REPRESENTATIONAL ANALYSIS AND DESIGN:

WHAT MAKES AN EFFECTIVE REPRESENTATION FOR LEARNING

PROBABILITY THEORY?

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ABSTRACT

That external representations have a pivotal role in problem solving is well understood in cognitive science. It is argued that the external representations used for conceptual learning are of comparable significance. The ontological, structural and functional properties of a representation in itself may substantially determine what is learnt and how easily it is done. Poor representations may create obstacles that exacerbate conceptual difficulties, whilst effective representations will promote coherent and well integrated networks of concepts for target domains. Six characteristics of effective representations derived from previous empirical work are considered in this paper. They are: (1) the integration of the levels of abstraction in the domain; (2) the provision of a globally homogenous but locally heterogeneous representation of concepts; (3) the integration of different perspectives or interpretations of the domain; (4) the provision of malleable expressions in the representation; (5) the use of compact procedures and (6) uniform procedures for manipulating the representation. Here the characteristics are used as guidelines for the design of a novel representation for a complex mathematical domain, specifically probability theory. The new representational system, Probability Space (PS) diagrams, uses geometric and spatial constraints to encode the laws of probability in such a way that each instantiation of a diagram (drawing) isomorphically represents the structure of a particular problem situation. PS diagrams are members of a class of diagrammatic representations called Law Encoding Diagrams (LEDs), which have been shown to support problem solving and to promote learning in science. Specific criteria for the design of effective LEDs were also used in the development of PS diagrams. PS diagrams and traditional algebra based approaches to learning probability theory are contrasted by presenting ideal solutions to a selection of difficult and counter intuitive problems. Predictions are made concerning how PS diagrams should ameliorate conceptual difficulties and thus promote learning, in contrast to the traditional approach.

INTRODUCTION

Understanding the role of internal and external representations in human cognition is one of the corner stones of cognitive science. In complex information processing approaches to the study of human problem solving the consideration of representations has been central (Newell and Simon, 1972). The choice of alternative representation may make problem solving harder (or easier) by over an order magnitude (e.g., Kotovsky, Hayes, and Simon, 1985). The relation between internal and external aspects of representations has been explicated (e.g., Tabachneck-Schijf, Leonardo and Simon, 1997; Zhang, 1997). The contrast between expert and novice performance in many domains can, in large part, be attributed to differences in the representations that are used for reasoning (e.g., Chi, Glaser and Farr, 1988; Ericsson and Smith, 1991, Koedinger and Anderson, 1990).

There has been much interest in diagrammatic representations (e.g., Glasgow, Narayanan, and Chandrasekaran, 1995; Peterson, 1996), because of the cognitive benefits that diagrams (often) confer compared to sentential or propositional representations (Larkin and Simon, 1987). Because of their ubiquity, there has been particular interest in the use and interpretation of charts and graphs (e.g., Pinker, 1990; Kosslyn, 1989; Cleveland and McGill, 1985; Carpenter and Shah, 1998; Cheng, Cupit, & Shadbolt, in press).

The role of multiple external representations in learning has been examined (Schwatz and Dreyfus, 1993; Ainsworth, Bibby, Wood, 1997). Providing a learner with more than one partially redundant representation of a domain may allow them to use a process of triangulation to discover general invariants of a domain that are independent of any particular representation. But this comes at the price of having to learn more than one representational system.

There have been formal conceptual analyses of the structure of representations or symbol systems in themselves, (e.g., Goodman, 1968), including those that infer psychological consequences from the analyses of the inherent nature of representations (Palmer, 1978; Stenning and Oberlander, 1995; Zhang and Norman, 1994a).

The conceptual analysis of the nature of external representations in themselves with the exploration of the cognitive implications is also the approach that is adopted here. By understanding the ontological, structural and functional properties of representational systems, the major variations in the difficulties experienced by users of different representations, for the same domain or conceptual field, can to a significant extent be explained. This follows the approach exemplified by Skemp (1986), who examined the problems of existing individual representations in the area of mathematics. Here, the analysis of representations is applied more generally to multiple representations and to science learning. The particular focus of this paper is on the role of representations in conceptual learning, with the claim that the representations used in learning can substantially affect what is learnt and how it is learnt. This extends Zhang and Norman's claims (Zhang, 1997; Zhang and Norman, 1994a, 1994b) about a form of representational determinism that occurs in problem solving by applying the same general notions and style of analysis to representations used in learning. They argue that the form of a representation determines what information can be perceived, what processes can be activated, and what structures (concepts) can be discovered. The last point is elaborated here: The ease of learning, the nature of the conceptual structures that develop and the problem solving procedures that are acquired are substantially determined by the nature of the representation(s) used during learning.

In particular, the paper will consider some formal characteristics of representations and the cognitive implications they have for problem solving and conceptual learning. The characteristics were in part derived from empirical studies of learning in complex scientific and mathematical domains (Cheng 1996a, 1996c, 1999b), conducted as part of a larger programme of research on the role of representations in problem solving, learning and discovery (Cheng and Simon, 1992, 1995; Cheng 1996a, 1996b). The six characteristics proposed are placed in

two categories that are concerned with: (1) the <u>semantic transparency/opacity</u> of representations; (2) the <u>plastic generativity</u> of expressions within the representations. The first set of characteristics address the way in which knowledge is expressed or encoded by representations in relation to the nature of the concepts of the domain, and the way that representations influence the structuring of networks of concepts. The second set deals with use and usability of representations for reasoning and problem solving, that is the procedures for manipulating the structure of expressions in the representation to access or derive information about the domain. A broad perspective is being adopted, in which it is assumed that assessment of the effectiveness of a representation must encompass considerations of the knowledge content of the domain and the procedures that are needed to operate on the representation to reason about the domain.

This work also differs from previous work in that complex representations of substantial mathematical and scientific domains are being considered, rather than simple puzzles or domains whose scope have been artificially restricted. Substantial domains are being considered for two reasons. First, to address issues of conceptual learning, domains that have rich conceptual content must be examined. Second, there are representational issues that are important in complex domains which are not readily apparent in relatively simple domains. Accounts of what make an effective representation based on the study of simple representations will not necessarily scale up to real world domains. The characteristics of effective representations proposed here are applicable to complex domains.

To demonstrate the generality and generativity of the characteristics they are adopted as general guidelines for the design of a novel representational system for probability theory - <u>Probability Space Diagrams</u>, PS diagrams. The scope and validity of PS diagrams is demonstrated by contrasting PS diagrams with traditional algebraic solutions to some complex and highly counter-intuitive problems from the domain. The sample solutions are also suggestive of the potential of PS diagrams for problem solving and learning. Thus, on the basis of each of the characteristics of effective representations, specific predictions are made about how PS diagrams may better support learning about probability than the conventional algebraic approach. Using the characteristics as the basis for such analyses and especially for the design of representations provides a stringent test of the coherence and utility of the characteristics.

The research that has led to the formulation of these characteristics of effective representations for learning will be outlined in the following subsection. This is followed by a brief discussion of why probability theory provides a good case study for the design of a new representation. The overall structure of the paper will then be outlined.

Law Encoding Diagrams

In work that studies the nature of representations themselves, it is advantageous to examine highly contrasting representations that are well differentiated in representational terms. This will better enable the tracing of variations in the effectiveness of particular representations for problem solving or learning back to different characteristics of the representations. The studies of diagrammatic versus propositional representations, noted above, do this to good effect. In the same vein, this paper is part of a programme of work that is contrasting an interesting class of representations, possessing some unusual properties, with traditional representations for learning in science and mathematics. The representations are <u>Law Encoding Diagrams</u>, LEDs (Cheng, 1996a, 1996b).

A Law Encoding Diagram is a representational system for a particular domain that uses geometric, topological or spatial constraints to capture the important relations or laws of the domain, in such a way that each instantiation (drawing) of a LED represents one instance of the phenomenon and one case of the laws of that domain. LEDs are substantial representational systems encompassing major scientific or mathematical topics (Cheng, 1996a, 1998b). They are more than simple visualizations of particular phenomena or concepts. LEDs have been found scattered throughout the history of science (Cheng, 1996b). New systems of LEDs have

also been invented for the purposes of the research on representations and are proposed as alternative representations for learning (Cheng, 1998a; 1999b). Here, a new system of LEDs for probability theory is introduced.

LEDs were used by some of the great figures in the history of science to make their original discoveries (Cheng, 1996b). Galileo's work on kinematics in the <u>Two New Sciences</u> and Newton's work on dynamics in the <u>Principia Mathematica</u> employed geometric diagrams that are LEDs. Huygen's discovery of momentum and energy conservation in the context of particle collisions also used LEDs. In the initial work on LEDs a computational model was built to study the ease with which Huygens could have made the discovery by searching for the right LED, in contrast to finding the laws using an algebraic approach (Cheng and Simon, 1995; Cheng, 1996b). Like many diagrammatic representations, the LED allowed different operators, regularity spotters and heuristics to be employed, which substantially reduced the amount of computation needed to make the discovery. In the context of discovery, at least, this is consistent with the claim about the importance of the nature of representation used and further suggests that LEDs may be effective for the acquisition of new laws or concepts.

The potential of LEDs for learning science and mathematics is being investigated. After relatively short sessions with a computer based learning environment that exploits LEDs for particle collisions, undergraduate science students were able to adopt novel diagrammatic approaches to successfully solve problems on which they had previously failed (Cheng, 1996a). Compared to a group who used an equivalent algebra based learning environment, the participants using LEDs obtained a better qualitative understanding (Cheng, 1994). Using LEDs specially invented for electricity (Cheng, 1998b), novices working on pen and paper exercises demonstrated substantially improved conceptual understanding of the domain compared to a matched group who used algebra (Cheng, 1999b). After 100 minutes of instruction the LED learners were solving problems that often tax undergraduate physicists. From the analysis of the participants' performance in the computer learning environments, and from their drawings and verbal protocols, the dramatic differences between the LEDs and algebra groups in the experiments can be attributed to the particular representation that they were using. A general representational determinacy effect clearly shows through in the studies.

The considerations in this paper deal largely with external aspects of representational systems, but this is not to deny the importance of the internal mental representations being acquired by learners. One of the central claims of this research is that the external representations used for learning substantially determine the structure of the internal representations being developed. In the case of LEDs, it is hypothesized that the external representation may produce well structured networks of schemas whose organization closely reflects the conceptual structure of the target domain. A theoretical framework of schemas that characterises the internal mental counterparts of LEDs has been proposed. The framework of schemas (i) integrates concrete cases with abstract knowledge and (ii) integrates different levels of knowledge about relations and interactions of the domain (Cheng, 1998a, 1999a). The LED schemas for concrete cases in the framework are similar to Koedinger and Anderson's (1990) diagrammatic configuration schemas, which are perceptual chunks that they used to explain the performance of expert geometry problem solvers. The other schemas provide the basis for an account of how conceptual knowledge from complex scientific domains is acquired and mentally structured.

These various lines of work on LEDs have provided insights that have led to the identification of some of the desirable formal "systemic" characteristics of representations that may effectively support reasoning, problem solving and learning. These characteristics will be described below and taken as guidelines for the development of a new representation for probability theory. The specific representational properties of LEDs, which underpin the desirable characteristics, have also been studied. They too are described below and have been used as criteria for the design of the new LEDs for probability theory.

Probability Theory

Necessarily, this paper is in part a conceptual investigation into the nature of the difficulties of learning probability theory, because the cognitive implication of the formal characteristics of representations are being addressed. Probability is good topic in this regard for a number of reasons. (a) It is a substantial mathematical domain with rich conceptual structure and using complex representational systems. (b) Research in probabilistic reasoning and instruction has identified and characterized many conceptual difficulties for learners and has tried various approaches to instruction in the domain (e.g., Kahneman, Slovic, & Tversky, 1982; Shimojo & Ichikawa, 1989; Austin, 1974; Fischbein & Schnarch, 1997; Garfield and Ahlgren; 1988 Shaughnessy, 1992). PS diagrams aim to address some of these difficulties. (c) Traditional approaches and some recent innovations to instruction in this domain have met with limited success (Shaughnessy, 1992). Among the novel approaches are visual models (Ichikawa, 1989; Armstrong, 1981; Dahlke & Fakler, 1981). The comparison of PS diagrams with that work provides an explanation of why those approaches have not substantially enhanced learning, often contrary to expectations. (d) Where particular approaches have successfully improved reasoning about probability (e.g., Cosmides, & Tooby, 1996; Gigerenzer & Hoffrage, 1995) presents a challenge to PS diagrams. Will the explanations for the benefits of the competing approaches be consistent with arguments underpinning the predictions about the potential of PS diagrams? (e) Probability is a complex topic covering classes of phenomena or situations, a wide variety of relations or laws, and a number of alternative perspectives or interpretations. Developing an alternative representation for this domain, which encompasses all these aspects, is a major test of the coherence and power of the general approach. (f) Probability is an important subject within mathematics and its application to diverse fields, spanning social science to engineering, means that any major improvements in the support of learning will be of value.

To gain an appreciation of the design challenge, consider the following problems. (Readers unfamiliar with the problem may find it instructive to do it and to compare their solutions with those considered below.)

Cab Problem. A cab was involved in a hit and run accident at night. There are two cab companies that operate in the city, a Blue Cab Company, and a Green Cab Company. It is known that 85% of the cabs in the city are green and 15% are blue. A witness at the scene identified the cab involved as a Blue Cab. The witness was tested under similar visibility conditions, and made correct blue or green identifications in 80% of the trial instances. What is the probability that the cab involved in the accident was a Blue Cab rather than a Green one?

Surprisingly and disappointingly, a large proportion of students who have studied probability are unable to answer the problem correctly (Shaughnessy, 1992). The students may, arguably, be ignoring the information about the base rate information (Kahneman, Slovic, and Tversky, 1982). But why do they do so when clearly the information was so deliberately included in the problem statement? An explanation in representational terms is provided by the comparison of the relative difficulties of obtaining solutions using PS diagrams and algebraic formulas that argues that the nature of the conventional symbolic approach may hinder the learner developing appropriate models for such problems. Various researchers have argued this and similar problems are hard to solve, because the problems are presented in Bayesian/epistemic terms (degrees of belief) rather than frequentist/aleotoric terms (e.g., Gigerenzer & Hoffrage 1995;Cosimdes & Tooby, 1996). The implications for learning of the extent that a representation supports different perspectives or ontologies will be one of the issues considered here. (The answer is 0.41 , not 0.15 nor 0.80 nor 1)

Consider another problem. (Again, readers may wish to attempt and reflect on how they solve this problem.)

Monty's Dilemma. During a certain game show, contestants are shown three closed doors. One of the doors has a big prize behind it, and the other two have gag gifts

behind them. The contestants are asked to pick a door, but not to open it. Then the game show host, Monty, opens one of the remaining doors and shows it to the contestant, always revealing a gag gift (as Monty knows what is behind each door). The contestants are then given the option to (1) stick with the original door, (2) choose between the two remaining closed doors, say, on the flip of a coin, or (3) switch to the other door.

The correct answers to this problem are highly counter intuitive for students naive in probability theory (Shaughnessy, 1992). For those knowledgeable in probability, clearly and simply explaining the answers to students is hard. Shafir (1994) argues that such probability problems are hard because they involve disjunctions through which people find it hard to think. An indication that a representation may be effective for learning about probability is if it can provide explanations of such problems that are easy to comprehend and that dispel the counter intuitive feeling of the solutions. (The probabilities of winning under each option are (1) 1/3 (not 1/2), (2) 1/2, (3) 2/3 (not 1/2).)

Further, the ways in which Monty's dilemma and the Cab problem are different should be explicable under an effective representation. Again, the comparison of solutions using the PS diagram versus those under a traditional algebraic approach support the thesis that different representations may have a substantial impact on the comprehensibility of a domain and hence how easy or difficult it is to learn.

Structure of the Paper

The next section summarizes probability theory and highlights the many aspects of the domain that must be covered by the PS diagrams. The section also includes some preliminary discussion as to why probability is hard to learn in respect of different classes of probabilistic situations that are often not distinguished clearly. The following two sections describe the general characteristics of effective representations and the specific criteria for the design of LEDs. The guidelines and criteria are used in the subsequent section in which Probability Space diagrams are introduced. Applications of the PS diagrams to hard questions such as the Cab problem and Monty's Dilemma come in the following section. The next section then compares PS diagrams with the traditional algebraic approach using the characteristics as a basis for making predictions about the potential of the systems for learning. The discussion section considers various generic representational issues, building on the representational analyses of the previous sections, to further elaborate the general thesis that the ontological, structural and functional properties of representations themselves significantly determine the structure of the knowledge that is acquired and how easily this occurs.

PROBABILITY THEORY

This section reviews aspects of probability theory that are found in most introductory texts on probability, or texts on applied mathematics for science and engineering, and even some subject specific texts (e.g., McColl, 1995; Kreyszig, 1983; Giarrantano and Riley, 1989). The review covers the knowledge requirements and conceptual problems that any system of representations for the domain must cover. The scope and validity of PS diagrams will be assessed below by the extent and coherence of its coverage. This section is also a summary of the various representations used in the "traditional algebraic approach", which will also be part of the analyses below.
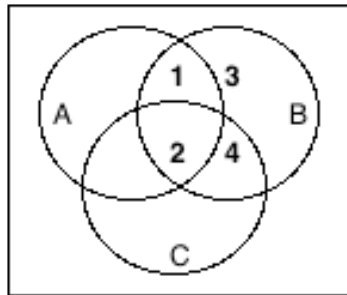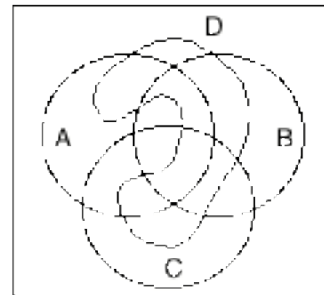
Figure 1. Venn diagram for three sets.

Figure 2. Venn diagram for four sets.

Set Theory

As set theory underpins much of probability theory a few selected items are discussed, which are of relevance to our representational considerations. The properties of sets includes commutativity, associativity, indempotence, distributivity, the law of the excluded middle, the law of contradiction, identity, absorption, de Morgan's laws, involution, equivalent and symmetrical difference. It is unusual to use many of these properties in the solution of probability problems, but any approach to probability should be able to consistently represent them. Just two examples are given here. The commutativity of sets is expressed by two formulas:

$$A \cup B = B \cup A, \tag{1a}$$

and
$$A \cap B = B \cap A, \tag{1b}$$

where A and B are sets and the symbols $\cap$ and $\cup$ are operators for intersection and union. The formulas for De Morgan's Laws give different views on the basic interaction and union relations in terms of membership and non-membership of the same sets;

$$\sim(A \cap B) = \sim A \cup \sim B \tag{2a}$$

and
$$\sim(A \cup B) = \sim A \cap \sim B . \tag{2b}$$

Venn diagrams are a convenient way to visualize and reason about sets and their relations. Figure 1 shows three intersecting sets. Figure 2 illustrates how by subdividing every area in Figure 1 a new diagram is produced that has a new set, D, which intersects every possible combination of the original sets.