Truth Diagrams: An Overview

Peter C-H. Cheng

Department of Informatics, University of Sussex, Brighton, BN1 9QH, UK

Email: p.c.h.cheng@sussex.ac.uk

Abstract. *Truth Diagrams*, TDs, are a new diagrammatic notation for propositional logic. TDs provide: (1) representations of logical states of affairs and relations; (2) operators on such relations; (3) a test of the validity of derivations. A proof of one of de Morgan's laws is given as an illustration of TDs.

Truth Diagrams (TDs) were invented as part of a programme of research on the *Representational Epistemic* approach to the study of how notational systems encode knowledge and the potential cognitive benefits that novel codifications of knowledge may confer [1-3]. The core principle of the approach claims that effective representational systems should directly encode the fundamental conceptual structure of their knowledge domains, using coherent notational schemes. TDs were designed as a further test of this idea. The purpose here is simply to give an informal overview of TDs by describing the derivation of one of de Morgan's laws: –(*P*vQ) |– –*P*&–*Q*.

The derivation is shown in Fig. 1. The sequent to be derived is stated at the top; columns C to H in row 1, or {C-H,1}. The derivation has three main parts: (a) the construction of the TD for the assumption of the sequent, to the left of Fig. 1 {A-C,2-6}; (b) the construction of the conclusion TD, on the right {H-J,2-6}; (c) a test of that the assumption and the conclusion constitutes a tautology, as required for a valid derivation, at the bottom of the diagram {F,7-8}.

TDs are configurations of lines and symbols that may be interpreted in three different ways. First, TDs may represent **logical states of affairs**. There are ten such TDs in Fig. 1; {A,2}, {C,2}}, {H,2}, {J,2}, {B,4}, {H,4}, {J,4}, {B,6}, {I,6}, and {F,8}. The formula for the state of affairs represented by a TD is shown by the underlined expression at the top of each TD; e.g., in {B,4} the relation is <u>*P*v*Q*</u>. A TD possess one or more variables identified by the letter(s) in the middle of each diagram; in the unary TD {A,2} the variable is *P*, and in the binary TD {B,4} they are *P* and *Q*. The position of the *end* of a line next to a variable represents a truth-values assignment to that variable: the top position stands for True and the bottom position stands for False. For example, the top left of {B,4} is *P*=T and the bottom right is *Q*=F, and the top of {A,2} is *P*=T and the bottom is *P*=F.

Within a TD, each line stands for particular set of truth-value assignments to the variables, depending on the position of the ends of the line. For example, in {B,4} the top horizontal line stands for (*P*=T, *Q*=T). The bottom (dashed) horizontal line represents (*P*=F, *Q*=F). The descending and ascending diagonals represents (*P*=T, *Q*=F) and (*P*=F, *Q*=T), respectively. The style of the line indicates whether the set of assignments is itself T or F. A solid line assigns True to the set and is called a *Tine*. A dashed line assigns False to the set and is called a *Faint*. For example, the descending diagonal Tine in {B,4} is (*P*=T, *Q*=F)=T and bottom Faint is (*P*=F, *Q*=F)=F. As all the lines of {F,8} are tines, all possible sets of assignments are true, so this TD stands for a tautology. The TDs in row {2} are simply unary variables, so by definition they have a Tine at the top and a Faint at the bottom.

The second interpretation is **TDs as operators**: {B,3}, {H,3}, {J,3}, {B,5} and {I,5} in Fig. 1, which are enclosed by dashed rectangles. The symbol above the TD identifies the operator. The bracketed formulas in the middle identify the argument TDs to which the operator is applied. In Fig. 1 the argument and result TDs are drawn just above and below the operator TD; e.g., {B,4} is the argument for operator {B,5} and the result is {B,6}. The configuration of the lines within an operator TD determines what the operator does. The position, either top or bottom, of the end of a line is an instruction to find either a Tine or a Faint, respectively, in the relevant argument TD; e.g., lines ending at the bottom left of {B,3} means locate a Faint in the TD for the expression [P]. In {B,5} the top position means find a Tine in the TD for [*P*v*Q*]. The two ends of a line in a binary operator constitute a pair of instructions to find the specified types of lines in each of the argument TDs; e.g., the ascending diagonal in {I,5} means find a Faint in the TD for [-*P*], {H,4}, and find a Tine in [-*Q*], {J,4}. The actual type of a line in an operator TD specifies the type of line to be drawn in the result TD; e.g., the negation operator {B,5} transforms all the Tines in {B,4} into Faints in {B,6}, as there is a top Faint in {B,5}; and vice versa for the Faint in {B,4} given the bottom Tine in {B,5}. The operator TD in {B,3} constructs {B,4} from all the four possible combinations of the Tines and Faints in {A,2} and {C,2}, with new Tines drawn whenever there is a Tine in either of the arguments.

On the left of the diagram, TDs for *P* and *Q* are disjunctively combined by the *or* operator {B,3} and the result is negated by {B,5}, to give a TD for –(*P*v*Q*), {B,6}. On the right, *P* and *Q* are individually negated before being conjunctively combined by the *and* operator {I,5}, to give –*P*&–*Q*, {I,6}.

The **tautology test** of the assumption and the conclusion of a derivation is the third interpretation of TDs. It applies the material implication TD in the solid rectangle {F,7} to {B,6} and {I,6}, in the manner of any operator. As there is a Tine in the same position in both {B,6} and {I,6}, a Tine is drawn in {F,8}, labeled 'TT'. Similarly, as Faints occur in all the same positions, Tines are also drawn for them in {F,8} 'FF', because the bottom line in {F,7} is a Tine. There are no combinations of a Tine in the assumption and a Faint in the conclusion, so no faints occur in {F,8}. Therefore, the test TD only has Tines, so it is a tautology and the derivation is thus valid.

The verbal descriptions of the interpretations of TDs given here do require some effort to follow, but graphical explanations in fuller treatments of TDs are easier to understand, because they can fully exploit the spatial structure of TDs.

Proofs that Truth Diagrams comprise a sound and complete system for propositional logic will be presented elsewhere, along with comparison of the relative benefits of TDs in comparison to the conventional formula notation and truth tables.

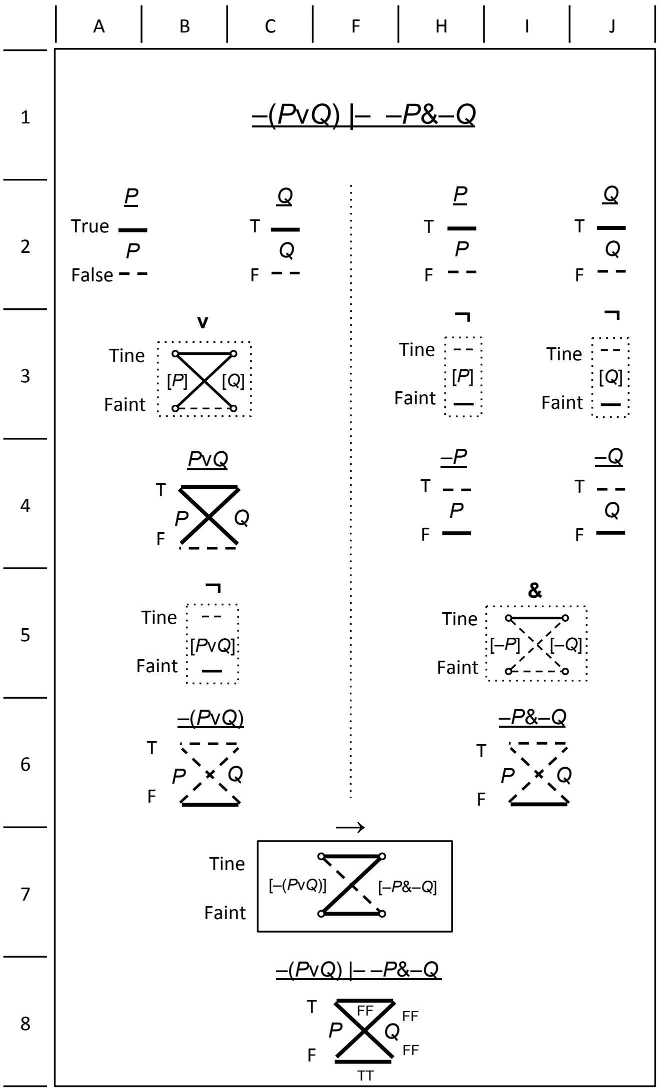

Fig. 1. TD derivation of one of de Morgan's Laws

References

1. Cheng, P. C-H. (2002). Electrifying diagrams for learning: principles for effective representational systems. Cognitive Science, 26(6), 685-736.
2. Cheng, P. C-H. (2011). Probably good diagrams for learning: Representational epistemic re-codification of probability theory Topics in Cognitive Science 3(3), 475-498.
3. Cheng, P. C-H., & Barone, R. (2007). Representing complex problems: A representational epistemic approach. In D. H. Jonassen (Ed.), Learning to solve complex scientific problems. (pp. 97-130). Mahmah, N.J.: Lawrence Erlbaum Associates.