Noise-Sustained Coherent Oscillation of Excitable Media in a Chaotic Flow

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Constructive effects of noise in spatially extended systems have been well studied in static reaction-diffusion media. We study a noisy two-dimensional Fitz Hugh–Nagumo excitable model under the stirring of a chaotic flow. We find a regime where a noisy excitation can induce a coherent global excitation of the medium and a noise-sustained oscillation. Outside this regime, noisy excitation is either diluted into homogeneous background by strong stirring or develops into noncoherent patterns at weak stirring. These results explain some experimental findings of stirring effects in chemical reactions and are relevant for understanding the effects of natural variability in oceanic plankton bloom.

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Constructive effects of noise in nonlinear systems, such as stochastic resonance (SR) [1], coherence resonance (CR) [2], noise-induced transitions [3], and noise-enhanced stability [4] have been a subject of great interest. Recently, the attention has been shifted to spatially extended systems [5–16]. Effects such as array-enhanced SR [8], array-enhanced CR [9], noise-enhanced signal propagation [10], noise-enhanced synchronization [11], or array-enhanced frequency and phase locking to weak signals [12] have been observed. In subexcitable reaction-diffusion media, noise-sustained wave propagation [13] or global oscillation [14] occurs due to multiplicative noise-induced transitions of the system to the excitable or the oscillatory regime [15]. Double noise effects have been demonstrated in systems subjected to both multiplicative and additive noises [16]. In these studies, the media are static, and the constructive effects are a consequence of the interplay between local excitation (switching) due to noise perturbation and propagation of excitation (wave) due to diffusion.

In nature and many engineering examples, the media, however, are not necessarily static, but quite on the contrary, may be subject to a motion, e.g., when stirred by a flow. This occurs especially in chemical reactions in a fluid environment [17], conversion of pollutants in atmospheric flows, or bloom of plankton in oceanic currents. Mixing due to chaotic advection [18] of the flow has strong influences on the pattern formation of excitable media [19]. Chaotic stirring and excitability are two features relevant to bloom of plankton, e.g., in ocean fertilization experiments [20], which can be described by an initial value problem of an excitable model subjected to turbulent ocean currents [21].

Noise, such as that from the natural variability, is inevitably present in this type of system, but its effects have not yet been addressed. In this Letter, we investigate the interplay among noise, excitability, diffusion, and mixing in excitable media advected by a chaotic flow, in a two-dimensional Fitz Hugh–Nagumo model described by the reaction-advection-diffusion equations:

$$\frac{\partial C_1}{\partial t} + \mathbf{v}(\mathbf{r}, t) \cdot \nabla C_1 = C_1(a - C_1)(C_1 - 1) - C_2 + D\nabla^2 C_1, \tag{1}$$

$$\frac{\partial C_2}{\partial t} + \mathbf{v}(\mathbf{r}, t) \cdot \nabla C_2 = \epsilon(C_1 - 3C_2) + \xi(\mathbf{r}, t) + D\nabla^2 C_2, \tag{2}$$

where a is the excitation threshold, D is the diffusion constant, and $\epsilon \ll 1$ is the time scale of the slow variable C_2. The spatially homogeneous system has a stable fixed point at $(C_1, C_2) = (0, 0)$, while it generates a large excursion to a maximum $C_1 \approx 1$ when perturbed over the threshold a. The noise $\xi(\mathbf{r}, t)$ is Gaussian white in space and time, satisfying $\langle \xi(\mathbf{r}, t)\xi(\mathbf{r}_1, t_1)\rangle = 2\Gamma\delta(\mathbf{r} - \mathbf{r}_1)\delta(t - t_1)$, with Γ being its intensity.

The flow is assumed to be imposed externally, with a velocity field $\mathbf{v}(\mathbf{r}, t)$ independent of the reaction:

$$v_x(x, y, t) = \frac{L}{T}\Theta\left(\frac{T}{2} - t \bmod T\right)\sin\left(\frac{2\pi y}{L} + \phi_i\right), \tag{3}$$

$$v_y(x, y, t) = \frac{L}{T}\Theta\left(t \bmod T - \frac{T}{2}\right)\sin\left(\frac{2\pi x}{L} + \phi_{i+1}\right), \tag{4}$$

where Θ is the Heaviside step function. This is a well-known standard model for mixing by chaotic advection [18]. The random phase ϕ_i in each half period makes the velocity field aperiodic to avoid transport barriers typically present in time-periodic flows [18]. Under the advection of such a flow, the fluid elements separate exponentially at a rate proportional to the stirring rate $\nu = 1/T$ [19]. The results below are not specific to this flow but should be characteristic to a class of unsteady laminar flows.

We consider the system on the unit square ($L = 1$) with doubly periodic boundaries in the weak diffusion case $L^2/(DT) \gg 1$. The parameters used in our simulations

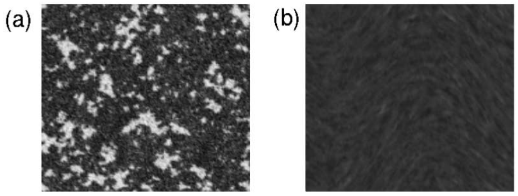

FIG. 1. Typical patterns of $C_1(x, y)$ of the motionless [(a) $\mathbf{v} \equiv \mathbf{0}$] and stirred [(b) $\nu = 0.08$] media, in the presence of noise with an intensity $\Gamma = 2^4 \times 10^{-10}$.

are $a = 0.25$, $D = 10^{-5}$, $\epsilon = 10^{-3}$. The system is integrated initially from the homogeneous steady state (HSS) $(C_1, C_2) = (0, 0)$, using a semi-Lagrangian scheme for the deterministic part; then an independent noise term $[\sqrt{2\Gamma\Delta t}/\Delta x]\eta_{ij}$, where η_{ij} is a random number from the normal distribution $N(0, 1)$, is added to $C_2(i, j, t)$ at each grid point (i, j) [5]. We fix $\Delta t = 0.05$ and $\Delta x = L/600$. We study the system behavior with respect to the noise intensity $\Gamma = 2^\alpha \times 10^{-10}$ and the stirring rate ν of the flow.

For the weak diffusion D considered here, noise alone cannot generate large-scale coherent behavior in the motionless media ($\mathbf{v} \equiv 0$). Many noise-induced excitation centers diffuse to form random patterns of small excited patches, as seen by a typical snapshot at a noise level with $\alpha = 4.0$ [Fig. 1(a)]. The number of excitation centers increases at larger noise intensities; however, the patches do not grow to a scale comparable to the domain size L^2. For the system subjected to the stirring of the flow ($\nu = 0.08$), the same noise intensity with $\alpha = 4.0$ cannot generate visible excitation patches [Fig. 1(b)]. Mixing of the flow tends to spread and dilute small excited centers before they can grow through diffusion as in the motionless media. The domain is almost uniform around HSS, as seen by a mean concentration $\langle C_1 \rangle \sim 10^{-3}$ in Fig. 2(a).

For a larger noise with $\alpha > 4.0$, excitation of the media occurs after a period of time. $\langle C_1 \rangle$ is now composed of a train of large spikes with almost periodic intervals T_I, as seen in Fig. 2(b) for $\alpha = 5.0$. The large spikes of $\langle C_1 \rangle \approx 1$ correspond to a coherent global excitation (CGE) of the whole domain, such that at all spatial points $C_1(x, y) \approx 1$, through the development of filaments. The typical process is shown in Fig. 3. At some moment, a strong enough local excitation survives the stirring. Instead of being diluted into the background, it is elongated along the trajectory of the flow and develops into filaments with a characteristic width resulting from a competition between the stirring and the diffusion [19]. The filaments become denser and denser to fill the whole domain. Later on, the whole domain starts to relax synchronously back to a close vicinity of HSS. The process repeats to generate a noise-sustained coherent oscillation of the mean field $\langle C_1 \rangle$.

To measure the degree of synchronization of the domain, we calculate the standard deviation σ_C of C_1 over the domain as a function of time:

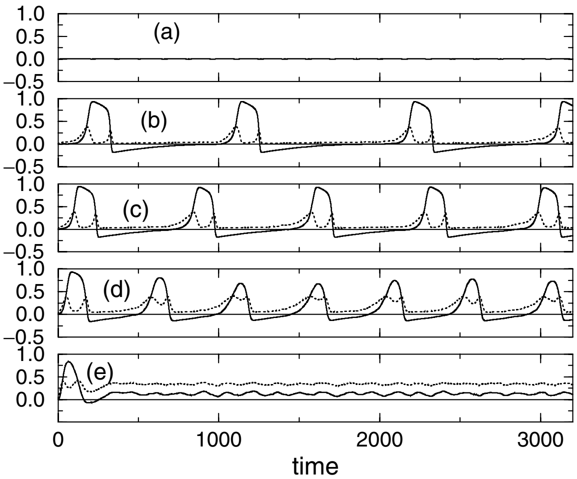

FIG. 2. Time series of mean value $\langle C_1 \rangle$ (solid lines) and standard deviation σ_C (dotted lines) of the concentration $C_1(x, y)$ at various noise intensities $\Gamma = 2^\alpha \times 10^{-10}$: (a) $\alpha = 4.0$, (b) $\alpha = 5.0$, (c) $\alpha = 6.0$, (d) $\alpha = 7.3$ and (e) $\alpha = 9.0$. The stirring rate of the flow is $\nu = 0.08$.

$$\sigma_C(t) = [\langle C_1(\mathbf{r}, t)^2 \rangle - \langle C_1(\mathbf{r}, t) \rangle^2]^{1/2}. \tag{5}$$

In Fig. 2(b) ($\alpha = 5.0$), it is seen that σ_C increases when the filaments grow till they fill the whole domain, where σ_C drops quickly to a quite small value $\sigma_C \sim 10^{-2}$ corresponding to a CGE of the whole domain. Such a well-expressed synchronization is lost temporally when parts of the domain change to $C_1 \approx -0.25$ quickly, while the others still have $C_1 \approx 0.75$. As a result, σ_C obtains a smaller peak at a rapid decrease of $\langle C_1 \rangle$. Note that in the noise-free media starting from a strong enough initial perturbation [19], σ_C also displays the first peak corresponding to the growth of the filaments; however, it decays monotonously to zero without exhibiting a second smaller peak. This noise-induced temporal inhomogeneity is homogenized quickly by stirring, and the synchronization is restored during the slow relaxation phase so that the whole domain approaches simultaneously back to

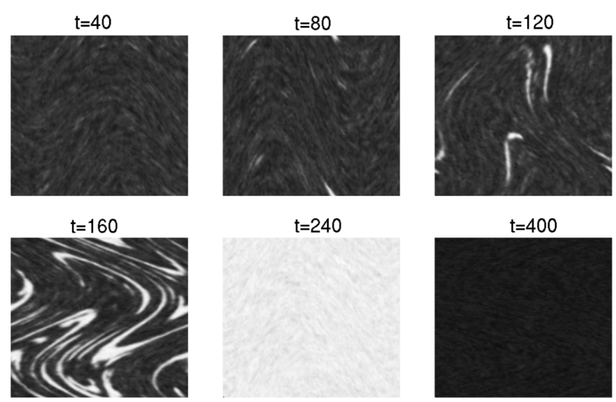

FIG. 3. Development of filaments and formation of a coherent global excitation for $\alpha = 5$. The stirring rate $\nu = 0.08$.

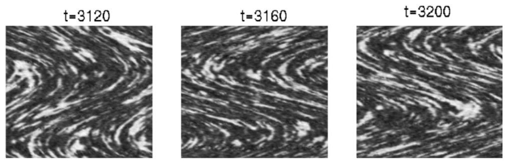

FIG. 4. Persistent noncoherent patterns at a large noise $\alpha = 9.0$. The stirring rate $\nu = 0.08$.

a close neighborhood of HSS to allow another round of excitation. At a larger noise, e.g., $\alpha = 6$ [Fig. 2(c)], a strong enough local excitation can develop within a shorter time, resulting in an earlier global excitation and shorter intervals T_I between the spikes of $\langle C_1 \rangle$.

However, for an even larger noise level, e.g., $\alpha = 7.3$ [Fig. 2(d)], after the first coherent global excitation from the initial HSS, not all the points of the domain relax simultaneously back to the close vicinity of the fixed point (0, 0). A small part of them become excited at early times before coming close to the fixed point. Later on, the filaments developed from excitations at different times cannot merge completely to form a fully CGE. The followed spikes are clearly lower than the first one, and the corresponding σ_C is clearly far away from zero.

With increasing noise intensity, more and more parts of the domain can be excited before coming back to the close vicinity of the fixed point. The coherence is lost, and there are no longer pronounced and large spikes in $\langle C_1 \rangle$ after the first one. At a rather strong intensity, e.g., $\alpha = 9.0$ [Fig. 2(e)], the whole domain does not achieve a CGE even from the initial HSS. The synchronization is totally destroyed and σ_C never comes close to zero again. The corresponding noncoherent patterns (Fig. 4) are composed of many random filamentlike strips.

The above three regimes, i.e., homogenization, coherent global excitation, and noncoherent excitation, can be manifested by the variance of the oscillation of $\langle C_1 \rangle$:

$$\mathrm{Var}(\langle C_1 \rangle) = \langle \langle C_1 \rangle^2 \rangle_t - \langle \langle C_1 \rangle \rangle_t^2, \tag{6}$$

where $\langle\,\rangle_t$ denotes average over time after a transient $t = 500$. As shown in Fig. 5(a) for fixed $\nu = 0.08$, $\mathrm{Var}(\langle C_1 \rangle)$ experiences an abrupt increase around $\alpha = 4.0$, and it keeps almost flat in the range of $\alpha \in (4.0, 7.0)$ for the CGE. The interspike interval T_I is the sum of the fixed duration of a single excitation cycle and the time necessary to produce a surviving excitation center. Close to the threshold, the stochastic waiting time is much longer and irregular, and therefore T_I also fluctuates. Otherwise the deterministic part dominates and there is a more regular (although not perfect) periodicity. The mean T_I decreases with increasing noise intensity Γ till too large noise prevents a CGE [Fig. 5(b)]. This behavior is similar to CR in zero-dimensional excitable elements [2] and in coupled arrays of such elements [9,11], where noise generates the most regular spike trains at some intermediate intensities.

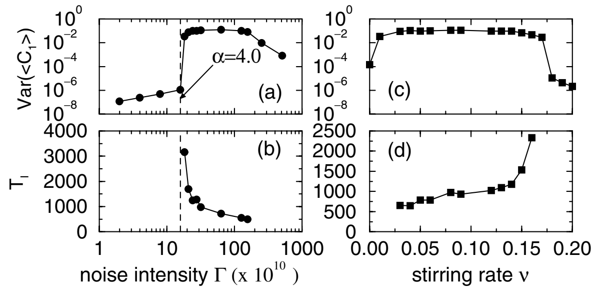

FIG. 5. Variance of $\langle C_1 \rangle$ [(a),(c)] and mean interspike interval T_I [(b),(d)] as a function of the noise intensity Γ (left: $\nu = 0.08$) and the stirring rate ν (right: $\Gamma = 2^5 \times 10^{-10}$).

Figure 6 shows the three regimes in the parameter space (α, ν). When the noise is rather small ($\alpha \lesssim 2.0$), a weak stirring ($\nu \sim 0.02$) already induces homogenization of the media. At even weaker stirring rates, noise excitation simply develops into noncoherent excitation, because such stirring rates do not support CGE even in noise-free media with strong enough initial perturbations [19]. For stronger noise levels, the regime of CGE appears. In this regime, noise is strong enough to induce a sufficiently large excitation center, but importantly, it does not affect much the growth of the filaments to form a CGE and then synchronized relaxation back to HSS for sustained coherent oscillations of the media. When the stirring rate is above the upper boundary, all noisy excitations are diluted and the domain is homogenized; while below the lower boundary weaker mixing is not enough to maintain a synchronized relaxation to HSS, resulting in a noncoherent excitation. Figures 5(c) and 5(d) depict the transitions via the varying stirring rate ν at the fixed noise level $\alpha = 5.0$. The interspike interval of $\langle C_1 \rangle$ increases on average and becomes clearly erratic when ν approaches the upper boundary of the CGE regime.

The mechanism underlying the CGE is the nonuniform sensitivity of the system to noisy perturbations. The dynamics of CGE can be divided into a stochastic and a deterministic component. There exists a minimal characteristic length scale $l_0 \sim \sqrt{(D/\nu)}$, over which there is fast

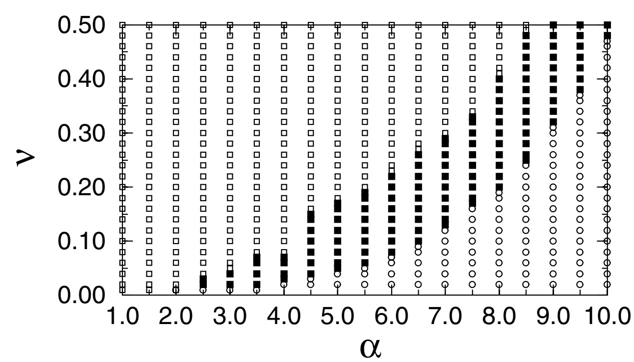

FIG. 6. Phase diagram of the system in the parameter space (α, ν). The symbols represent homogenization (□), coherent global excitation (■), and noncoherent excitation (○).

homogenization by mixing. A region of size l_0 is homogenized by diffusion on a time scale $\tau \sim l_0^2/D \sim 1/\nu$. The noise level controls the density of superthreshold perturbations received within a patch of size l_0 during the time τ. If the noise is not strong enough, the density of points with superthreshold perturbations is too small, and the patch becomes subthreshold after the homogenization and it decays. If the noise level is sufficiently large, then the amount of perturbations within the patch during the time τ is sufficient to stay above the threshold after the homogenization, forming a survived excitation center. Afterwards, the dynamics becomes essentially deterministic as it is dominated by the excitable reaction as in the noise-free media. At a stronger stirring rate ν, a larger noise level is required to generate denser and more frequent superthreshold perturbations, so that sufficient simultaneous threshold crossings occur within a smaller patch of l_0 over a shorter time τ to form an excitation center for CGE. A rather sharp transition (upper boundary in Fig. 6) is thus observed as a result of the homogenization of the perturbations over the finite length scale l_0. The coherence is lost when the dynamics is strongly influenced by noise during the relaxation period.

These results provide an explanation for experimentally observed stirring effects in the excitable Belousov-Zhabotinsky reaction [22]. Fluctuations of local reaction rates or heat release can induce oscillations of larger and more erratic periods with increasing stirring rate, which become quenched at strong stirring rates [22].

Noisy fluctuations are key elements in the dynamics of ecosystems [23,24]. The birth and death processes of individuals are intrinsically stochastic. The interaction of oceanic zooplankton with fish, which are far from being uniformly distributed, also introduces randomness [24]. Recently, Vilar *et al.* showed that fluctuations *and* turbulent stirring in the standard prey-predator models, around a stable state, are able to account for field observation of oceanic plankton patchiness at mesoscales [24]. This regime corresponds to the vicinity of HSS of Eqs. (1) and (2). The excitability is more relevant for plankton bloom situations [21,25] (mainly in spring and summer) similar to that induced by the ocean fertilization experiments [20]. Furthermore, several parameters can change irregularly in space and time. One of them is the depth of the mixed layer. Within this layer there is strong vertical mixing; therefore, its depth controls the average amount of light received and thus the growth rate of the phytoplankton. Another parameter crucial for the phytoplankton productivity is the iron concentration, so that adding small amounts of iron to the surface of the ocean can induce a mesoscale bloom [20]. Elevated iron concentrations have been observed in surface waters of the equatorial Pacific after rain [26]. There are also important regular changes of the parameters associated with the seasonal cycle. The combined effects of the fluctuations and the regular forcing may lead to a resonant response of the excitable media in the flow, which is under investigation in the context of oceanic plankton bloom.

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