtaining the subsequent closure properties which follow from Theorem 4.1, Lemma 5.6, and Theorem 5.7.

Theorem 6.7 *The S-replete sets are closed under lifting, dependent products of families indexed over a set, and equalisers. If Set is impredicative then S-replete sets are closed under arbitrary dependent products.*

Finally, we observe that S-repleteness is the most restrictive notion of predomain containing S.

7 Domains

The well-complete and S-replete *S-spaces* studied so far are "synthetic" analogues of "classical" *predomains*, i.e. partial orders closed under limits of ω-chains. As in classical domain theory for construction of canonical fixpoints one has to restrict attention to *domains*, i.e. predomains having a least element. Next we give an "order-free" reformulation of the notion of domain.

Definition 7.1 *A* focal L-algebra *is a map $\alpha : L\,A \longrightarrow A$ satisfying $\alpha \circ \eta_A = \mathsf{id}_A$. The element $\alpha(\perp_A)$ will be referred to as $\perp_\alpha$.*

If $\alpha : L\,A \longrightarrow A$ and $\beta : L\,B \longrightarrow B$ are focal L-algebras then a homomorphism from α to β *is a map $h : A \longrightarrow B$ with $h \circ \alpha = \beta \circ L\,h$.*

Notice that *a priori* focal L-algebra structure on a set need not be unique. For S-spaces, however, uniqueness of focal L-algebra structure is guaranteed by the following two axioms.

Axiom 7 *For any S-space A the map $c_A = [\perp_A, \eta_A] : 1 + A \longrightarrow L\,A$ is an S-epi.*

Intuitively, that means that for any S-space A maps from $L\,A$ to S are uniquely determined by their behaviour on the image of η_A and the element $\perp_A$.

Axiom 8 *If $f(\perp) = \top$ for some $f : S \longrightarrow S$ then $f(u) = \top$ for all $u \in S$.*

So $\perp$ is the least element of S w.r.t. "information ordering" (*c.f.* Def. 9.2).

We next show that a focal L-algebra structure on an S-space A gives rise to a least element of A w.r.t. "information ordering".

Lemma 7.1 *Let $\alpha : L\,A \longrightarrow A$ be a focal L-algebra structure on an S-space A. If $p \in S(A)$ with $p(\bot_\alpha) = \top$ then $p(a) = \top$ for all $a \in A$.*

Proof. Let $p \in S(A)$ with $p(\bot_\alpha) = \top$. Let $a \in A$. Define $h_a : S \longrightarrow LA$ as $h_a(u) \triangleq \langle u, \lambda p : \mathsf{def}(u).a \rangle$ for which we have $h_a(\bot) = \bot_A$ and $h_a(\top) = \eta_A(a)$. Thus, we have

$$(p \circ \alpha \circ h_a)(\bot) = p(\alpha(\bot_A)) = p(\bot_\alpha) = \top$$

$$(p \circ \alpha \circ h_a)(\top) = p(\alpha(\eta_A(a))) = p(a)$$

which by Axiom 8 implies that $p(a) = \top$.

Now we get uniqueness of focal L-algebra structure for S-spaces.

Theorem 7.2 *There is at most one focal L-algebra structure on an S-space A.*

Proof. Suppose α_1 and α_2 are focal L-algebra structures on an S-space A. As A is an S-space and c_A is S-epic by Axiom 7 it suffices to show that $\alpha_1 \circ c_A = \alpha_2 \circ c_A$. As $\alpha_1 \circ \eta_A = \mathsf{id}_A = \alpha_2 \circ \eta_A$ it even suffices to show that $\alpha_1(\bot_A) = \alpha_2(\bot_A)$. But by the previous Lemma 7.1 we have for all $p \in S(A)$ that $p(\alpha_1(\bot_A)) = \top$ iff $p(\alpha_2(\bot_A)) = \top$. As A is an S-space this implies $\alpha_1(\bot_A) = \alpha_2(\bot_A)$.

Theorem 7.3 *Focal S-spaces are closed under dependent products, equalisers of L-algebra morphisms and under lifting.*

Proof. By straightforward construction of the required structure maps and direct verification. The focal structure on $L\,A$ is given by $\mu_A : L\,L\,A \longrightarrow L\,A$ classifying the partial map $(\eta_{LA} \circ \eta_A, \mathsf{id}_A)$.

Remark. The map μ_A can be defined for arbitrary sets A and together with η gives rise to the so-called *lifting monad* (L, η, μ) on Set.

An appropriate notion of *domain* is now given by *well-complete focal S-spaces* or the full subcategory of *replete focal S-spaces.*

8 Fixpoints

For any focal L-algebra $\alpha : L\,A \longrightarrow A$ over a complete A any map $f : A \longrightarrow A$ admits a *canonical fixpoint* which can be constructed *à la Kleene* analogous to classical domain theory *c.f.* [1, 16]. Since completeness encompasses well-completeness and S-repleteness this can be applied to all notions of domains considered in this paper.

Lemma 8.1 *Let $\alpha : LA \longrightarrow A$ be a focal L-algebra. For any $f : A \longrightarrow A$ there is a canonical map $\mathsf{kl}_{\alpha,f} : \omega \longrightarrow A$ called* Kleene chain of f (w.r.t. α)*, such that*

$$\mathsf{kl}_{\alpha,f}(\mathit{step}(n)) = f^n(\bot_\alpha)$$

for all $n \in \mathbb{N}$.
Accordingly, if A is complete for the unique map $\overline{\mathsf{kl}_{\alpha,f}} : \overline{\omega} \longrightarrow A$ with $\mathsf{kl}_{\alpha,f} = \overline{\mathsf{kl}_{\alpha,f}} \circ \iota$ it holds that $\overline{\mathsf{kl}_{\alpha,f}}(\overline{\mathit{step}}(n)) = f^n(\bot_\alpha)$ for all $n \in \mathbb{N}$.

Theorem 8.2 *([16, 1])*
Let A be a complete set and $\alpha : LA \longrightarrow A$ be a focal L-algebra structure on A. Then for any endomap $f : A \longrightarrow A$ the element

$$\mathsf{fix}_\alpha(f) \triangleq \overline{\mathsf{kl}_{\alpha,f}}(\infty) \in A$$

is a fixpoint of f, *i.e.* $f(\mathsf{fix}_\alpha(f)) = \mathsf{fix}_\alpha(f)$.

9 Domain-Theoretic Induction Principles

For the purposes of *program verification* one needs proof principles allowing one to prove properties of canonical fixpoint such as *Fixpoint Induction*, *Computational Induction* and *Park Induction* in Classical Domain Theory.

We now will derive "synthetic" analogues of these domain-theoretic induction principles which is a somewhat delicate task in GSDT as one has to avoid reference to order-theoretic notions.

As in classical domain theory induction principles are valid only for a certain class of so-called *admissible* predicates. Classically, a predicate P on a cpo A is called *admissible* iff P is closed under suprema of ascending chains. The straightforward "synthetic" analogue of "classical" admissibility would be to require that P is complete, i.e. orthogonal to ι (because the very idea of the synthetic approach is to replace ascending $\mathbb{N}$-indexed chains by ω-chains).

Alas, this straightforward choice would enforce a *reformulation* already of fixpoint induction as illustrated by the following theorem.

Theorem 9.1 *("Synthetic" Fixpoint Induction)*
Let A be complete, $f : A \longrightarrow A$ and $\alpha : LA \longrightarrow A$ be a focal L-algebra. If $m : P \rightarrowtail A$ is complete and also a subalgebra of f_α, i.e. $f_\alpha \circ L(m)$ factors through m, then $\overline{\mathsf{kl}_{\alpha,f}}$ factors through m and therefore $\mathsf{fix}_\alpha(f) \in P$.

That P is a sub-L-algebra of f_α explicitly reads as follows

$$\forall u{:}S.\forall a : \mathsf{def}(u) \longrightarrow A.\, (\forall p : \mathsf{def}(u).P(a(p))) \Rightarrow P(\alpha(\langle u, f \circ a\rangle))$$

which would be equivalent to the premiss of classical fixpoint induction if it were sufficient to consider only the cases $u = \top$ and $u = \bot$. But such case analysis is not available as the proposition $\forall u{:}S.\, u = \top \vee u = \bot$ is not valid in the Kleene realisability model which is sort of a standard model guiding our intuitions [3].

We will next give an improved definition of "admissible predicate" for which fixpoint induction and the other induction principles will be valid in their classical form.

[3] However, the weaker statement $\forall u{:}S.\, \neg\neg(u = \top \vee u = \bot)$ is valid in the Kleene realisability model and this guarantees the validity of $P(\bot) \wedge P(\top) \Rightarrow \forall u{:}S.\, P(u)$ provided the predicate P is $\neg\neg$*-closed, i.e.* $\forall u{:}S.\, \neg\neg P(u) \Rightarrow P(u)$. Of course, this is only useful if sufficiently many predicates P on S are $\neg\neg$-closed which is the case in the Kleene realisability model and has been exploited axiomatically in the first author's Thesis [12, 13] by postulating *Markov's Principle*.

9.1 Admissible Predicates

Classically, an admissible predicate on predomain A is a subset $P \subseteq A$ such that for any ω-chain $a : \omega \longrightarrow A$ its image is contained in P iff the image of $a \circ \mathsf{step} = \overline{a} \circ \overline{\mathsf{step}}$ is contained in P, *i.e.* iff P contains all "finite approximations" $a(\mathsf{step}(n))$. This can be reformulated as an orthogonality condition as follows.

Definition 9.1 *A subobject $m_P : P \rightarrowtail A$ of a completes et A is called* admissible *iff $\overline{\mathsf{step}} \perp m_P$.*

The relationship between completeness and admissibility is given by the following lemma whose proof is straightforward.

Lemma 9.2 *A subobject $m_P : P \rightarrowtail A$ of a complete set A is admissible iff P is complete and $\mathsf{step} \perp m_P$.*

For equality on complete S-spaces being admissible we need the following theorem whose proof is essentially due to Alex Simpson, *c.f.* [17], where, however, it has been used for quite a different purpose.

Theorem 9.3 *The maps $\mathsf{step} : \mathbb{N} \longrightarrow \omega$ and $\overline{\mathsf{step}} : \mathbb{N} \longrightarrow \overline{\omega}$ are S-epic.*

Proof. As $S(\iota)$ is an isomorphism and $\overline{\mathsf{step}} = \iota \circ \mathsf{step}$ it is sufficient to show that step is an S-epi.

For this purpose let $m : P \rightarrowtail \omega$ be the greatest subobject of ω through which step factors by an S-epi. Let $e : \mathbb{N} \longrightarrow P$ be the unique map with $\mathsf{step} = m \circ e$. Of course, e is an S-epi. Notice that

$$\phi \circ L(\mathsf{step}) \circ c_{\mathbb{N}} = \phi \circ c_\omega \circ (1 + \mathsf{step}) = \mathsf{step} \circ [0, s]$$

as $L(\mathsf{step}) \circ c_{\mathbb{N}} = c_\omega \circ (1 + \mathsf{step})$ by naturality of c and $\phi \circ c_\omega \circ (1 + \mathsf{step}) = \mathsf{step} \circ [0, s]$ by definition of step.

Now consider the following diagram

$$\begin{tikzcd}
 & L(\mathbb{N}) \arrow[rr, "L(\mathsf{step})"] \arrow[d, equal] & & L(\omega) \arrow[d, equal] \\
1 + \mathbb{N} \arrow[r, "c_{\mathbb{N}}"] & L(\mathbb{N}) \arrow[r, "L(e)"] & L(P) \arrow[r, tail, "L(m)"] \arrow[d, two heads] & L(\omega) \arrow[d, "\phi"] \\
\mathbb{N} \arrow[u, "{[0,s]^{-1}}"] \arrow[rr, "e'"'] & & \phi(L(P)) \arrow[r, tail, "m'"'] & \omega
\end{tikzcd}$$

where m' is the image of $\phi \circ L(m)$ and e' is the unique map with $\mathsf{step} = m' \circ e'$.

By Axiom 7 it follows that $c_{\mathbb{N}}$ and $L(e)$ are S-epic (the latter as $L(e) \circ c_{\mathbb{N}} = c_P \circ e$ and both $c_{\mathbb{N}}$ and $c_P \circ e$ are S-epic). Thus, the map e' is an S-epi as it arises as a composition of S-epis. Therefore, m' is contained in m from which it follows that P is a sub-L-algebra of ϕ via m. As ϕ is an initial L-algebra the L-algebra morphism m is an isomorphism and therefore step is an S-epi.

Now we can establish the usual closure properties for admissible predicates.

Theorem 9.4 *Equality on complete S-spaces is admissible.*

Proof. Let A be a complete S-space. We have to show that the equality on A as given by the subobject $\delta_A \triangleq \langle \mathsf{id}_A, \mathsf{id}_A \rangle$ is orthogonal to $\overline{\mathsf{step}}$.
Suppose $f = \langle f_1, f_2 \rangle : \overline{\omega} \longrightarrow A \times A$ and $g : \mathbb{N} \longrightarrow A$ with $f \circ \overline{\mathsf{step}} = \delta_A \circ g$. But then $f_1 \circ \overline{\mathsf{step}} = g = f_2 \circ \overline{\mathsf{step}}$ entailing $f_1 = f_2$ as A is an S-space and $\overline{\mathsf{step}}$ is an S-epi by Theorem 9.3.

Well known "abstract nonsense" about orthogonality classes gives rise to the following closure properties of admissible predicates.

Theorem 9.5 *Admissible predicates are closed under arbitrary intersections and, therefore, in particular under conjunction and universal quantification. They are also stable under substitution.*

9.2 Computational Induction and Fixpoint Induction

Now one can prove validity of computational induction for admissible predicates.

Theorem 9.6 *(Computational Induction)*
Let A be a complete set and $\alpha : LA \longrightarrow A$ a focal L-algebra. Then for any map $f : A \longrightarrow A$ and admissible predicate P on A it holds that $P(\mathsf{fix}_\alpha(f))$ provided $\forall n{:}\mathbb{N}.\, P(f^n(\perp_\alpha))$.

Proof. By Lemma 8.1 we know that $\overline{\mathsf{kl}_{\alpha,f}}(\overline{\mathsf{step}}(n)) = f^n(\perp_\alpha)$. Thus, by assumption $\forall n{:}\mathbb{N}.\, P(f^n(\perp_\alpha))$ the map $\overline{\mathsf{kl}_{\alpha,f}} \circ \overline{\mathsf{step}}$ factors through $m_P : P \rightarrowtail A$ and, therefore, by admissibility of P the map $\overline{\mathsf{kl}_{\alpha,f}}$ factors through m_P. Thus, in particular $\mathsf{fix}_\alpha(f) = \overline{\mathsf{kl}_{\alpha,f}}(\infty) \in P$.

Fixpoint Induction follows from Computational Induction by the usual argument.

Theorem 9.7 (Fixpoint Induction for Admissible Predicates)
Let A be a complete set and $\alpha : LA \longrightarrow A$ be a focal L-algebra. Then for any map $f : A \longrightarrow A$ and admissible predicate P on A it holds that $P(\mathsf{fix}_\alpha(f))$ provided $P(\perp_\alpha)$ and $\forall x : A.\, P(x) \Rightarrow P(f(x))$.

9.3 Park Induction

In Classical Domain Theory the principle of "Park induction"[4] says that the canonical fixpoint of a (continuous) endofunction $f : A \longrightarrow A$ on a domain A is the least pre-fixpoint of f. Of course, the notions "least" and "pre-fixpoint" refer to the *"information ordering"* $\sqsubseteq_A$ *on* A to be defined next.

Definition 9.2 *For any set* A *the binary relation*

$$x \sqsubseteq_A y \;\; :\Leftrightarrow \forall p : S(A).\; p(x) = \top \Rightarrow p(y) = \top$$

is called information preorder *on* A.

Obviously, $\sqsubseteq_A$ is reflexive and transitive. Moreover, $\sqsubseteq_A$ is antisymmetric, i.e. a partial order, iff A is an S-space.

Of course, every function between sets is monotonic in the sense that it preserves the information preorder. Furthermore, we have the following two easy lemmas where the second follows immediately from Theorem 9.5 by the logical form of its definition.

Lemma 9.8 *Let* A *be a complete set and* $a \in A$ *then the predicate* $\sqsubseteq_A$ *is an admissible binary predicate on* A.

Now validity of Park induction for complete S-spaces can be proved as usual.

Theorem 9.9 *(Park Induction)*
Let A *be a complete focal* S*-space. If* $f : A \longrightarrow A$ *and* $a \in A$ *with* $f(a) \sqsubseteq_A a$ *then* $\mathsf{fix}(f) \sqsubseteq_A a$ *where* $\mathsf{fix}(f)$ *is* $\mathsf{fix}_\alpha(f)$ *for the unique focal* L*-algebra structure* α *on* A.

Proof. Let $a \in A$ with $f(a) \sqsubseteq a$. By Theorem 9.5 $P(x) \equiv x \sqsubseteq_A a$ is an admissible predicate on A. By Lemma 7.1 the domain A has a least element $\bot := \alpha(\bot_A)$, i.e. $P(\bot)$. As f is monotonic we get that $P(x)$ implies $P(f(x))$ for all $x \in A$. Now, by fixpoint induction it follows that $P(\mathsf{fix}_\alpha)$. Thus, we have $\mathsf{fix}(f) \sqsubseteq_A a$.

10 Domain Equations

Recursive types are given by domain equations $D \cong F(D, D)$ where F is a (necessarily internal) functor $F : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$ and $\mathcal{C}$ is the (necessarily internal) category of domains and L-algebra morphisms, i.e. *strict* maps[5].

[4] Park induction is useful for proving that a recursively defined function $f \triangleq \mathsf{fix}(\Phi) : A \longrightarrow B$ diverges for some argument $a \in A$, i.e. that $f(a) = \bot_B$. Usually, this is achieved by exhibiting a function $g : A \longrightarrow B$ with $\Phi(g) \sqsubseteq_{B^A} g$ and $g(a) = \bot_B$.

[5] If A and B are S-spaces with focal algebra structure α and β, respectively, then $f : A \longrightarrow B$ is an algebra morphism, i.e. $f \circ \alpha = \beta \circ L\,f$, iff $f \circ \alpha \circ c_A = \beta \circ L\,f \circ c_A$ which in turn is equivalent to $f(\bot_\alpha) = \bot_\beta$.

Following a suggestion of A. Simpson (personal communication, March 1997) one may establish the existence of solutions of domain equations provided the *universe* Set *is impredicative.*

First one shows that $\mathcal{C}$ is *algebraically compact*, i.e. for any functor $F : \mathcal{C} \longrightarrow \mathcal{C}$ there exists an an initial F-algebra α whose inverse is a terminal F-coalgebra. By Freyd's [3] the category $\mathcal{C}^{op} \times \mathcal{C}$ is algebraically compact, too. Thus, any internal mixed-variant functor $F : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$ admits a canonical solution $D \cong F(D, D)$ by taking the initial/terminal algebra for the functor $F^{\S} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}^{op} \times \mathcal{C}$ where $F^{\S}(\_1, \_2) = \langle F(\_2, \_1), F(\_1, \_2)\rangle$.

Algebraic compactness of $\mathcal{C}$ can be shown as follows. Let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be a functor. As $\mathcal{C}$ is small complete the category of F-algebras is small complete, too. The initial F-algebra is constructed as the least sub-algebra of the weakly initial F-algebra $\phi : FP \longrightarrow P$ where $P := \Pi_{\alpha:FA\longrightarrow A} A$ and for any F-algebra $\alpha : FA \longrightarrow A$ the α-st component of ϕ is given by $\pi_\alpha \circ \phi = \alpha \circ F(\pi_\alpha)$. Thus, we have $\pi_\alpha : \phi \longrightarrow \alpha$. This construction makes *essential use of impredicativity of* Set *as* ϕ *is defined by quantification over* all F*-algebras including* ϕ *itself.* Now let $\alpha : FA \longrightarrow A$ be an initial F-algebra. For showing that α^{-1} is a terminal F-coalgebra (*c.f.* [3]) let $\beta : B \longrightarrow FB$ be some F-coalgebra. An F-coalgebra morphism $h : B \longrightarrow A$ from α^{-1} to β is obtained as the canonical solution of the fixpoint equation $h = \alpha \circ Fh \circ \beta$ (which exists as the set $[B \circ\!\!\longrightarrow A]$ of strict maps from B to A is a domain). It remains to show that this $h : \beta \longrightarrow \alpha^{-1}$ is unique as an F-coalgebra morphism. Let e be the canonical solution of $e = \alpha \circ Fe \circ \alpha^{-1}$. By initiality of α any of its endomorphisms is equal to id_A and, therefore, $e = \mathsf{id}_A$. Now if $k : \beta \longrightarrow \alpha^{-1}$ is an F-coalgebra morphism then we get $h = e \circ k$ by fixpoint induction. Thus, we get $h = k$.

Such "cheating" is not possible in classical domain theory due to the absence of a non-trivial impredicative universe Set. Instead, canonical solutions of domain equations are constructed explicitly as inverse limits, *c.f.* [18]. This classical construction can be performed also in a special version of SDT suitable for Domain Theory á la Scott, *c.f.* [12] for a proof checked mechanically within the LEGO system. In general SDT, however, this is not possible anymore as not every ascending $\mathbb{N}$-indexed chain appears as restriction of an ω-chain. However, in [17] A. Simpson has suggested to avoid this problem by considering *inverse limits of* ω*-diagrams* instead of $\mathbb{N}$-diagrams. It remains to be checked whether his method will work in our type-theoretic framework as well.

11 Markov's Principle and its Consequences

In the intended models of our theory, namely the *realisability models*, for all types X equality is $\neg\neg$-closed, i.e. $\forall x, y{:}X.\neg\neg(x = y) \Rightarrow x = y$. For the purposes of GSDT it suffices to assume that equality on S is $\neg\neg$-closed which is equivalent to def being $\neg\neg$-closed. For the case when S is the collection of all Σ^0_1-propositions this axiom is known as *Markov's Principle*, a terminology we adopt also for our more general situation.

In presence of Markov's Principle one may easily verify the following powerful proof principles used intrinsically in [12, 13].

Lemma 11.1 *In presence of Markov's Principle for any type X and $\neg\neg$-closed predicate P on LX we have that $\forall z : LX.\, P(z)$ iff $P(\perp_X)$ and $\forall x : X.P(\eta_X(x))$.*

The following Theorem shows that Axiom 7 can be derived from Markov's Principle.

Theorem 11.2 *Under Markov's Principle $p = q$ is equivalent to $p \circ \eta_X = q \circ \eta_X$ and $p(\perp_X) = q(\perp_X)$ for all types X and maps $p, q : LX \longrightarrow S$.*

Proof. Follows from Lemma 11.1 by taking the predicate $\lambda z{:}LX.\, p(z) = q(z)$ which is $\neg\neg$-closed by Markov's Principle.

12 Conclusion and Future Work

We have presented a logical approach to general SDT avoiding any reference to external notions. Any realisability structure as considered in [9] provides a model of the logic and axioms employed in this paper. However, it is not clear how the sheaf models of Fiore and Rosolini [2] can be considered as models of our theory as it is unknown whether they admit an appropriate universe Set (though there cannot exist an *impredicative* universe Set !).

In [12] extensional S-spaces have been studied as an axiomatic analogue of the *extensional pers* of Freyd et. al. [5], however, under the stronger assumptions of [6, 21] and Markov's Principle. It seems to be worthwhile investigating to which extent the theory of extensional S-spaces can be developed under the weaker assumptions of GSDT.

Furthermore, there is need for a more detailed account of domain equations when Set is not impredicative.

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